ECE 821
Optimal Control and Variational Methods
Lecture Notes

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1 Introduction

Optimal control theory is the study of dynamic systems, where an "input function" is sought to minimize a given "cost function". The input and state of the system may be constrained in a variety of ways. In most applications, a general solution is desired that establishes the optimal input as a function of the system's initial condition. This, in turn, leads to a "feedback" formulation of the solution.

Let us consider a simple example.

Example 1.1 *A cart with mass $m = 1$ is free to slide without friction in one dimension along a horizontal surface. A force $u(t) \in \mathbb{R}$ may be applied in the direction of motion at time t. Letting $x(t)$ be the position of the mass, the system is described by the differential equation*

$$\ddot{x} = u. \tag{1.1}$$

We wish to steer the cart from the initial state

$$x(0) = a, \quad \dot{x}(0) = b$$

to the final state

$$x(1) = \dot{x}(1) = 0 \tag{1.2}$$

while minimizing the cost

$$J = \int_0^1 u^2 dt. \tag{1.3}$$

It turns out that the optimal input is

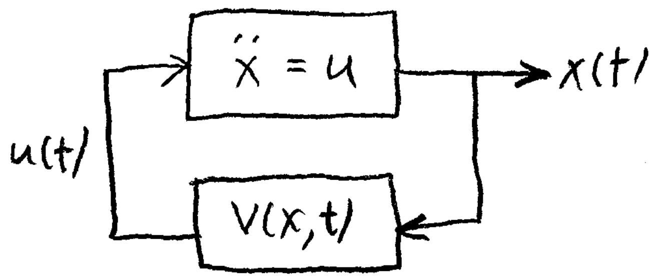
$$u^*(t) = 6(2a + b)t - 2(3a + 2b), \tag{1.4}$$

which leads to the corresponding solution

$$x^*(t) = (2a + b)t^3 - (3a + 2b)t^2 + bt + a, \tag{1.5}$$

of (1.1). The input u^* can be implemented as feedback according to

$$v\left(x, \dot{x}, t\right) = \frac{4(t-1)\dot{x} - 6x}{(t-1)^2}. \tag{1.6}$$



Note that (1.4)-(1.6) are consistent, since

$$v\left(x^*(t), \dot{x}^*(t), t\right) = \frac{4(t-1)\dot{x}^*(t) - 6x^*(t)}{(t-1)^2} = u^*(t)$$

(after considerable algebra). The feedback system is depicted below.

Numerous control problems based on the same differential equation (1.1) may be formulated by choosing different cost functions J and constraints.

	Cost	Constraints	Comments
1)	$\int_0^{t_f} u^2 dt$	$x(t_f) = \dot{x}(t_f) = 0$	energy optimal
2)	$\int_0^{\infty} (x^2 + u^2)\, dt$	–	optimal regulation
3)	t_f	$x(t_f) = \dot{x}(t_f) = 0$ $\lvert u(t)\rvert \leq 1$	time optimal
4)	$t_f + x^2(t_f) + \dot{x}^2(t_f)$ $+ \int_0^{t_f} (u^2 - 1)^2 dt$	–	"soft constraints"
5)	t_f	$x(t_f) = \dot{x}(t_f) = 0$ $\lvert u(t)\rvert \leq 1, \quad \lvert x(t)\rvert \leq M$	state constraint

The methods covered in this course are "variational" in nature. This means they involve derivatives on some level. Hence, we must assume all functions are "smooth". Typically, this means that they are either once or twice continuously differentiable. Consequently, this approach (as it is known today) is not appropriate for problems with switching or other kind of discontinuous nonlinearity. Such problems are theoretically difficult and lie at the frontier of control research.

The techniques we will study can be viewed as various generalizations of Lagrange multipliers. Thus, as a warm-up to optimal control theory, we will first review Lagrange multipliers in Euclidean space $\mathbb{R}^n$. Then we will advance to general vector spaces and develop the basic principles of the calculus of variations from the viewpoint of elementary functional analysis. The final step to optimal control is to impose a differential equation, such at (1.1), as a constraint on x and u. Many such problems (Example 1.1 and table entries 1, 2, and 4) can be solved using extended Lagrange multiplier techniques. However, we will see that there are additional problems (table entries 3 and 5), which do not fit well into any classical framework. For these we will introduce the Pontryagin Maximum Principle. Although the Maximum Principle maintains the flavor of Lagrange multipliers, it goes beyond ordinary functional analytic techniques.

The study of the calculus of variations began in 1696 when Johann Bernoulli posed the "brachistochrone" problem. In this problem, one wishes to find the curve connecting two given points in space such that a mass sliding without friction under the influence of gravity will move from the first point to the second in minimum time. Several mathematicians of the era, including Newton, responded with solutions, leading to the further development of the subject. Problems in optimal control did not receive attention until the 1950's, when emerging technologies motivated the unification of variational calculus with differential equations. A major breakthrough occurred in 1956 when the Russian mathematician Pontryagin published his "Maximum Principle".

2 Finite-Dimensional Optimization

Reference: Bartle, Section 42

2.1 Background

2.1.1 Euclidean Spaces

Let $\mathbb{R}^n$ be the set of all real n-dimensional vectors – i.e.

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \,\middle|\, x_1, \ldots, x_n \in \mathbb{R} \right\}.$$

Addition and scalar multiplication are carried out entrywise:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix},$$

$$\alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

Let

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \longleftarrow \; i\text{th entry} \\ \\ \\ \\ \end{matrix}$$

be the jth *unit vector. For any x, we may write*

$$x = \sum_{i=1}^{n} x_i e_i.$$

Matrices may also be considered members of Euclidean space. If M is $m \times n$, we write $M \in \mathbb{R}^{m \times n}$. Addition and scalar multiplication are carried out entrywise. Listing the columns of a matrix

$$M = \left[\begin{array}{ccc} \mu_1 & \cdots & \mu_n \end{array} \right],$$

we may identify M with the vector

$$v = \left[\begin{array}{c} \mu_1 \\ \vdots \\ \mu_n \end{array} \right] \in \mathbb{R}^{nm}. \tag{2.1}$$

2.1.2 Norms

A *norm on* $\mathbb{R}^n$ is any function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ such that for every $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

1) $\|x\| \geq 0$ with equality iff $x = 0$ (positive definite)
2) $\|\alpha x\| = |\alpha| \, \|x\|$ (scaling)
3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Here are some common norms:

Example 2.1 *1)* $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$ $(1 \leq p < \infty)$
2) Setting $p = 1$ *yields* $\|x\|_1 = \sum_{i=1}^n |x_i|$.
3) Setting $p = 2$ *yields* $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^T x}$. *(Euclidean norm)*
4) $\|x\|_\infty = \max_i |x_i|$
5) If $\|\cdot\|$ *is any norm and* T *is a square, nonsingular matrix, then* $\|x\|_T = \|Tx\|$ *is also a norm.*

The triangle inequality leads to other useful inequalities.

Theorem 2.1 *Let* $x, y \in \mathbb{R}^n$.
1) $\|x + y\| \geq |\|x\| - \|y\||$
2) $\|x\| - \|x + y\| \leq \|y\| \leq \|x\| + \|x + y\|$

Proof. 1) From the triangle inequality,

$$\|x\| = \|(x + y) - y\| \leq \|x + y\| + \|y\|,$$

so

$$\|x\| - \|y\| \leq \|x + y\|.$$

Interchanging x and y yields

$$\|x\| - \|y\| \geq -\|x + y\|,$$

which yields

$$|\|x\| - \|y\|| \leq \|x + y\|.$$

2) From 1),

$$-\|x + y\| \leq \|x\| - \|y\| \leq \|x + y\|.$$

Solve for $\|y\|$ on each side. ■

The next example shows that two norms on $\mathbb{R}^n$ may be related in a simple way.

Example 2.2 *1)* $\|x\|_\infty = \|\sum_{i=1}^n x_i e_i\|_\infty \le \sum_{i=1}^n |x_i| \, \|e_i\|_\infty = \sum_{i=1}^n |x_i| = \|x\|_1$
2) $\|x\|_1 = \sum_{i=1}^n |x_i| \le n \max_i |x_i| = n \, \|x\|_\infty$

Example 2.2 may be generalized as follows.

Theorem 2.2 *(Two-Norm Theorem) For any two norms* $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *on* $\mathbb{R}^n$*, there exists* $M < \infty$ *such that*

$$\|x\|_a \le M \, \|x\|_b$$

for every $x \in \mathbb{R}^n$*.*

Here are some other important inequalities.

Theorem 2.3 *(Cauchy-Schwarz Inequality) For any* $x, y \in \mathbb{R}^n$*,*

$$\left|x^T y\right| \le \|x\|_2 \, \|y\|_2 \, .$$

Proof. Luenberger, Section 2.10, Theorem 1 ■

Theorem 2.4 *For any norm* $\|\cdot\|$ *, there exists* $M < \infty$ *such that*

$$\left|x^T y\right| \le M \, \|x\| \, \|y\|$$

for every $x, y \in \mathbb{R}^n$*.*

Proof. By the two norm theorem, there exists M such that

$$\|x\|_2 \le \sqrt{M} \, \|x\|$$

for every x. From the Cauchy-Schwarz inequality,

$$\left|x^T y\right| \le \|x\|_2 \, \|y\|_2 \le \left(\sqrt{M} \, \|x\|\right) \left(\sqrt{M} \, \|y\|\right) = M \, \|x\| \, \|y\| \, .$$

■

For any norm, $x^* \in \mathbb{R}^n$, and $R > 0$ we may define the *ball centered at* x^* *with radius* R to be

$$B(x^*, R) = \left\{ x \,\middle|\, \|x - x^*\| < R \right\}.$$

Theorem 2.5 *For any norms* $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *there exists* $M < \infty$ *such that*

$$B_b\left(x, \frac{R}{M}\right) \subset B_a(x, R)$$

for every $x \in \mathbb{R}^n$ *and* $R > 0$*.*

Proof. By the two norm theorem, there exists $M < \infty$ such that

$$\|y - x\|_a \le M \, \|y - x\|_b$$

for every $x, y \in \mathbb{R}^n$. Hence,

$$\|y - x\|_b < \frac{R}{M}$$

implies

$$\|y - x\|_a \le M \|y - x\|_b < R.$$

■

The following fact will be useful later: For any $h \in \mathbb{R}^n$,

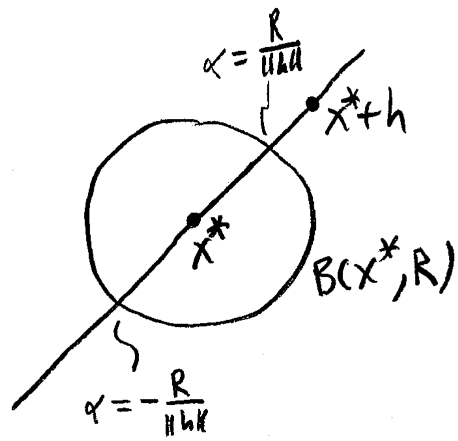
$$x^* + \alpha h \in B(x^*, R)$$

iff

$$|\alpha| \|h\| = \|\alpha h\| = \|(x^* + \alpha h) - x^*\| < R.$$

In other words, the line $x^* + \alpha h$ intersects $B(x^*, R)$ at those points corresponding to

$$\alpha \in \left(-\frac{R}{\|h\|}, \frac{R}{\|h\|}\right).$$



2.1.3 Matrix Norms

A matrix $M \in \mathbb{R}^{m\times n}$ may be thought of as a point in the Euclidean space $\mathbb{R}^{mn}$ by stacking the columns of M to form a vector. In this way, any vector norm can be applied to matrices in $\mathbb{R}^{m\times n}$. An alternative approach is to choose any vector norm $\|\cdot\|$ and define the corresponding *induced norm*

$$\|M\| = \max_{\|x\|=1} \|Ax\|$$

on $\mathbb{R}^{m\times n}$. One can show that the maximum always exists and that $\|M\|$ satisfies the three axioms of a norm. In addition, induced matrix norms enjoy the property

$$\|MN\| \le \|M\| \|N\|,$$

where N is any other matrix. In particular,

$$\|Mx\| \le \|M\| \|x\|$$

for any $x \in \mathbb{R}^n$.

2.2 Unconstrained Optimization in $\mathbb{R}^n$

2.2.1 Extrema

To discuss optimization, we need some basic definitions. A *cost function* is any function $J : \mathbb{R}^n \to \mathbb{R}$. We say J *achieves a global minimum at* x^* if $J(x^*) \le J(x)$ for every $x \in \mathbb{R}^n$. The vector x^* is a *point of global minimum.* For any norm $\|\cdot\|$, we say J *achieves a local minimum at* x^* (relative to a norm $\|\cdot\|$) if there exists $\varepsilon > 0$ such that $J(x^*) \le J(x)$ for every x satisfying $\|x - x^*\| < \varepsilon$. Note that this is the same as J achieving a global minimum on the ball $B(x^*, \varepsilon)$. We say the global minimum of J is *strict* if $J(x^*) < J(x)$ for every $x \in \mathbb{R}^n$. A local minimum is *strict* if there exists $\varepsilon > 0$ such that J has a strict global minimum on $B(x^*, \varepsilon)$. Note that a point of strict global minimum is unique. In the other cases (local or non-strict), x^* may not be unique. Similar definitions may be stated for maxima (global, local, and strict). In all cases, we say J achieves an *extremum* at x^*.

Although the definition of a local minimum appears to depend on the choice of norm, we can use the two norm theorem to prove that this is not the case.

Theorem 2.6 *If J has a local extremum at x^* relative to some norm, then J has a local extremum (of the same type) relative to every norm.*

Proof. We will prove the result for local minima. Other types of local extrema can be handled similarly. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms, and suppose J has a local minimum relative to $\|\cdot\|_a$. Then there exists $\varepsilon > 0$ such that $J(x^*) \le J(x)$ for $x \in B_a(x^*, \varepsilon)$. By Theorem 2.5,

$$B_b\left(x^*, \frac{\varepsilon}{M}\right) \subset B_a(x^*, \varepsilon),$$

so $J(x^*) \le J(x)$ for $x \in B_b\left(x^*, \frac{\varepsilon}{M}\right)$, making x^* a point of local minimum relative to $\|\cdot\|_b$. ■

2.2.2 Jacobians

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Writing f in detail,

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{bmatrix}.$$

Suppose all the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist at some $x^* \in \mathbb{R}^n$. Then we may define the *Jacobian* matrix

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{x=x^*}.$$

One can easily check that the usual rules of calculus apply (with certain modifications):

$$\frac{\partial}{\partial x}(f + g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} \quad (g : \mathbb{R}^n \to \mathbb{R})$$

$$\frac{\partial}{\partial x}(Mf) = M\frac{\partial f}{\partial x} \quad (M \text{ an } n \times m \text{ constant matrix}) \tag{2.2}$$

$$\frac{\partial}{\partial x}\left(f^T g\right) = g^T\frac{\partial f}{\partial x} + f^T\frac{\partial g}{\partial x} \tag{2.3}$$

$$\frac{\partial}{\partial y} f(g(y)) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \qquad (g : \mathbb{R}^m \to \mathbb{R}^n)$$

Expressions (2.2)-(2.3) lead to the special cases

$$\frac{\partial}{\partial x}(Mx) = M\frac{\partial x}{\partial x} = M \cdot I = M,$$

$$\frac{\partial}{\partial x}\left(x^T M x\right) = (Mx)^T \frac{\partial x}{\partial x} + x^T \frac{\partial}{\partial x}(Mx) = x^T\left(M^T + M\right).$$

2.2.3 Critical Points

If, for some point $x^* \in \mathbb{R}^n$, $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ exists and equals 0, we say x^* is a *critical point* of J. The following theorem generalizes a familiar fact from calculus.

Theorem 2.7 *If J achieves a local extremum at x^* and $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ exists, then x^* is a critical point of J.*

Proof. Suppose J achieves a local minimum at x^*. Then $J(x^*) \leq J(x)$ for $x \in B(x^*, \varepsilon)$. Let

$$J_i(\alpha) = J(x^* + \alpha e_i).$$

Since

$$x^* + \alpha e_i \in B(x^*, \varepsilon)$$

for

$$\alpha \in \left(-\frac{\varepsilon}{\|e_i\|}, \frac{\varepsilon}{\|e_i\|}\right), \tag{2.4}$$

we obtain

$$J_i(0) = J(x^*) \leq J(x^* + \alpha e_i) = J_i(\alpha)$$

on the interval (2.4). In other words, J_i achieves a local minimum at $\alpha = 0$. From calculus,

$$\left.\frac{\partial J}{\partial x_i}\right|_{x^*} = \left.\frac{dJ_i}{d\alpha}\right|_{\alpha=0} = 0.$$

Since i was arbitrary,

$$\left.\frac{\partial J}{\partial x}\right|_{x^*} = \left[\begin{array}{ccc} \left.\frac{\partial J}{\partial x_1}\right|_{x^*} & \cdots & \left.\frac{\partial J}{\partial x_n}\right|_{x^*}\end{array}\right] = 0.$$

The same argument works for local maxima. ■

A critical point x^* is called a *saddle point* if for every $\varepsilon > 0$ there exist $x, y \in B(x^*, \varepsilon)$ such that

$$J(x) < J(x^*) < J(y).$$

In other words, a saddle point is any critical point where J does not achieve a local extremum. From the two norm theorem, saddle points do not depend on the choice of norm. For $n = 1$, a saddle point is called an *inflection point*.

Example 2.3 *Let* $n = 2$ *and*

$$J(x) = \|x\|_2^4 - \|x\|_2^2 = \left(x_1^2 + x_2^2\right)^2 - \left(x_1^2 + x_2^2\right).$$

Then

$$\frac{\partial J}{\partial x} = 4\left(x_1^2 + x_2^2 - \frac{1}{2}\right)\left[\begin{array}{cc} x_1 & x_2 \end{array}\right],$$

so the critical points are $x = 0$ *and the points on the circle*

$$x_1^2 + x_2^2 = \frac{1}{2}.$$

Example 2.4 *Let*

$$J(x) = 4\|x\|_2^6 - 6\|x\|_2^4 + 3\|x\|_2^2.$$

Then

$$\frac{\partial J}{\partial x} = 24\left(x_1^2 + x_2^2 - \frac{1}{2}\right)^2\left[\begin{array}{cc} x_1 & x_2 \end{array}\right],$$

so the critical points are the same as in Example 2.3.

2.2.4 Hessians

For $J : \mathbb{R}^n \to \mathbb{R}$, the Jacobian $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ is a $1 \times n$ matrix (a row vector). Suppose there exists $\varepsilon > 0$ such that the Jacobian $\frac{\partial J}{\partial x}$ exists at every $x \in B(x^*, \varepsilon)$. Taking the transpose, we obtain the function $\left(\frac{\partial J}{\partial x}\right)^T : B(x^*, \varepsilon) \to \mathbb{R}^n$. If each entry of $\left(\frac{\partial J}{\partial x}\right)^T$ is differentiable at x^*, we may again take the Jacobian. This defines the *Hessian* matrix

$$\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*} = \frac{\partial}{\partial x}\left(\frac{\partial J}{\partial x}\right)^T\Bigg|_{x^*} = \left[\begin{array}{ccc} \left.\frac{\partial^2 J}{\partial x_1^2}\right|_{x^*} & \cdots & \left.\frac{\partial^2 J}{\partial x_1 \partial x_n}\right|_{x^*} \\ \vdots & & \vdots \\ \left.\frac{\partial^2 J}{\partial x_n \partial x_1}\right|_{x^*} & \cdots & \left.\frac{\partial^2 J}{\partial x_n^2}\right|_{x^*} \end{array}\right].$$

Here are some useful identities. Let $x, y \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$.

$$\frac{\partial^2}{\partial x^2}\left(y^T x\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(y^T x\right)\right)^T = \frac{\partial y}{\partial x} = 0$$

$$\frac{\partial^2}{\partial x^2}\left(x^T M x\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(x^T M x\right)\right)^T = \frac{\partial}{\partial x}\left(\left(M + M^T\right)x\right) = M + M^T$$

2.2.5 Definite Matrices

We say an $n \times n$ matrix P is *positive semidefinite* (written $P \geq 0$) if $x^T P x \geq 0$ for every $x \in \mathbb{R}^n$ and *positive definite* ($P > 0$) if $x^T P x > 0$ for $x \neq 0$. Similarly, P is *negative semidefinite* ($P \leq 0$) if $x^T P x \leq 0$ for every $x \in \mathbb{R}^n$ and *negative definite* ($P < 0$) if $x^T P x < 0$ for $x \neq 0$.

An $n \times n$ matrix Q is *symmetric* if $Q^T = Q$. For an arbitrary $n \times n$ matrix P, we may define the symmetric matrix

$$Q = \frac{1}{2}\left(P + P^T\right)$$

and rewrite the form

$$x^T P x = \frac{1}{2}\left(x^T P x + \left(x^T P x\right)^T\right) = x^T Q x.$$

Thus $P \geq 0$ iff $Q \geq 0$, etc. Typically, the Hessian $\frac{\partial^2 J}{\partial x^2}$ is symmetric, since $\frac{\partial^2 J}{\partial x_i \partial x_j} = \frac{\partial^2 J}{\partial x_j \partial x_i}$. However, this is not always the case.

Example 2.5 *Let*

$$J(x) = \begin{cases} 0, & x = 0 \\ \frac{x_1 x_2\left(x_1^2 - x_2^2\right)}{x_1^2 + x_2^2}, & else \end{cases}.$$

Then

$$\left.\frac{\partial J}{\partial x_1}\right|_{x_1=0} = -x_2, \quad \left.\frac{\partial J}{\partial x_2}\right|_{x_2=0} = x_1,$$

$$\left.\frac{\partial^2 J}{\partial x_1 \partial x_2}\right|_{x=0} = -1, \quad \left.\frac{\partial J}{\partial x_2 \partial x_1}\right|_{x=0} = 1,$$

$$\left.\frac{\partial^2 J}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Shortly we will state exact conditions under which $\frac{\partial^2 J}{\partial x^2}$ is symmetric.

Definiteness of a symmetric matrix can be described in terms of its eigenvalues.

Theorem 2.8 *Let Q be a symmetric matrix.*
1) The eigenvalues λ of Q are all real.
2) $Q \geq 0$ iff every eigenvalue satisfies $\lambda \geq 0$.
3) $Q > 0$ iff every eigenvalue satisfies $\lambda > 0$.
4) $Q \leq 0$ iff every eigenvalue satisfies $\lambda \leq 0$.
5) $Q < 0$ iff every eigenvalue satisfies $\lambda < 0$.
6) $Q > 0$ iff every leading principal minor of Q is positive.
7) $Q \geq 0$ iff every principal minor of Q is nonnegative.
8) $Q \leq 0$ iff $-Q \geq 0$
9) $Q < 0$ iff $-Q > 0$

Proof. 1) Consider any eigenvalue λ of Q and the corresponding eigenvector x. Then $Qx = \lambda x$ and

$$\lambda \|x\|_2^2 = \lambda x^T x = x^T Q x.$$

Hence,

$$\lambda = \frac{x^T Q x}{\|x\|_2^2} \tag{2.5}$$

is real.

2)-5) (Necessity) The sign of λ in (2.5) is inherited from Q.

8)-9) follow from the observation

$$x^T Q x = -x^T(-Q)x.$$

The remaining results require more advance matrix theory. ■

If Q is symmetric, Theorem 2.8, parts 2)-3) imply $Q > 0$ iff $Q \geq 0$ and $\det Q \neq 0$. Similarly, 4)-5) imply $Q < 0$ iff $Q \leq 0$ and $\det Q \neq 0$.

Example 2.6 *Consider the symmetric matrix*

$$Q = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The 3 *leading principal minors are*

$$m_1 = 4,$$
$$m_2 = \det \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix} = 3,$$
$$m_3 = \det Q = 2,$$

so

$$Q > 0.$$

Example 2.7 *Let*

$$Q = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}.$$

$Q \not\geq 0$ *since*

$$m_1 = -2.$$

Now consider

$$-Q = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$m_1 = 2,$$
$$m_2 = \det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = 1,$$

but

$$m_3 = \det(-Q) = -\det Q = 0,$$

so $Q \not< 0$. *The remaining (non-leading) principal minors are* 1, 1,

$$\det \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = 1,$$

$$\det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0.$$

Hence, $Q \leq 0$.

2.2.6 Continuity and Continuous Differentiability

Convergence $x \to x^*$ in $\mathbb{R}^n$ means $\|x - x^*\| \to 0$ for some norm. In view of the two-norm theorem, convergence holds relative to a particular norm iff $\|x - x^*\|_\infty \to 0$. But this is the same as $|x_i - x_i^*| \to 0$ for every i. Hence, $x \to x^*$ means $x_i \to x_i^*$ for $i = 1, \ldots, n$ – i.e. convergence is entrywise. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous at* x^* if $x \to x^*$ implies $f(x) \to f(x^*)$. If f is continuous at every x, we say f is *continuous* and write $f \in C^0$. Note that every norm $\|\cdot\|$ is a continuous function, since $x \to x^*$ implies

$$|\|x\| - \|x^*\|| \leq \|x - x^*\| \to 0,$$

so

$$\|x\| \to \|x^*\|.$$

For $n = 1$, the existence of $\left.\frac{df}{dx}\right|_{x^*}$ implies continuity of f at x^* by

$$\lim_{x \to x^*} (f(x) - f(x^*)) = \lim_{x \to x^*} (x - x^*) \frac{f(x) - f(x^*)}{x - x^*} \to 0 \cdot \left.\frac{df}{dx}\right|_{x^*} = 0.$$

Unfortunately, this fact breaks down for $n > 1$.

Example 2.8 *Let* $n = 2$, $x^* = 0$, *and*

$$f(x) = \begin{cases} \frac{x_1}{x_2}, & x_2 \neq 0 \\ 0, & x_2 = 0 \end{cases}.$$

Since $f \equiv 0$ *on each axis,*

$$\left.\frac{\partial f}{\partial x}\right|_{x^*} = \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{x^*} & \left.\frac{\partial f}{\partial x_2}\right|_{x^*} \end{bmatrix} = 0.$$

But every point on the line $x_2 = cx_1$ *yields*

$$f(x) = \frac{1}{c},$$

so every limit in $\mathbb{R}$ *is achievable as* $x \to 0$. *This makes* f *discontinuous at* $x = 0$.

If $\frac{\partial f}{\partial x}$ exists for all $x \in \mathbb{R}^n$ and the function $\frac{\partial f}{\partial x} : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ is continuous, then we say f is *continuously differentiable* and write $f \in C^1$.

Theorem 2.9 *If* $f \in C^1$, *then* $f \in C^0$.

Proof. Bartle, Theorem 41.2. ■

For $m = 1$, if the Hessian $\frac{\partial^2 J}{\partial x^2}$ exists for all $x \in \mathbb{R}^n$ and the function $\frac{\partial^2 J}{\partial x^2} : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is continuous, then we say J is *twice continuously differentiable* and write $J \in C^2$. If $J \in C^2$, then $\frac{\partial J}{\partial x} \in C^1$. From Theorem 2.9, we conclude that $\frac{\partial J}{\partial x} \in C^0$. Hence,

$$C^2 \subset C^1 \subset C^0.$$

It is worth noting that many familiar functions are in C^2. For example, polynomials, sinusoids, exponentials, and compositions of these functions all belong to C^2.

Theorem 2.10 *If* $J \in C^2$, *then* $\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*}$ *is symmetric for every* $x^* \in \mathbb{R}^n$.

Proof. Bartle, Theorem 40.8. ■

2.2.7 Second Derivative Conditions

Various conditions can be developed in terms of the Hessian matrix to distinguish the different kinds of critical points. First, some necessary conditions.

Theorem 2.11 *Suppose $J \in C^2$.*
1) If J achieves a local minimum at $x^* \in \mathbb{R}^n$, then $\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*} \geq 0$.
2) If J achieves a local maximum at $x^* \in \mathbb{R}^n$, then $\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*} \leq 0$.

There are also sufficient conditions.

Theorem 2.12 *Suppose $x^* \in \mathbb{R}^n$ is a critical point of J and that $J \in C^2$.*
1) If $\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*} > 0$, then J achieves a strict local minimum at x^*.
2) If $\left.\frac{\partial^2 J}{\partial x^2}\right|_{x^*} < 0$, then J achieves a strict local maximum at x^*.

Theorems 2.11 and 2.12 will be proven later in a more general context. The following examples show that, while these results can yield useful information, they do not always resolve the critical points completely.

Example 2.9 *We revisit Example 2.3.*

$$J(x) = \|x\|_2^4 - \|x\|_2^2$$

consists of polynomials, so $J \in C^2$. Then

$$\frac{\partial J}{\partial x} = 4\left(x_1^2 + x_2^2 - \frac{1}{2}\right)\left[\begin{array}{cc} x_1 & x_2 \end{array}\right],$$

$$\frac{\partial^2 J}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial J}{\partial x}\right)^T = \left[\begin{array}{cc} 12x_1^2 + 4x_2^2 - 2 & 8x_1x_2 \\ 8x_1x_2 & 4x_1^2 + 12x_2^2 - 2 \end{array}\right].$$

The critical points are $x = 0$ and $x_1^2 + x_2^2 = \frac{1}{2}$. In the first case,

$$\left.\frac{\partial^2 J}{\partial x^2}\right|_{x=0} = \left[\begin{array}{cc} -2 & 0 \\ 0 & -2 \end{array}\right] < 0.$$

According to Theorem 2.12, part 2), $x = 0$ achieves a strict local maximum. For the second case,

$$\left.\frac{\partial^2 J}{\partial x^2}\right|_{x_2 = \pm\sqrt{\frac{1}{2} - x_1^2}} = \left[\begin{array}{cc} 8x_1^2 & \pm 8x_1\sqrt{\frac{1}{2} - x_1^2} \\ \pm 8x_1\sqrt{\frac{1}{2} - x_1^2} & 4 - 8x_1^2 \end{array}\right].$$

Since $x_1 \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$, the principal minors satisfy

$$8x_1^2 \geq 0,$$

$$4 - 8x_1^2 \geq 0,$$

$$\det\left(\frac{\partial^2 J}{\partial x^2}\right) = 0.$$

Hence, $\frac{\partial^2 J}{\partial x^2} \geq 0$. From Theorem 2.11, each critical point on the circle is either a local minimum or a saddle point.

Example 2.10 *Working from Example 2.4,*

$$J(x) = 4\|x\|_2^6 - 6\|x\|_2^4 + 3\|x\|_2^2,$$

$$\frac{\partial J}{\partial x} = 24\left(x_1^2 + x_2^2 - \frac{1}{2}\right)^2 \left[\begin{array}{cc} x_1 & x_2 \end{array}\right],$$

$$\frac{\partial^2 J}{\partial x^2} = \left[\begin{array}{cc} 24\left(5x_1^2 + x_2^2 - \frac{1}{2}\right)\left(x_1^2 + x_2^2 - \frac{1}{2}\right) & 96x_1x_2\left(x_1^2 + x_2^2 - \frac{1}{2}\right) \\ 96x_1x_2\left(x_1^2 + x_2^2 - \frac{1}{2}\right) & 24\left(x_1^2 + 5x_2^2 - \frac{1}{2}\right)\left(x_1^2 + x_2^2 - \frac{1}{2}\right) \end{array}\right].$$

The critical points are given by $x = 0$ and $x_1^2 + x_2^2 = \frac{1}{2}$. In the first case,

$$\left.\frac{\partial^2 J}{\partial x^2}\right|_{x=0} = \left[\begin{array}{cc} 6 & 0 \\ 0 & 6 \end{array}\right] > 0,$$

so $x = 0$ achieves a strict local minimum. But

$$\left.\frac{\partial^2 J}{\partial x^2}\right|_{x_1^2+x_2^2=\frac{1}{2}} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right]$$

provides no information.

To fully resolve the critical points in Examples 2.9 and 2.10, one must resort to other (sometimes ad hoc) methods. In Example 2.9, it can be shown that the maximum at $x = 0$ is merely local and that every point on the circle achieves a global minima. Similarly in Example 2.10, $x = 0$ achieves a strict global minimum, while the circle consists entirely of saddle points.

2.3 Constrained Optimization in $\mathbb{R}^n$

2.3.1 Constrained Extrema

Let $\Omega \subset \mathbb{R}^n$. We say J achieves a *constrained global minimum at x^* subject to Ω* if $x^* \in \Omega$ and $J(x^*) \leq J(x)$ for every $x \in \Omega$. Ω is the *constraint set.* J achieves a *constrained local minimum at x^* subject to* Ω if there exists $\varepsilon > 0$ such that $J(x^*) \leq J(x)$ for every $x \in \Omega \cap B(x^*, \varepsilon)$. The minimum is *strict* if $J(x^*) < J(x)$ for $x = x^*$. Similar definitions can be stated for constrained maxima.

For a vector

$$v = \left[\begin{array}{c} v_1 \\ \vdots \\ v_m \end{array}\right] \in \mathbb{R}^m,$$

we write $v \geq 0$ if $v_i \geq 0$ for every i – i.e. v lies in the "first orthant". Similarly, we may write $v > 0$, $v \leq 0$, $v < 0$. The constraint set Ω is often defined by an equation or inequality involving a function $g : \mathbb{R}^n \to \mathbb{R}^m$. For example, we may set

$$\Omega = \left\{x \in \mathbb{R}^n \,\middle|\, g(x) = 0\right\},$$

$$\Omega = \left\{x \in \mathbb{R}^n \,\middle|\, g(x) \geq 0\right\},$$

$$\Omega = \left\{x \in \mathbb{R}^n \,\middle|\, g(x) > 0\right\}.$$

Theorem 2.13 *If J achieves an unconstrained extremum at $x^* \in \Omega$, then J achieves a constrained extremum (of the same type) at x^* subject to Ω.*

Proof. Suppose J achieves a global minimum at x^*. Then $J(x^*) \leq J(x)$ for $x \in \mathbb{R}^n$, so $J(x^*) \leq J(x)$ for $x \in \Omega$. For a local minimum, $J(x^*) \leq J(x)$ for $x \in B(x^*, \varepsilon)$, so $J(x^*) \leq J(x)$ for $x \in B(x^*, \varepsilon) \cap \Omega$. Other kinds of extrema are handled similarly. ■

2.3.2 Open Sets

The converse to Theorem 2.13 is obviously false in general. However, the converse statement does hold for local extrema, if we impose an additional assumption on Ω : A set Ω is *open* if for every $x \in \Omega$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega$. Note that, since $B(x, \varepsilon)$ depends on the choice of norm, the definition of an open set also appears to depend on the norm. Fortunately, the two norm theorem dispels this fear.

Theorem 2.14 *If Ω is open relative to some norm on $\mathbb{R}^n$, then Ω is open relative to every norm.*

Proof. Suppose Ω is open relative to $\|\cdot\|_a$ and let $x \in \Omega$. Then there exists $\varepsilon > 0$ such that

$$B_a(x, \varepsilon) \subset \Omega.$$

For any other norm $\|\cdot\|_b$, Theorem 2.5 guarantees that there exists $M < \infty$ such that

$$B_b\left(x, \frac{\varepsilon}{M}\right) \subset B_a(x, \varepsilon) \subset \Omega.$$

Since x was arbitrary, Ω is open relative to $\|\cdot\|_b$. ■

Example 2.11 *It is easy to show that $B(x, \varepsilon)$ (using any norm) and*

$$(0, \infty)^m = \left\{ v \in \mathbb{R}^m \,\middle|\, v > 0 \right\}$$

are open sets.

Theorem 2.15 *If Ω_1 and Ω_2 are open, then so are $\Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2$.*

Proof. If $x \in \Omega_1 \cup \Omega_2$, then $x \in \Omega_i$ for some i. Since Ω_i is open, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset \Omega_i \subset \Omega_1 \cup \Omega_2.$$

Now let $x \in \Omega_1 \cap \Omega_2$ and $\varepsilon_1, \varepsilon_2 > 0$ be such that $B(x, \varepsilon_i) \subset \Omega_i$ for each i. Then

$$B\left(x, \min_i \varepsilon_i\right) = B(x, \varepsilon_1) \cap B(x, \varepsilon_2) \subset \Omega_1 \cap \Omega_2.$$

■

The *inverse image* of a set $U \subset \mathbb{R}^m$ under f is

$$f^{-1}(U) = \left\{ x \,\middle|\, f(x) \in U \right\}.$$

Open sets provide an alternative description of continuity.

Theorem 2.16 *g is continuous iff $g^{-1}(U)$ is open for every open $U \subset \mathbb{R}^m$.*

It is useful to note that the definitions of continuity and continuous differentiability apply perfectly well to functions $g : U \to \mathbb{R}^m$ for any open U.

Now we return to optimization. Compare the following result with Theorem 2.13.

Theorem 2.17 *If $\Omega \subset \mathbb{R}^n$ is open, $x^* \in \Omega$, and J achieves a constrained local extremum at x^* subject to Ω, then J achieves an unconstrained local extremum (of the same type) at x^*.*

Proof. If J achieves a local minimum at x^*, then $J(x^*) \leq J(x)$ for every $x \in \Omega \cap B(x^*, \varepsilon)$. Since Ω and $B(x^*, \varepsilon)$ are open, so is $\Omega \cap B(x^*, \varepsilon)$. Hence, there exists $\delta > 0$ such that

$$B(x^*, \delta) \subset \Omega \cap B(x^*, \varepsilon)$$

and $J(x^*) \leq J(x)$ for $x \in B(x^*, \delta)$. The other types of local extrema are handled similarly. ■

It is worth noting at this point that, for a problem with an open constraint set Ω, the cost function J **need only be defined on** Ω. This is because all analytic arguments applied so far carry over perfectly well to $J : \Omega \to \mathbb{R}$. Although this idea may seem to be the same as merely restricting x to Ω, there actually is a generalization here: In some problems, $|J(x)| \to \infty$ as x tends to the boundary of Ω. In this case, J cannot be extended smoothly to all of $\mathbb{R}^n$.

Example 2.12 *Let*

$$J(x) = \frac{1}{\sqrt{1 - x^2}}$$

and $\Omega = (-1, 1)$. Note that J cannot be extended continuously outside $(-1, 1)$. The critical points are given by

$$J'(x) = \frac{x}{(1 - x^2)^{\frac{3}{2}}} = 0$$

or $x^* = 0$. The Hessian is

$$J''(x) = \frac{1 + 2x^2}{(1 - x^2)^{\frac{5}{2}}} > 0,$$

so x^* is a strict local minimum.

2.3.3 Strict Inequality Constraints

If g is continuous, then Theorem 2.16 guarantees that the constraint set

$$\Omega = g^{-1}\left((0, \infty)^m\right) = \left\{ x \,\middle|\, g(x) > 0 \right\}$$

is open. By Theorems 2.13 and 2.17, if $g(x^*) > 0$, then J achieves a constrained local extremum at x^* subject to $g(x) > 0$ iff J achieves an unconstrained local extremum at x^*. Hence, all previous results involving the Jacobian and Hessian carry over to this case.

Example 2.13 *As in Examples 2.3 and 2.9,*

$$J(x) = \|x\|_2^4 - \|x\|_2^2$$

achieves a strict local maximum at $x^* = 0$. If we set

$$g(x) = x_1^2 + x_2^2 - \frac{1}{2},$$

then $g(x^*) > 0$, so J achieves a strict constrained local maximum at $x^* = 0$ subject to $g(x) > 0$. However, setting

$$g(x) = \left(x_1 - \frac{1}{2\sqrt{2}}\right)^2 + x_2^2 - \frac{1}{8}$$

places $x^* \notin \Omega$, since $g(0) = 0$. The other critical points of J lie on the circle with radius $\frac{1}{2\sqrt{2}}$, in which case $g(x) \leq 0$. Hence, J has no critical points satisfying $g(x) > 0$ and, therefore, no constrained extrema.

2.3.4 Equality Constraints and Lagrange Multipliers

Equality constraints of the form $g(x) = 0$ may be handled using the technique of "Lagrange multipliers". For given J and g, define the *Lagrangian*

$$L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R},$$

according to

$$L(x, \mu, \lambda) = \mu J(x) - \lambda^T g(x).$$

The central idea is that, for x in the constraint set

$$\Omega = \left\{ x \,\middle|\, g(x) = 0 \right\},$$

we obtain

$$L(x, 1, \lambda) = J(x). \tag{2.6}$$

The amazing fact is that much can be learned about the **constrained** behavior of J by studying the **unconstrained** behavior of L. Furthermore, the Lagrangian idea can be generalized enormously, making it applicable to problems in the calculus of variations and optimal control.

We begin by establishing the main necessary condition for a constrained extremum x^*. We say a vector $x^* \in \Omega$ is *regular* if $\left.\frac{\partial g}{\partial x}\right|_{x^*}$ has rank m.

Theorem 2.18 *If $x^* \in \Omega$ is regular, then there exists $\varepsilon > 0$ such that $g(B(x^*, \varepsilon))$ is open.*

Proof. Bartle, Theorem 41.7. ■

Theorem 2.19 *(Lagrange Multipliers) Let $J, g \in C^1$. If J achieves a constrained local extremum at $x^* \in \mathbb{R}^n$ subject to $g(x) = 0$, then there exist $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^m$, not both zero, such that*

$$\left.\frac{\partial L}{\partial x}\right|_{(x^*, \mu, \lambda)} = 0. \tag{2.7}$$

If x^* is regular, then we may set $\mu = 1$.

Proof. Let

$$f(x) = \begin{bmatrix} J(x) \\ g(x) \end{bmatrix}.$$

Then $f \in C^1$ and

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial J}{\partial x} \\ \frac{\partial g}{\partial x} \end{bmatrix}.$$

If x^* is a regular point of f, then Theorem 2.18 states that $f(B(x^*,\varepsilon))$ is open. Since

$$f(x^*) \in f(B(x^*,\varepsilon)),$$

there exists $\delta > 0$ such that

$$B(f(x^*),\delta) \subset f(B(x^*,\varepsilon)).$$

But

$$\begin{bmatrix} J(x^*) + \alpha \\ 0 \end{bmatrix} = f(x^*) + \alpha e_1 \in B(f(x^*),\delta)$$

for

$$\alpha \in \left(-\frac{\delta}{\|e_1\|}, \frac{\delta}{\|e_1\|} \right),$$

so there exist $x, y \in B(x^*,\varepsilon)$ such that

$$g(x) = g(y) = 0,$$

$$J(x) < J(x^*) < J(y).$$

This contradicts the assumption that J achieves a constrained local extremum at x^*. Hence, x^* is not a regular point of f, so $\frac{\partial f}{\partial x}\Big|_{x^*}$ has linearly dependent rows. Thus there exist μ and λ, not both zero, such that

$$\frac{\partial L}{\partial x}\Big|_{(x^*,\mu,\lambda)} = \begin{bmatrix} \mu & \lambda^T \end{bmatrix} \begin{bmatrix} \frac{\partial J}{\partial x}\Big|_{x^*} \\ \frac{\partial g}{\partial x}\Big|_{x^*} \end{bmatrix} = 0. \tag{2.8}$$

If x^* is a regular point of g, then $\frac{\partial g}{\partial x}\Big|_{x^*}$ has linearly independent rows. If $\mu = 0$, then $\lambda \neq 0$ and

$$\lambda^T \frac{\partial g}{\partial x}\Big|_{x^*} = 0,$$

which is a contradiction. Hence, $\mu \neq 0$. Dividing (2.8) by μ and redefining μ and λ yields a solution with $\mu = 1$. ■

If x^* is regular, it is common practice to write $L(x^*,\lambda)$, rather than $L(x^*,1,\lambda)$. If x^* is not regular, then the rows of $\frac{\partial g}{\partial x}\Big|_{x^*}$ are linearly dependent. Hence, (2.7) may be solved by setting $\mu = 0$ and choosing any $\lambda \neq 0$ such that

$$\lambda^T \frac{\partial g}{\partial x}\Big|_{x^*} = 0.$$

This merely determines a line on which λ must reside, yielding no information about x^*. In contrast, regularity of x^* forces a relationship between J and g:

$$\frac{\partial J}{\partial x}\Big|_{x^*} = \lambda^T \frac{\partial g}{\partial x}\Big|_{x^*}. \tag{2.9}$$

Combining (2.9) with $g(x^*) = 0$ yields $n+m$ equations in $n+m$ variables. We conclude that **Lagrange multipliers provide useful information if and only if x^* is regular**. We say a vector $x^* \in \Omega$ is a *critical point of* L if x^* is regular and (2.9) holds for some $\lambda \in \mathbb{R}^m$.

Example 2.14 *Let $n = 2$, $m = 1$,*

$$J(x) = x_1,$$

$$g(x) = x_1^2 + x_2^2 - 1.$$

Then

$$\frac{\partial g}{\partial x} = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \neq 0$$

for points on the circle $g(x) = 0$. Hence, every point in the constraint set is regular. Equation (2.9) becomes

$$\begin{bmatrix} 1 & 0 \end{bmatrix} = \lambda \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}.$$

Since $\lambda = 0$ leads to a contradiction, we must have $x_2 = 0$. Then the constraint forces $x_1 = \pm 1$, which implies $\lambda = \pm\frac{1}{2}$.

Example 2.15 *Let*

$$J(x) = x_1^2 + x_2^2,$$

$$g(x) = x_1 x_2.$$

Then

$$\frac{\partial g}{\partial x} = \begin{bmatrix} x_2 & x_1 \end{bmatrix},$$

so every point is regular, except $x = 0$. For the regular points,

$$\begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_2 & x_1 \end{bmatrix},$$

and the constraint $g(x) = 0$ implies either $x_1 = 0$ or $x_2 = 0$ (but not both). In the former case, $2x_2 = 0$, which is a contradiction. In the latter case, $2x_1 = 0$, which is a also contradiction. The analysis rules out constrained extrema, except at $x = 0$.

2.3.5 Second Derivative Conditions

Theorem 2.20 *Let x^* be regular with $g(x^*) = 0$. If $L(\cdot, \lambda)$ achieves an unconstrained local extremum at x^* for some $\lambda \in \mathbb{R}^m$, then J achieves a constrained local extremum (of the same type) at x^* subject to $g(x) = 0$.*

Proof. By Theorem 2.13, L also achieves a constrained local extremum of the same type at x^* subject to $g(x) = 0$. As we noted in (2.6), $L(x, \lambda) = J(x)$, so J inherits the constrained extremum from L. ■

If $J, g \in C^2$, Theorem 2.20 allows us to apply the sufficient conditions from Theorem 2.12 to $L(\cdot, \lambda)$ by taking the second derivative

$$\frac{\partial^2 L}{\partial x^2} = \frac{\partial J}{\partial x^2} - \lambda^T \frac{\partial^2 g}{\partial x^2}.$$

Since constrained local extrema of J can only occur at critical points, it makes sense to find these first, along with the corresponding λ, and test each $L(\cdot, \lambda)$ at each x^*. Other values of x^* and λ are ruled out by Theorems 2.19 and 2.20.

Example 2.16 *Working from Example 2.14, we examine the critical points*

$$x^* = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}$$

and corresponding Lagrange multipliers $\lambda = \pm 1$. *This yields*

$$L(x, \lambda) = x_1 \mp \left(x_1^2 + x_2^2 - 1\right)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 1 \mp 2x_1 & \mp 2x_2 \end{bmatrix},$$

$$\frac{\partial^2 L}{\partial x^2} = \begin{bmatrix} \mp 2 & 0 \\ 0 & \mp 2 \end{bmatrix}.$$

In the first case, J *achieves a strict constrained local maximum at* x^**, while the second case yields a minimum.*

Unfortunately, the converse of Theorem 2.20 is not true: Even if J achieves a constrained extremum at x^*, x^* may be a saddle point of $L(\cdot, \lambda)$.

Example 2.17 *Let* $n = 2$, $m = 1$,

$$J(x) = x_1^3,$$

$$g(x) = x_1^2 + x_2^2 - 1.$$

Since

$$\frac{\partial g}{\partial x} = 2 \begin{bmatrix} x_1 & x_2 \end{bmatrix},$$

every $x \in \Omega$ *is regular. The Lagrangian*

$$L(x, \lambda) = x_1^3 - \lambda\left(x_1^2 + x_2^2 - 1\right)$$

has Jacobian

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 3x_1^2 - 2\lambda x_1 & -2\lambda x_2 \end{bmatrix},$$

so the critical points are

$$x = \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix}, \quad \lambda = 0,$$

$$x = \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix}, \quad \lambda = \pm \frac{3}{2}.$$

Examination of J *reveals that it achieves a strict global maximum at*

$$x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

But the Hessian of

$$L\left(x^*, \frac{3}{2}\right) = x_1^3 - \frac{3}{2}\left(x_1^2 + x_2^2 - 1\right)$$

at x^* *is*

$$\left.\frac{\partial^2 L}{\partial x^2}\right|_{x^*} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}.$$

By Theorem 2.11, x^* *is a saddle point of* $L\left(x, \frac{3}{2}\right)$*, not a maximum.*

In view of Example 2.17, we conclude that applying the second derivative necessary conditions of Theorem 2.11 to L may lead to incorrect results. The following table summarizes the cases where the second derivative necessary (Theorem 2.11) and sufficient (Theorem 2.12) conditions may be applied.

Applicability of Second Derivative Conditions

	Unconstrained	Constrained
Necessary	Yes	No
Sufficient	Yes	Yes

2.3.6 Non-Strict Inequality Constraints

The third kind of constraint we will examine is the non-strict inequality:

$$\Omega = \left\{ x \in \mathbb{R}^n \,\middle|\, g(x) \geq 0 \right\}.$$

Writing

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix},$$

we note that for each $x \in \Omega$ the functions g_i may evaluate to 0 or a positive number. One way to handle such problems is to decompose Ω into several pieces. To do so, consider all partitions of $\{1, \ldots, m\}$. That is, choose any subset

$$\pi \subset \{1, \ldots, m\}$$

and let

$$\pi_+ = \{1, \ldots, m\} - \pi.$$

From combinatorics, there are 2^m possible ways in which this can be done. For each partition, let

$$g_\pi = \begin{bmatrix} g_{i_1} \\ g_{i_2} \\ \vdots \end{bmatrix}; \quad i_1, i_2, \ldots \in \pi,$$

$$g_{\pi_+} = \begin{bmatrix} g_{j_1} \\ g_{j_2} \\ \vdots \end{bmatrix}; \quad j_1, j_2, \ldots \in \pi_+,$$

$$\Omega_\pi = \left\{ x \in \mathbb{R}^n \,\middle|\, g_\pi(x) = 0 \right\},$$

$$\Omega_{\pi_+} = \left\{ x \in \mathbb{R}^n \,\middle|\, g_{\pi_+}(x) > 0 \right\}.$$

Then Ω may be written as the disjoint union

$$\Omega = \bigcup_\pi \left(\Omega_\pi \cap \Omega_{\pi_+} \right). \tag{2.10}$$

Compare the next result to Theorem 2.17.

Theorem 2.21 *If g is continuous, $x^* \in \Omega_\pi \cap \Omega_{\pi_+}$, and J achieves a constrained local extremum at x^* subject to Ω, then J achieves a constrained local extremum (of the same type) at x^* subject to Ω_π.*

Proof. We will prove the result for non-strict local minima. The other types of local extrema are handled similarly. Suppose $J(x^*) \leq J(x)$ for every $x \in \Omega \cap B(x^*, \varepsilon)$. Since g is continuous, so is g_{π_+} and

$$\Omega_{\pi_+} = g_{\pi_+}^{-1}\left([0,\infty)^r\right)$$

is open. Hence, $\Omega_{\pi_+} \cap B(x^*, \varepsilon)$ is open, so there exists $\delta > 0$ such that

$$B(x^*, \delta) \subset \Omega_{\pi_+} \cap B(x^*, \varepsilon).$$

Consequently,

$$\Omega_\pi \cap B(x^*, \delta) \subset \Omega_\pi \cap \Omega_{\pi_+} \cap B(x^*, \varepsilon) \subset \Omega \cap B(x^*, \varepsilon),$$

so $J(x^*) \leq J(x)$ for

$$x \in \Omega_\pi \cap B(x^*, \delta).$$

■

Theorem 2.21 tells us that we can apply Lagrange multipliers in a piecemeal fashion by letting π range over all partitions. Setting

$$L_\pi(x, \lambda) = J(x) - \lambda^T g_\pi,$$

we say $x^* \in \Omega_\pi$ is a *critical point of* L if x^* is a critical point of L_π.

Theorem 2.22 *(Kuhn-Tucker Theorem) Let $J, g \in C^1$. If $x^* \in \Omega_\pi$ and J achieves a constrained local extremum at x^* subject to $g(x) \geq 0$, then x^* is a critical point of L. If the extremum is a constrained local minimum (maximum) at x^*, then $\lambda \geq 0$ ($\lambda \leq 0$).*

Proof. Apply Theorem 2.21 and Lagrange multipliers. The sign of λ is proven in Bartle, Corollary 42.13. ■

In this setting, $x^* \in \Omega_\pi \cap \Omega_{\pi_+}$ is *regular* if $\left.\frac{\partial g_\pi}{\partial x}\right|_{x^*}$ has rank r_π. As before, regularity allows us to set $\mu = 1$. For non-strict inequality constraints, λ plays the same role as the Hessian in that its sign provides a necessary condition that distinguishes between minima and maxima. For regular points, the critical point equation reduces to

$$\left.\frac{\partial J}{\partial x}\right|_{x^*} = \lambda^T \left.\frac{\partial g_\pi}{\partial x}\right|_{x^*}.$$

A sufficient condition may also be stated. Compare Theorem 2.23 and Theorem 2.20.

Theorem 2.23 *Let $x^* \in \Omega_\pi$ be regular.*
1) If $L_\pi(\cdot, \lambda)$ achieves an unconstrained local minimum at x^* for some $\lambda \geq 0$, then J achieves a constrained local minimum (of the same type) at x^* subject to $g(x) \geq 0$.
2) If $L_\pi(\cdot, \lambda)$ achieves an unconstrained local maximum at x^* for some $\lambda \leq 0$, then J achieves a constrained local minimum (of the same type) at x^* subject to $g(x) \geq 0$.

Proof. 1) By Theorem 2.13, L also achieves a constrained local extremum of the same type at x^* subject to $g(x) \geq 0$. For all $x \in \Omega$,

$$\lambda^T g_\pi(x) = \sum \lambda_i g_{\pi i}(x) \geq 0.$$

Hence, there exists $\varepsilon > 0$ such that, for any $x \in \Omega \cap B(x^*, \varepsilon)$,

$$\begin{aligned} J(x^*) &= L_\pi(x^*, \lambda) + \lambda^T g_\pi(x^*) \\ &= L_\pi(x^*, \lambda) \\ &\leq L_\pi(x, \lambda) \\ &\leq L_\pi(x, \lambda) + \lambda^T g_\pi(x) \\ &= J(x). \end{aligned}$$

2) Similar to 1). ■

Combining the last three results, we may approach problems with non-strict inequality constraints as follows. 1) Decompose Ω as in (2.10). 2) Apply the Kuhn-Tucker theorem to find critical points x^* and corresponding values of λ. 3) Based on λ and the Hessian for each x^*, apply Theorem 2.23 in an attempt to prove that J achieves a minimum or maximum at x^*.

Example 2.18 *Let* $n = 2$,

$$J(x) = 2x_1 - x_2,$$

$$g(x) = \begin{bmatrix} -x_1^2 + x_2 \\ x_1 - x_2^2 \end{bmatrix}.$$

Case I: $\pi = \phi$

In this case, the constraint is $g(x^*) > 0$ *determines an open set, so we simply apply*

$$\frac{\partial J}{\partial x} = \begin{bmatrix} 2 & -1 \end{bmatrix} = 0,$$

which yields no critical point.

Case II: $\pi = \{1\}$

The equality constraint is

$$g_\pi(x) = -x_1^2 + x_2,$$

$$\frac{\partial g_\pi}{\partial x} = \begin{bmatrix} -2x_1 & 1 \end{bmatrix},$$

so every point is regular. We need to solve

$$\begin{bmatrix} 2 & -1 \end{bmatrix} = \lambda \begin{bmatrix} -2x_1 & 1 \end{bmatrix},$$

$$-x_1^2 + x_2 = 0.$$

This leads to

$$x_1 = x_2 = 1.$$

But then

$$x_1 - x_2^2 = 0,$$

which violates $g_{\pi_+}(x) > 0$. Again, this case yields no critical point.

Case III: $\pi = \{2\}$

$$g_\pi(x) = x_1 - x_2^2$$

$$\frac{\partial g_\pi}{\partial x} = \begin{bmatrix} 1 & -2x_2 \end{bmatrix}$$

Every point is regular.

$$\begin{bmatrix} 2 & -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 & -2x_2 \end{bmatrix}$$

$$x_1 - x_2^2 = 0$$

$$x_1 = \frac{1}{16}, \quad x_2 = \frac{1}{4}, \quad \lambda = 2$$

There is one critical point, which satisfies the necessary condition for a constrained local minimum. The Hessian is

$$\frac{\partial^2 L_\pi}{\partial x^2} = \frac{\partial}{\partial x}\left(\begin{bmatrix} 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & -2x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix},$$

which yields no information.

Case IV: $\pi = \{1, 2\}$

In this case, $g_\pi = g$. Solving $g(x) = 0$ leads to only two solutions:

$$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The Jacobian

$$\frac{\partial g_\pi}{\partial x} = \begin{bmatrix} -2x_1 & 1 \\ 1 & -2x_2 \end{bmatrix}$$

is nonsingular in both cases, so both points are regular. For the first point, solving

$$\begin{bmatrix} 2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} -2x_1 & 1 \\ 1 & -2x_2 \end{bmatrix}$$

yields

$$\lambda = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

so x^* is a saddle. The second point yields

$$\lambda = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

so x^* satisfies the necessary condition for a constrained local maximum. The Hessian is

$$\frac{\partial^2 L_\pi}{\partial x^2} = \frac{\partial}{\partial x}\left(\begin{bmatrix} 2 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} -2x_1 & 1 \\ 1 & -2x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},$$

which yields no information.

2.3.7 Mixed Constraints

More generally, an optimization problem can have a combination of equality, non-strict inequality, and strict inequality constraints:

$$\Omega = \left\{ x \in \mathbb{R}^n \,\middle|\, g_1(x) = 0, \quad g_2(x) \geq 0, \quad g_3(x) > 0 \right\}.$$

It should be transparent at this point, that such problems can be handled by decomposing the set $\{g_2 \geq 0\}$ into pieces and applying Lagrange multipliers to each $\{g_1 = g_{2\pi} = 0\}$. Then the Lagrangian is

$$L(x, \lambda_1, \lambda_2) = J(x) - \lambda_1^T g_1(x) - \lambda_2^T g_{2\pi}(x).$$

The Hessian of $\begin{bmatrix} g_1 \\ g_{2\pi} \end{bmatrix}$ along with sign of λ_2 correlate with the type of extremum. J need only be defined on the open set $\{g_3 > 0\}$.

3 Calculus of Variations

References: Luenberger, Chapter 7; Gelfand and Fomin, Chapters 1-3

3.1 Background

3.1.1 Vector Spaces

A *real linear space* or *vector space* is a set X along with two operations $+$ (vector addition) and $\cdot$ (scalar multiplication) such that, for any $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

1) $x + y = y + x$
2) $(x + y) + z = x + (y + z)$
3) $\alpha(x + y) = \alpha x + \alpha y$
4) $(\alpha + \beta)x = \alpha x + \beta x$
5) $(\alpha\beta)x = \alpha(\beta x)$
6) $1 \cdot x = x$

Furthermore, there must exist a vector $0 \in X$ such that, for any $x \in X$,

7) $0 \cdot x = 0$
8) $0 + x = x$

A vector space X that is not $\mathbb{R}^n$ for some n is said to be *infinite dimensional.* This is because no basis for X contains finitely many vectors.

Example 3.1 *The following constructions are easily shown to be vector spaces by checking the 8 axioms.*
1) $\mathbb{R}^n$ *for* $n = 1, 2, 3, \ldots$
2) $X = \{x : [0,1] \to \mathbb{R}\}$ *using pointwise operations*

$$(x + y)(t) = x(t) + y(t),$$

$$(\alpha x)(t) = \alpha x(t).$$

3) $X = \left\{ x : [0,1] \to \mathbb{R} \,\middle|\, x \in C^0 \right\}$ *using pointwise operations.*
4) $X = \left\{ x : [0,1] \to \mathbb{R} \,\middle|\, x \in C^k \right\}$ *using pointwise operations. (*C^k *means* k *times continuously differentiable.)*
5) $X = \left\{ x : [0,1] \to \mathbb{R} \,\middle|\, \int_{-\infty}^{\infty} x^2(t)\, dt < \infty \right\}$ *using pointwise operations. Such functions are said to belong to* L^2.
6) $X = \left\{ x : [0,1] \to \mathbb{R} \,\middle|\, \int_{-\infty}^{\infty} |x(t)|^p\, dt < \infty \right\}$ *using pointwise operations. Such functions are said to belong to* L^p.
7) Replace $[0,1]$ *in 2)-7) by any interval (finite or infinite) in* $\mathbb{R}$.
8) Replace $\mathbb{R}$ *in 2)-7) by* $\mathbb{R}^n$. *In 6) and 7) the integrals become*

$$\int_{-\infty}^{\infty} \|x(t)\|_2^2\, dt < \infty,$$

$$\int_{-\infty}^{\infty} \|x(t)\|_p^p\, dt < \infty.$$

9) For any vector spaces X *and* Y, *consider the Cartesian product*

$$X \times Y = \left\{ (x,y) \,\middle|\, x \in X, \quad y \in Y \right\}.$$

Then $X \times Y$ *is a vector space using the operations*

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$\alpha(x,y) = (\alpha x, \alpha y).$$

The zero vector is $0 = (0,0)$. *An important special case is*

$$X^2 = X \times X.$$

Cases 2)-8) are examples of *function spaces.*

3.1.2 Norms

Norms on an arbitrary vector space X are formally defined as for $\mathbb{R}^n$:

1) $\|x\| \geq 0$ with equality iff $x = 0$ (positive definite)
2) $\|\alpha x\| = |\alpha| \, \|x\|$ (scaling)
3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Certain norms make use of the following fact.

Theorem 3.1 *If* $x : [0,1] \to \mathbb{R}$ *is continuous, then there exist* $t_1, t_2 \in [0,1]$ *such that* $x(t_1) \leq x(t) \leq x(t_2)$ *for every* $t \in [0,1]$. *In other words,* x *achieves a maximum and minimum.*

Proof. Bartle, Theorem 22.6. ■

We say a function x is *bounded* if there exists $M < \infty$ such that $|x(t)| < M$ for every t. It follows from Theorem 3.1 that a continuous function defined on $[0,1]$ must be bounded.

Example 3.2 *Here are some examples of common norms. In each case, we assume* $x : [0,1] \to \mathbb{R}$.
1) C^0

$$\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|.$$

2) C^1

$$\|x\|_{C^1} = \max_{0 \le t \le 1} |x(t)| + \max_{0 \le t \le 1} \left| \dot{x}(t) \right|.$$

3) C^k

$$\|x\|_{C^k} = \sum_{i=1}^{k} \max_{0 \le t \le 1} \left| x^{(i)}(t) \right|.$$

4) L^2

$$\|x\|_2 = \sqrt{\int_0^1 x^2 dt}.$$

5) L^p

$$\|x\|_p = \left(\int_0^1 |x|^p \, dt \right)^{\frac{1}{p}}.$$

The definition of convergence in X carries over from $\mathbb{R}^n$. The notation $x \to x^*$ means $\|x - x^*\| \to 0$. Unfortunately, the two norm theorem does not hold for infinite-dimensional spaces.

Example 3.3 *Let* $X =$ *continuous functions* $x : [0,1] \to \mathbb{R}$, *and consider the two norms*

$$\|x\|_\infty = \max_{0 \le t \le 1} |x(t)|,$$

$$\|x\|_2 = \sqrt{\int_0^1 x^2 dt}.$$

The functions $x_n(t) = e^{-nt}$ *have norms*

$$\|x_n\|_\infty = 1,$$

$$\|x_n\|_2 = \sqrt{\int_0^1 e^{-2nt} dt} = \sqrt{\frac{1 - e^{-2n}}{2n}} \to 0$$

as $n \to \infty$. *If*

$$\|x_n\|_\infty \le M \|x_n\|_2$$

for some $M < \infty$, *then*

$$M \ge \frac{\|x_n\|_\infty}{\|x_n\|_2} \to \infty,$$

which is a contradiction.

3.1.3 Functionals

A *functional* on X is any function $f : X \to \mathbb{R}$. Normally, X will be a function space, so a functional is a "function of functions". A functional is *linear* if

1) $f(\alpha x) = \alpha f(x)$ (homogeneous)
2) $f(x+y) = f(x) + f(y)$ (additive)

for every $x, y \in X$ and $\alpha \in \mathbb{R}$. Note that any homogeneous function satisfies

$$f(0) = f(0 \cdot x) = 0 \cdot f(x) = 0.$$

A functional $K : X^2 \to \mathbb{R}$ is *bilinear* if it is linear in each argument (with the other fixed). Finally, a functional $f : X \to \mathbb{R}$ is *quadratic* if there exists a bilinear functional K such that

$$f(x) = K(x, x)$$

for every $x \in X$. Note that any quadratic functional satisfies

$$f(\alpha x) = K(\alpha x, \alpha x) = \alpha K(x, \alpha x) = \alpha^2 K(x, x) = \alpha^2 f(x),$$

$$f(0) = f(0 \cdot x) = 0^2 \cdot f(x) = 0.$$

As in $\mathbb{R}^n$, we say a functional f is *continuous* if $x \to x^*$ implies $f(x) \to f(x^*)$.

Theorem 3.2 *Let f be a functional on $X = \mathbb{R}^n$.*
1) f is linear iff there exists $v \in \mathbb{R}^n$ such that $f(x) = v^T x$ for every $x \in X$.
2) f is quadratic iff there exists $P \in \mathbb{R}^{n \times n}$ such that $f(x) = x^T P x$ for every $x \in X$.
3) If f is linear or quadratic, then it is continuous.

Proof. 1) Suppose f is linear, and let

$$v = \begin{bmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{bmatrix}.$$

Then

$$v^T x = \sum_{i=1}^{n} x_i f(e_i) = f\left(\sum_{i=1}^{n} x_i e_i\right) = f(x).$$

Conversely, if $f(x) = v^T x$, then

$$f(\alpha x) = v^T(\alpha x) = \alpha\left(v^T x\right) = \alpha f(x),$$

$$f(x+y) = v^T(x+y) = v^T x + v^T y = f(x) + f(y).$$

2) Now suppose f is quadratic. Then $f(x) = K(x, x)$ for some bilinear K. Setting

$$P = \begin{bmatrix} K(e_1, e_1) & \cdots & K(e_1, e_n) \\ \vdots & & \vdots \\ K(e_n, e_1) & \cdots & K(e_n, e_n) \end{bmatrix},$$

we obtain

$$\begin{aligned}
x^T P x &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j K(e_i, e_j) \\
&= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{n} x_j K(e_i, e_j) \right) \\
&= \sum_{i=1}^{n} x_i K \left(e_i, \sum_{j=1}^{n} x_j e_j \right) \\
&= K \left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} x_j e_j \right) \\
&= K(x, x).
\end{aligned}$$

Conversely, if $f(x) = x^T P x$, set $K(x, y) = x^T P y$. Then

$$K(\alpha x, y) = (\alpha x)^T P y = \alpha \left(x^T P y \right) = \alpha K(x, y),$$

$$K(x + y, z) = (x + y)^T P z = x^T P z + y^T P z = K(x, z) + K(y, z),$$

so K is linear in its first argument. By a similar calculation, K is linear in its second argument, making K bilinear. Since $f(x) = K(x, x)$, f is quadratic.

3) From Theorem 2.4, $x \to x^*$ implies

$$\left| v^T (x - x^*) \right| \leq M \|v\| \|x - x^*\| \to 0,$$

$$\begin{aligned}
\left| (x - x^*)^T P (x - x^*) \right| &\leq M \|x - x^*\| \|P(x - x^*)\| \\
&\leq M \|P\| \|x - x^*\|^2 \\
&\to 0
\end{aligned}$$

(using an appropriate matrix norm). Hence, both functionals are continuous at every $x^* \in X$. ■

Unfortunately, linear and quadratic functions on infinite-dimensional spaces X may not be continuous.

Example 3.4 *1) Let $X = C^1$ functions $x : [0, 1] \to \mathbb{R}$, $\|x\| = \|x\|_\infty$, and $f(x) = \dot{x}(0)$. Then*

$$f(\alpha x) = \frac{d}{dt} (\alpha x) \Big|_{t=0} = \alpha \dot{x}(0) = \alpha f(x),$$

$$f(x + y) = \frac{d}{dt} (x + y) \Big|_{t=0} = \dot{x}(0) + \dot{y}(0) = f(x) + f(y),$$

so f is linear. Consider

$$x_\varepsilon(t) = \varepsilon \sin \frac{t}{\varepsilon}$$

and note that

$$\|x_\varepsilon\|_\infty = \varepsilon \to 0$$

as $\varepsilon \to 0^+$. *But*

$$\dot{x}_\varepsilon(t) = \cos\frac{t}{\varepsilon},$$

$$f(x_\varepsilon) = \dot{x}_\varepsilon(0) = 1 \not\to 0 = f(0).$$

Hence, f *is discontinuous at* $x = 0$.
2) Under the conditions of 1), let $f(x) = \dot{x}^2(0)$. *By similar arguments,* f *is discontinuous at* $x = 0$.

For linear functionals, it suffices to check continuity at any one point (e.g. $x = 0$).

Theorem 3.3 *If* f *is a linear functional on a normed linear space* X *and* f *is continuous at some* $x^* \in X$, *then* f *is continuous on* X.

Proof. Suppose f is linear and continuous at x^*. Let $y^* \in X$, $y \to y^*$, and

$$z = y - y^* + x^*.$$

Then

$$\|z - x^*\| = \|y - y^*\| \to 0,$$

so $z \to x^*$. Hence,

$$f(y) - f(y^*) = f(y - y^*) = f(z - x^*) = f(z) - f(x^*) \to 0,$$

so f is continuous at y^*. Since y^* was arbitrary, f is continuous on X. ■

3.2 Unconstrained Optimization in X

3.2.1 Extrema

The definitions of the various kinds of extrema carry over verbatim to infinite-dimensional spaces. A typical optimization problem on X is the same as for $\mathbb{R}^n$ – i.e. to find the extrema of J on X subject to some constraints. We will see that the many of the methods we have already encountered (Lagrangians, Jacobians, Hessians, etc.) all generalize to functionals on normed linear spaces.

Unfortunately, not all results that hold in $\mathbb{R}^n$ carry over to infinite-dimensional spaces. One such result is the two norm theorem. Example 3.3 can be extended to show that the set of local extrema may depend on the choice of norm.

Example 3.5 *Let*

$$J(x) = \|x\|_\infty - \|x\|_\infty^2$$

and $x^* = 0$. *For* $x \in B_\infty(x^*, 1)$ *with* $x \neq 0$,

$$\|x\|_\infty < 1,$$

$$\|x\|_\infty^2 < \|x\|_\infty,$$

$$J(x^*) = 0 < J(x),$$

so J *achieves a strict local minimum at* x^* *relative to* $\|\cdot\|_\infty$. *As in Example 3.3, let* $x_n(t) = e^{-nt}$. *Then*

$$\|x_n\|_2 \to 0$$

as $n \to \infty$, *so*

$$2x_n \in B_2(x^*, \varepsilon)$$

for any $\varepsilon > 0$ *and sufficiently large* n. *But*

$$J(x_n) = \|2x_n\|_\infty - \|2x_n\|_\infty^2 = 2 - 4 = -2$$

for all n, *so*

$$J(x^*) > J(2x_n),$$

contradicting the definition of a local minimum. In fact, x^* *is not a local extremum of any type relative to* $\|\cdot\|_2$, *since the constant functions* $\frac{1}{n} \to 0$, *but*

$$J\left(\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n^2} > J(x^*).$$

On the other hand, if J achieves a global extremum at x^*, then J restricted to any ball $B(x^*\varepsilon)$ achieves a global extremum at x^*, regardless of the choice of norm. Hence, every global extremum is local relative to every norm. The choice of norm is not relevant when searching for global extrema.

3.2.2 Differentiation of Functionals

Let $\|\cdot\|$ be any norm on X. A functional J is *Gateaux differentiable* at $x \in X$ if there exists a functional $\delta J(x) : X \to \mathbb{R}$ such that

$$\frac{J(x + \alpha h) - J(x)}{\alpha} \to \delta J(x) h$$

as $\alpha \to 0$ for every $h \in X$. The functional $\delta J(x)$ is the *Gateaux derivative of* J *at* x. ($\delta J(x)$ is also called the *first variation* of J.) Note that the Gateaux derivative is the same as the first *directional derivative*

$$\delta J(x) h = \frac{d}{d\alpha} J(x + \alpha h)\Big|_{\alpha=0}.$$

A functional J is *Frechet differentiable at* $x \in X$ (relative to $\|\cdot\|$) if there exists a continuous linear functional $J'(x) : X \to \mathbb{R}$ such that

$$\frac{J(x+h) - J(x) - J'(x) h}{\|h\|} \to 0$$

as $h \to 0$. The functional $J'(x)$ is the *Frechet derivative of* J *at* x. Note that linearity and continuity are not part of the definition of $\delta J(x)$ as they are with $J'(x)$. Indeed, the next example demonstrates that $\delta J(x)$ may not have either property.

Example 3.6 *Let* $X = \mathbb{R}^2$, $x^* = 0$, *and*

$$J(x) = \begin{cases} \frac{x_1^2}{x_2}, & x_2 \neq 0 \\ 0, & x_2 = 0 \end{cases}.$$

Then $J(x^*) = 0$ *and*

$$J(x^* + \alpha h) = \begin{cases} \alpha \frac{h_1^2}{h_2}, & h_2 \neq 0 \\ 0, & h_2 = 0 \end{cases}.$$

For $h_2 = 0$,

$$\frac{J(x^* + \alpha h) - J(x^*)}{\alpha} = 0 = J(h).$$

For $h_2 \neq 0$,

$$\frac{J(x^* + \alpha h) - J(x^*)}{\alpha} = \frac{h_1^2}{h_2} = J(h).$$

Hence,

$$\delta J(0)\, h = J(h).$$

But

$$J\left(\left[\begin{array}{c} 1 \\ 0 \end{array}\right]\right) + J\left(\left[\begin{array}{c} 0 \\ 1 \end{array}\right]\right) = 0 \neq 1 = J\left(\left[\begin{array}{c} 1 \\ 1 \end{array}\right]\right),$$

so $\delta J(0)$ *is not linear. Also,*

$$J\left(\left[\begin{array}{c} \alpha \\ \alpha^2 \end{array}\right]\right) = 1 \not\to 0 = J(0)$$

as $\alpha \to 0$, *so* $\delta J(0)$ *is not continuous.*

All Gateaux derivatives do share part of the definition of linearity.

Theorem 3.4 *If* $\delta J(x)$ *exists, then it is homogeneous.*

Proof. If $\alpha = 0$,

$$\delta J(x)(\alpha h) = \lim_{\beta \to 0} \frac{J(x + \beta(\alpha h)) - J(x)}{\beta} = \lim_{\beta \to 0} \frac{J(x) - J(x)}{\beta} = \alpha \delta J(x)\, h.$$

For $\alpha \neq 0$,

$$\delta J(x)(\alpha h) = \lim_{\beta \to 0} \frac{J(x + \beta(\alpha h)) - J(x)}{\beta} = \alpha \lim_{\beta \to 0} \frac{J(x + \alpha\beta h) - J(x)}{\alpha\beta} = \alpha \delta J(x)\, h.$$

■

The advantage of Gateaux derivatives is that they are easy to calculate. The following fact provides a convenient method of finding Frechet derivatives.

Theorem 3.5 *If* $J'(x)$ *exists, then so does* $\delta J(x)$, *and the two functionals coincide.*

Proof. Choose any $h \in X$. If $h = 0$, then

$$\frac{J(x + \alpha h) - J(x)}{\alpha} = 0 = J'(x)\, h.$$

For $h \neq 0$, note that $\alpha h \to 0$ as $\alpha \to 0^+$, so

$$\frac{J(x + \alpha h) - J(x) - J'(x)(\alpha h)}{\|\alpha h\|} \to 0.$$

Multiplication by $\|h\|$ yields

$$\frac{J(x + \alpha h) - J(x)}{\alpha} - J'(x)\, h \to 0,$$

so

$$J'(x)h = \lim_{\alpha \to 0^+} \frac{J(x+\alpha h) - J(x)}{\alpha} = \delta J(x,h).$$

■

In particular, Theorem 3.5 implies that, whenever it exists, the Frechet derivative is unique. Note that the definition of $\delta J(x)$ does not depend on the choice of norm. Hence, all norms under which the Frechet derivative exists yield the same functional $J'(x) = \delta J(x)$. If the directional derivative is not continuous and linear, Theorem 3.5 implies that $J'(x)$ does not exist for any norm.

Compare the following result to Theorem 2.9.

Theorem 3.6 *If f is Frechet differentiable at $x \in X$, then f is continuous at x.*

Proof. Frechet differentiability implies

$$f(x+h) - f(x) - f'(x)h = \|h\| \frac{f(x+h) - f(x) - f'(x)h}{\|h\|} \to 0$$

as $h \to 0$. Since $f'(x)$ is continuous, $f'(x)h \to 0$, so $f(x+h) \to f(x)$. ■

Let us look at some simple, but common, special cases.

Theorem 3.7 *If f is constant, then it is Frechet differentiable with $\delta f(x) = 0$ for every $x \in X$.*

Proof. The result follows from the simple observation

$$\frac{f(x+h) - f(x)}{\|h\|} = 0.$$

■

Theorem 3.8 *Suppose f is linear.*
1) $\delta f(x) = f$ for every $x \in X$.
2) If f is continuous, then it is Frechet differentiable for every $x \in X$.

Proof. 1)

$$\frac{f(x+\alpha h) - f(x)}{\alpha} = \frac{f(x) + \alpha f(h) - f(x)}{\alpha} = f(h)$$

as $\alpha \to 0$.

2)

$$\frac{f(x+h) - f(x) - f(h)}{\|h\|} = \frac{f(x) + f(h) - f(x) - f(h)}{\|h\|} = 0.$$

■

Theorem 3.9 *If f is quadratic and continuous, then there exists $M < \infty$ such that $|f(x)| < M$ for every $x \in B(0,1)$.*

Proof. Since f is continuous, $f^{-1}(B(0,1))$ is open. Hence, there exists $\varepsilon > 0$ such that $|f(x)| < 1$ for every $x \in B(0,\varepsilon)$. Set

$$M = \frac{1}{\varepsilon^2}.$$

For $x = 0$,

$$|f(x)| = 0 < M.$$

For $x \in B(0,1) - \{0\}$, let

$$y = \frac{\varepsilon}{\|x\|} x.$$

Then $\|y\| = \varepsilon$, so

$$|f(x)| = \frac{\|x\|^2}{\varepsilon^2} |f(y)| < \frac{\|x\|^2}{\varepsilon^2} = M.$$

■

Theorem 3.10 *Suppose f is quadratic, K is bilinear, and $f(x) = K(x,x)$.*
1) $\delta f(x) h = K(x,h) + K(h,x)$ for every $x \in X$.
2) If K is continuous (on X^2), then f is Frechet differentiable.

Proof. 1) The Gateaux derivative is the limit

$$\begin{aligned} \frac{f(x+\alpha h) - f(x)}{\alpha} &= \frac{K(x+\alpha h, x+\alpha h) - K(x,x)}{\alpha} \\ &= \frac{\alpha K(x,h) + \alpha K(h,x) + \alpha^2 K(h,h)}{\alpha} \\ &= K(x,h) + K(h,x) + \alpha K(h,h) \\ &\to K(x,h) + K(h,x) \end{aligned}$$

as $\alpha \to 0$.

2) $\delta f(x)$ is obviously linear. If K is continuous, then $\delta f(x)$ is continuous for every x. From Theorem 3.9,

$$\begin{aligned} \frac{f(x+h) - f(x) - \delta f(x) h}{\|h\|} &= \frac{K(x+h, x+h) - K(x,x) - (K(x,h) + K(h,x))}{\|h\|} \\ &= \frac{K(h,h)}{\|h\|} \\ &= \|h\| K\left(\frac{h}{\|h\|}, \frac{h}{\|h\|}\right) \\ &= \|h\| f\left(\frac{h}{\|h\|}\right) \\ &< \|h\| M \\ &\to 0 \end{aligned}$$

as $h \to 0$, so $f'(x) = \delta f(x)$. ■

Theorem 3.11 *Differentiation is invariant under linear combinations.*

Proof. Let f and g be differentiable in any sense and consider $\phi = af + bg$, where $a, b \in \mathbb{R}$. For the first Gateaux derivative,

$$\frac{\phi(x+\alpha h) - \phi(x)}{\alpha} = a\frac{f(x+\alpha h) - f(x)}{\alpha} + b\frac{g(x+\alpha h) - g(x)}{\alpha} \to a\delta f(x) h + b\delta g(x) h.$$

For the first Frechet derivative,

$$\begin{aligned}
&\frac{\phi(x+h)-\phi(x)-(a\delta f(x)h+b\delta g(x)h)}{\|h\|}\\
&=a\frac{f(x+h)-f(x)-\delta f(x)h}{\|h\|}+b\frac{g(x+h)-g(x)-\delta g(x)h}{\|h\|}\\
&\to 0.
\end{aligned}$$

For the second Gateaux derivative,

$$\begin{aligned}
&\frac{\phi(x+\alpha h)-\phi(x)-(a\delta f(x)h+b\delta g(x)h)}{\alpha^2}\\
&=a\frac{f(x+\alpha h)-f(x)-\alpha\delta f(x)h}{\alpha^2}+b\frac{g(x+\alpha h)-g(x)-\alpha\delta g(x)h}{\alpha^2}\\
&\to a\delta^2 f(x)h+b\delta^2 g(x)h.
\end{aligned}$$

For the second Frechet derivative,

$$\begin{aligned}
&\frac{\phi(x+h)-\phi(x)-(a\delta f(x)h+b\delta g(x)h)-\frac{1}{2}\left(a\delta^2 f(x)h+b\delta^2 g(x)h\right)}{\|h\|^2}\\
&=a\frac{f(x+h)-f(x)-\delta f(x)h-\frac{1}{2}\delta^2 f(x)h}{\|h\|^2}+b\frac{g(x+h)-g(x)-\delta g(x)h-\frac{1}{2}\delta^2 g(x)h}{\|h\|^2}\\
&\to 0.
\end{aligned}$$

■

In view of Theorem 3.11, any sum of a constant, linear, and quadratic functional can be handled similarly.

3.2.3 The Case $X=\mathbb{R}^n$

Let us examine how our new definitions of the derivative relate to $X=\mathbb{R}^n$.

Theorem 3.12 *If $X=\mathbb{R}^n$ and $\delta J(x^*)$ exists, then $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ exists.*

Proof. By definition,

$$\left.\frac{\partial J}{\partial x_i}\right|_{x^*}=\left.\frac{d}{d\alpha}J(x^*+\alpha e_i)\right|_{\alpha=0}$$

for every i, so $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ exists. ■

Unfortunately, $\delta J(x^*)$ and $\left.\frac{\partial J}{\partial x}\right|_{x^*}$ may not coincide in the sense that

$$\delta J(x^*)h\neq\left.\frac{\partial J}{\partial x}\right|_{x^*}h$$

for certain h. One such functional is given in Example 3.6, where

$$\left.\frac{\partial J}{\partial x}\right|_{x^*}=0,\quad \delta J(0)=J\neq 0.$$

For finite-dimensional optimization, these issues are resolved when $J\in C^1$.

Theorem 3.13 *Let $X = \mathbb{R}^n$. If $J \in C^1$, then $J'(x)$ exists for every $x \in X$ and*

$$J'(x^*)\,h = \delta J(x^*)\,h = \frac{\partial J}{\partial x}\Big|_{x^*} h$$

for every $h \in \mathbb{R}^n$.

Proof. Bartle, Theorem 39.9 and Corollary 39.7. ■

3.2.4 Differentiation Examples

Let us examine some typical derivative calculations.

Example 3.7 *Let $X = C^1$ functions $x : [0,1] \to \mathbb{R}$ and*

$$J(x) = \int_0^1 \left(x^2 + \dot{x}^2\right) dt.$$

Then

$$\begin{aligned}
\delta J(x)\,h &= \frac{d}{d\alpha} J(x + \alpha h)\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \int_0^1 \left((x + \alpha h)^2 + \left(\dot{x} + \alpha \dot{h}\right)^2 \right) dt\Big|_{\alpha=0} \\
&= \int_0^1 \frac{d}{d\alpha} \left((x + \alpha h)^2 + \left(\dot{x} + \alpha \dot{h}\right)^2 \right)\Big|_{\alpha=0} dt \\
&= 2 \int_0^1 \left(xh + \dot{x}\dot{h} \right) dt.
\end{aligned}$$

Example 3.8 *Let $X =$ continuous functions $x : [0,1] \to \mathbb{R}$ and*

$$J(x) = \int_0^1 \frac{x^2}{1 + x^2} dt.$$

Then

$$\begin{aligned}
\delta J(x)\,h &= \frac{d}{d\alpha} \int_0^1 \frac{(x + \alpha h)^2}{1 + (x + \alpha h)^2} dt\Big|_{\alpha=0} \\
&= \int_0^1 \frac{2xh\left(1 + x^2\right) - x^2 (2xh)}{\left(1 + x^2\right)^2} dt \\
&= 2 \int_0^1 \frac{xh}{\left(1 + x^2\right)^2} dt.
\end{aligned}$$

We will prove later that the functionals in Examples 3.7 and 3.8 are Frechet differentiable.

3.2.5 Critical Points

For unconstrained optimization problems, the necessary condition for an extremum can be stated directly in terms of the Gateaux derivative. We say $x^* \in X$ is a *critical point* of J if $\delta J(x^*)$ exists and $\delta J(x^*)h = 0$ for every $h \in X$. In this case, we write $\delta J(x^*) = 0$.

Theorem 3.14 *If J achieves a local extremum at $x^* \in X$ and $\delta J(x^*)$ exists, then x^* is a critical point of J.*

Proof. Let $h \in X$. If J achieves a local extremum at x^*, then $\alpha = 0$ is a local extremum of $J(x^* + \alpha h)$. Hence,

$$\delta J(x^*)h = \frac{d}{d\alpha} J(x^* + \alpha h)\Big|_{\alpha=0} = 0.$$

■

Example 3.9 *Let $X = \mathbb{R}^n$ and $J \in C^2$. According to Theorem 3.13, local extrema can only be achieved at the solutions of $\frac{\partial J}{\partial x}\Big|_{x^*} = 0$.*

3.2.6 Euler's Equation

Now we can state the simplest "calculus of variations" problem. Let $X = C^1$ functions $x : [0,1] \to \mathbb{R}$ with $\|\cdot\| = \|\cdot\|_{C^1}$, and

$$J(x) = \int_0^1 F\left(x, \dot{x}, t\right) dt,$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ lies in C^1. We wish to find all local extrema. In many problems of this type, F depends only on the first two arguments, and we say F is *time-invariant*. Otherwise, it is *time-varying*.

The Gateaux derivative of J is

$$\begin{aligned}\delta J(x^*)h &= \frac{d}{d\alpha}\int_0^1 F\left(x^* + \alpha h, \dot{x}^* + \alpha \dot{h}, t\right) dt\Big|_{\alpha=0} \\ &= \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} h + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \dot{h}\right) dt.\end{aligned}$$

Note that, as the Jacobians of F are taken, x, $\dot{x}$, and t are treated as **independent real variables**.

The next result is central to the analysis of integral cost functionals.

Theorem 3.15 *(Fundamental Lemma of the Calculus of Variations) Suppose $f : [0,1] \to \mathbb{R}$ is continuous and*

$$\int_0^1 fh dt = 0$$

for every C^2 function $h : [0,1] \to \mathbb{R}$ with $h(0) = h(1) = 0$. Then $f \equiv 0$.

Proof. Suppose $f \not\equiv 0$. Then there exist $a, b \in [0,1]$ with $a \neq b$ such that $f(t) \neq 0$ and has constant sign for $t \in [a,b]$. Choose h with the same sign as f on (a,b) and $f(t) = 0$ elsewhere. Then

$$\int_0^1 fh dt = \int_a^b fh dt > 0.$$

In view of the contradiction, $f \equiv 0$. ■

By using the fundamental lemma, we can characterize the critical points more explicitly.

Theorem 3.16 *x^* is a critical point of J iff*

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} = \frac{d}{dt}\left(\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right) \tag{3.1}$$

for every $t \in [0,1]$ and

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} = \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} = 0. \tag{3.2}$$

Proof. (Sufficient) Substituting (3.1)-(3.2) into (3.3) yields $\delta J(x^*)h = 0$.
(Necessary) Integration by parts yields

$$\delta J(x^*)h = \int_0^1 \left(\left.\frac{\partial F}{\partial x}\right|_{x^*} - \frac{d}{dt}\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right) h\,dt + \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} h(1) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} h(0). \tag{3.3}$$

Restricting attention to h with $h(0) = h(1) = 0$, the fundamental lemma implies

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} - \frac{d}{dt}\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*} = 0.$$

Hence, for arbitrary h,

$$\delta J(x^*)h = \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} h(1) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} h(0).$$

For any $v \in \mathbb{R}^n$, we may choose h such that $h(0) = v$ and $h(1) = 0$, yielding

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} v = 0.$$

Hence,

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} = 0.$$

By a similar argument,

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} = 0.$$

■

Condition (3.1) is called *Euler's equation* or the *Euler-Lagrange equation.* Equations (3.2) are *boundary conditions.* Any solution of Euler's equation is an *extremal.* In calculating (3.1), we may apply the chain rule to obtain the second-order differential equation

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} = \left.\frac{\partial^2 F}{\partial \dot{x}\partial x}\right|_{x^*} \dot{x}^* + \left.\frac{\partial^2 F}{\partial \dot{x}^2}\right|_{x^*} \ddot{x}^* + \left.\frac{\partial^2 F}{\partial \dot{x}\partial t}\right|_{x^*}. \tag{3.4}$$

Euler's equation will often be nonlinear, making it difficult to obtain a closed-form solution. In some cases, numerical solution is the only possibility.

We first examine some problems where Euler's equation is linear.

Example 3.10 *Let*

$$F\left(x, \dot{x}\right) = x^2 + x\dot{x} + \dot{x}^2.$$

Then

$$\frac{\partial F}{\partial x} = 2x + \dot{x},$$
$$\frac{\partial F}{\partial \dot{x}} = x + 2\dot{x},$$

so Euler's equation is

$$2x + \dot{x} = \frac{d}{dt}\left(x + 2\dot{x}\right) = \dot{x} + 2\ddot{x}$$

or, equivalently,

$$\ddot{x} = x. \tag{3.5}$$

The extremals are all the functions of the form

$$x^*(t) = ae^t + be^{-t}.$$

Applying the boundary conditions, we obtain

$$\dot{x}^*(t) = ae^t - be^{-t},$$

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*} = x^* + 2\dot{x}^* = 3ae^t - be^{-t},$$
$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*, t=0} = 3a - b = 0,$$
$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*, t=1} = 3ae - be^{-1} = 0.$$

In matrix form,

$$\begin{bmatrix} 3 & -1 \\ 3e & -e^{-1} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

But

$$\det \begin{bmatrix} 3 & -1 \\ 3e & -e^{-1} \end{bmatrix} = 3\left(e - e^{-1}\right) = 7.05,$$

so $a = b = 0$*, making* $x^* \equiv 0$ *the only critical point.*

Example 3.11 *Let* $J(x)$ *be the arc length of the graph of* x *:*

$$ds = \sqrt{dt^2 + dx^2} = \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$J(x) = \int_0^1 \sqrt{1 + \dot{x}^2}\, dt$$

Then

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1 + \dot{x}^2}}.$$

Euler's equation is

$$0 = \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = \frac{\ddot{x}}{\left(1+\dot{x}^2\right)^{\frac{3}{2}}}$$

or

$$\ddot{x} = 0.$$

The extremals are

$$x^*(t) = a + bt.$$

The boundary conditions yield

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} = \left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} = \frac{b}{\sqrt{1+b^2}} = 0$$

or $b = 0$. *The critical points are the constants* $x^* \equiv a$.

Example 3.12 *Let* $y, z \in C^1$ *and*

$$F\left(x, \dot{x}\right) = \left(x - y(t)\right)^2 + \left(\dot{x} - z(t)\right)^2.$$

J *expresses the trade-off between making* x *close to* y *and* $\dot{x}$ *close to* z. *Then*

$$\frac{\partial F}{\partial x} = 2\left(x - y(t)\right),$$

$$\frac{\partial F}{\partial \dot{x}} = 2\left(\dot{x} - z(t)\right).$$

Euler's equation is

$$2(x-y) = \frac{d}{dt}\left(2\left(\dot{x} - z\right)\right) = 2\left(\ddot{x} - \dot{z}\right)$$

or

$$\ddot{x} - x = \dot{z} - y.$$

The boundary conditions are

$$\dot{x}(0) = z(0),$$

$$\dot{x}(1) = z(1).$$

One way to proceed is to set $w = \dot{x}$ *and apply state-space theory:*

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\dot{z} - y\right)$$

$$\begin{aligned} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} &= \exp\left(t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} + \int_0^t \exp\left((t-\tau) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\dot{z}(\tau) - y(\tau)\right) d\tau \\ &= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \sinh(t-\tau) \\ \cosh(t-\tau) \end{bmatrix} \left(\dot{z}(\tau) - y(\tau)\right) d\tau \end{aligned}$$

$$w(0) = \dot{x}(0) = z(0)$$

$$w(1) = \dot{x}(1) = z(1)$$

$$w(1) = \begin{bmatrix} \sinh 1 & \cosh 1 \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} + \int_0^1 (\cosh(1-\tau)) \left(\dot{z}(\tau) - y(\tau)\right) d\tau$$

$$\begin{aligned} x(0) &= \frac{w(1) - (\cosh 1) w(0) - \int_0^1 (\cosh(1-\tau)) \left(\dot{z}(\tau) - y(\tau)\right) d\tau}{\sinh 1} \\ &= \frac{z(1) - (\cosh 1) z(0) - \int_0^1 (\cosh(1-\tau)) \left(\dot{z}(\tau) - y(\tau)\right) d\tau}{\sinh 1} \end{aligned}$$

$$x^*(t) = \begin{bmatrix} \cosh t & \sinh t \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \sinh(t-\tau) \\ \cosh(t-\tau) \end{bmatrix} \left(\dot{z}(\tau) - y(\tau)\right) d\tau$$

For example, setting $y(t) = t$ and $z(t) = 0$ yields

$$x^*(t) = \frac{e^{1-t} - e^t}{e+1} + t.$$

If Euler's equation is nonlinear and F is time-invariant, then the problem can often be made easier through a simple trick. Here the cost is

$$J(x) = \int_0^1 F\left(x, \dot{x}\right) dt. \tag{3.6}$$

Theorem 3.17 *(Beltrami Identity) If J has the form (3.6) and x^* is a critical point of J, then there exists $a \in \mathbb{R}$ such that*

$$F\left(x^*, \dot{x}^*\right) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*} \dot{x}^* = a \tag{3.7}$$

Proof. Applying calculus,

$$\begin{aligned} \frac{d}{dt}\left(F - \frac{\partial F}{\partial \dot{x}}\dot{x}\right) &= \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial \dot{x}}\ddot{x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\dot{x} - \frac{\partial F}{\partial \dot{x}}\ddot{x} \\ &= \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right)\dot{x}. \end{aligned} \tag{3.8}$$

For $x = x^*$, Euler's equation guarantees

$$\frac{d}{dt}\left(F\left(x^*, \dot{x}^*\right) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*} \dot{x}^*\right) = 0.$$

Then (3.7) follows by integration. ■

Note that equation (3.7) is a **first-order** differential equation. If we can solve for

$$\dot{x} = f(x),$$

then we may apply "separation of variables"

$$\int \frac{1}{f(x)} dx = t + b,$$

assuming the integral can be calculated in closed-form. Then we are faced with an equation

$$g(x) = t + b,$$

which must be solved for x. In applying this technique, one should keep in mind that expression (3.8) vanishes for any constant x. Hence, these functions will always show up as solutions of (3.7), but may not be solutions of Euler's equation.

Example 3.13 *Let*

$$F\left(x, \dot{x}\right) = x\sqrt{1 + \dot{x}^2}.$$

Euler's equation is

$$\sqrt{1 + \dot{x}^2} = \frac{d}{dt}\frac{x\dot{x}}{\sqrt{1 + \dot{x}^2}} = \frac{x\ddot{x} + \dot{x}^4 + \dot{x}^2}{\left(1 + \dot{x}^2\right)^{\frac{3}{2}}},$$

$$x\ddot{x} - \dot{x}^2 - 1 = 0,$$

which is hard. Applying the Beltrami identity,

$$x\sqrt{1 + \dot{x}^2} - \frac{x\dot{x}^2}{\sqrt{1 + \dot{x}^2}} = a$$

or

$$\frac{x}{\sqrt{1 + \dot{x}^2}} = a.$$

Solving for $\dot{x}$*, we obtain*

$$\dot{x} = \frac{\sqrt{x^2 - a^2}}{a}.$$

$x \equiv a$ *is a solution for every* $a \in \mathbb{R}$*. For* $a \not\equiv 0$*, separation of variables yields*

$$\int \frac{a}{\sqrt{x^2 - a^2}} dx = t + b + a \ln a.$$

(The constant of integration is written $b + a \ln a$ *to make the solution look better.) Integrating, we obtain*

$$a \ln\left(x + \sqrt{x^2 - a^2}\right) = t + b + a \ln a,$$

$$x = \frac{a}{2}\left(\exp\left(\frac{t+b}{a}\right) + \exp\left(-\frac{t+b}{a}\right)\right) = a \cosh\left(\frac{t+b}{a}\right).$$

Each extremal is a ***catenary****.*

3.2.7 Extensions

One may use the above techniques to derive Euler's equation and boundary conditions for a variety of similar problems:
1)

$$X = C^2 \text{ functions } x : [0,1] \to \mathbb{R}^n$$

Here we need to choose a norm on $\mathbb{R}^n$ and define

$$\|x\| = \max_t \|x(t)\| + \max_t \left\|\dot{x}(t)\right\| + \max_t \left\|\ddot{x}(t)\right\|.$$

In view of the two norm theorem, every norm on $\mathbb{R}^n$ yields essentially the same results.

$$J(x) = \int_0^1 F\left(x, \dot{x}\right) dt$$

$$\frac{\partial F}{\partial x}\Big|_{x^*} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*}\right)$$

$$\frac{\partial F}{\partial x}\Big|_{x=x^*,t=0} = \frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} = 0$$

The partial derivatives are actually $1 \times n$ Jacobians.
2)

$$X = C^3 \text{ functions } x : [0,1] \to \mathbb{R}$$

$$J(x) = \int_0^1 F\left(x, \dot{x}, \ddot{x}, t\right) dt$$

$$\begin{aligned}
\delta J(x)h &= \frac{d}{d\alpha}\int_0^1 F\left(x + \alpha h, \dot{x} + \alpha \dot{h}, \ddot{x} + \alpha \ddot{h}, t\right) dt\Big|_{\alpha=0} \\
&= \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} h + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \dot{h} + \frac{\partial F}{\partial \ddot{x}}\Big|_{x^*} \ddot{h}\right) dt
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\delta J(x)h = &\int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right)\right) h dt \\
&+ \left(\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right)\Big|_{t=1}\right) h(1) - \left(\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} - \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right)\Big|_{t=0}\right) h(0) \\
&+ \frac{\partial F}{\partial \ddot{x}}\Big|_{x=x^*,t=1} \dot{h}(1) - \frac{\partial F}{\partial \ddot{x}}\Big|_{x=x^*,t=0} \dot{h}(0)
\end{aligned}$$

By the fundamental lemma,

$$\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*}\right) + \frac{d^2}{dt^2}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right) = 0,$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} = \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right)\Big|_{t=0},$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} = \frac{d}{dt}\left(\frac{\partial F}{\partial \ddot{x}}\Big|_{x^*}\right)\Big|_{t=1},$$
$$\frac{\partial F}{\partial \ddot{x}}\Big|_{x=x^*,t=0} = \frac{\partial F}{\partial \ddot{x}}\Big|_{x=x^*,t=1} = 0.$$

3) Let

$$X = \ C^2 \text{ functions } x : [0,\infty) \to \mathbb{R} \text{ such that } x,\ \dot{x},\ \text{and } \ddot{x} \text{ are bounded},$$
$$\|x\| = \sup_t |x(t)| + \sup_t \left|\dot{x}(t)\right| + \sup_t \left|\ddot{x}(t)\right|,$$
$$J(x) = \int_0^\infty e^{-t} F\left(x, \dot{x}\right) dt,$$

where $F \in C^2$. From integration by parts,

$$\begin{aligned}\delta J(x)\, h &= \int_0^\infty e^{-t}\left(\frac{\partial F}{\partial x}\Big|_{x^*} h + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \dot{h}\right) dt \\ &= \int_0^\infty \left(e^{-t}\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(e^{-t}\frac{\partial F}{\partial \dot{x}}\Big|_{x^*}\right)\right) h dt + \lim_{t\to\infty}\left(e^{-t}\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*}\right) - \frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} h(0).\end{aligned}$$

Since x and $\dot{x}$ are bounded and $\frac{\partial F}{\partial \dot{x}}$ is continuous in x and $\dot{x}$, $\frac{\partial F}{\partial \dot{x}}$ is bounded (in t), so

$$\lim_{t\to\infty}\left(e^{-t}\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*}\right) = 0,$$
$$\delta J(x)\, h = \int_0^\infty e^{-t}\left(\frac{\partial F}{\partial x}\Big|_{x^*} + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*}\right)\right) h dt - \frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} h(0).$$

An extension of the fundamental lemma (Theorem 3.15) is required.

Theorem 3.18 *Suppose $f : [0,\infty) \to \mathbb{R}$ is continuous and*

$$\int_0^\infty f h dt = 0$$

for every $h \in X$ with $h(0) = 0$. Then $f \equiv 0$.

Proof. Same as for Theorem 3.15. ■

Theorem 3.18 yields

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0,$$
$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} = 0.$$

3.2.8 Second Derivatives

Reference: Gelfand and Fomin, Chapter 5

Let J be a functional on X having first Gateaux derivative $\delta J(x)$ for some $x \in X$. We say J is *twice Gateaux differentiable at* x if there exists a functional $\delta^2 J(x)$ such that

$$\frac{J(x+\alpha h) - J(x) - \alpha \delta J(x) h}{\alpha^2} \to \frac{1}{2}\delta^2 J(x) h \tag{3.9}$$

as $\alpha \to 0$. $\delta^2 J(x)$ is called the *second Gateaux derivative of* J *at* x. ($\delta^2 J(x)$ is also called the *second variation* of J.) Now suppose J has first Frechet derivative $J'(x)$ for some $x \in X$. We say J is *twice Frechet differentiable at* x (relative to a norm $\|\cdot\|$) if there exists a continuous bilinear functional K such that

$$\frac{J(x+h) - J(x) - J'(x) h - \frac{1}{2}K(h,h)}{\|h\|^2} \to 0 \tag{3.10}$$

as $h \to 0$. $J''(x) = K(x,x)$ is called the *second Frechet derivative of* J *at* x. Note that $J''(x)$ is quadratic.

If the *second directional derivative*

$$\frac{d^2}{d\alpha^2} J(x+\alpha h)\Big|_{\alpha=0}$$

exists for every $h \in X$, then it can be used to find $\delta^2 J(x)$.

Theorem 3.19 *(Taylor's Theorem) If $f : \mathbb{R} \to \mathbb{R}$ is k times differentiable at x, then*

$$\frac{f(x+h) - \sum_{i=0}^{k-1} \frac{h^i}{i!} f^{(i)}(x)}{h^k} \to \frac{1}{k!} f^{(k)}(x)$$

as $h \to 0$.

Theorem 3.20 *If the second directional derivative of J exists at $x \in X$, then so does $\delta^2 J(x)$ and the two functionals coincide.*

Proof. Let

$$f(\alpha) = J(x+\alpha h).$$

Applying Taylor's theorem to $k=2$,

$$\frac{f(\alpha) - f(0) - \alpha f'(0) - \frac{\alpha^2}{2} f''(0)}{\alpha^2} \to 0,$$

so

$$\frac{f(\alpha) - f(0) - \alpha f'(0)}{\alpha^2} \to \frac{1}{2} f''(0). \tag{3.11}$$

But

$$f'(0) = \delta J(x) h,$$

so comparing (3.9) and (3.11) yields the desired result. ■

Existence of the second directional derivative is actually stronger than existence of the second Gateaux derivative.

Example 3.14 *Let $X = \mathbb{R}$ and*

$$J(x) = \begin{cases} x^3 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

For $h \neq 0$,

$$\frac{J(\alpha h) - J(0)}{\alpha} = \alpha^2 h^3 \sin \frac{1}{\alpha h} \to 0$$

as $\alpha \to 0$, so

$$\delta J(0) h = \frac{d}{d\alpha} J(\alpha h) \Big|_{\alpha=0} = 0.$$

Furthermore,

$$\frac{J(\alpha h) - J(0) - \alpha \delta J(0) h}{\alpha^2} = \alpha h^3 \sin \frac{1}{\alpha h} \to 0,$$

so

$$\delta^2 J(0) h = 0.$$

Hence, the Gateaux derivative exists at $x = 0$ and equals the 0 functional. To find the second directional derivative, first calculate the first directional derivative as a function of α :

$$\frac{d}{d\alpha} J(\alpha h) = h^3 \frac{d}{d\alpha} \left(\alpha^3 \sin \frac{1}{\alpha h} \right) = 3\alpha^2 h^3 \sin \frac{1}{\alpha h} - \alpha h^2 \cos \frac{1}{\alpha h}$$

But

$$\frac{3\alpha^2 h^3 \sin \frac{1}{\alpha h} - \alpha h^2 \cos \frac{1}{\alpha h}}{\alpha} = \alpha h^3 \sin \frac{1}{\alpha h} - h^2 \cos \frac{1}{\alpha h}$$

does not converge as $\alpha \to 0$. Hence, $\frac{d}{d\alpha} J(\alpha h)$ is not differentiable at $\alpha = 0$ and the second directional derivative does not exist.

Theorem 3.21 *If $J''(x)$ exists, then so does $\delta^2 J(x)$ and the two functionals coincide.*

Proof. Choose any $h \in X$. If $h = 0$, then $J''(x) h = 0$, since $J''(x)$ is quadratic. Also,

$$\delta^2 J(x) h = \frac{d^2}{d\alpha^2} J(x) \Big|_{\alpha=0} = 0.$$

For $h \neq 0$, note that $\alpha h \to 0$ as $\alpha \to 0$, so

$$\frac{J(x + \alpha h) - J(x) - J'(x)(\alpha h) - \frac{1}{2} J''(x)(\alpha h)}{\|\alpha h\|^2} \to 0.$$

Multiplication by $\|h\|^2$ yields

$$\frac{J(x + \alpha h) - J(x) - \alpha J'(x) h - \frac{\alpha^2}{2} J''(x) h}{\alpha^2} \to 0. \tag{3.12}$$

By Theorem 3.5, $J'(x) = \delta J(x)$, so Taylor's theorem yields

$$\frac{J(x + \alpha h) - J(x) - \alpha J'(x) h - \frac{\alpha^2}{2} \delta^2 J(x) h}{\alpha^2} \to 0. \tag{3.13}$$

Dividing (3.13) by α^2 and subtracting from (3.12), we obtain

$$\frac{1}{2}\left(\delta^2 J(x)h - J''(x)h\right) - f(\alpha) \to 0.$$

Hence,

$$J''(x)h = \delta^2 J(x)h.$$

■

In a typical application of Theorems 3.20 and 3.21, one begins by finding the second directional derivative of J using ordinary calculus. This gives $\delta^2 J(x)$ as well as a candidate for $J''(x)$. Existence of $J''(x)$ must then be established by applying the definition (3.10).

Let us look at some special cases.

Theorem 3.22 *If f is constant, then it is twice Frechet differentiable with $f''(x) = 0$ for every $x \in X$.*

Proof. The result follows from

$$\frac{f(x+h) - f(x)}{\|h\|^2} = 0.$$

■

Theorem 3.23 *Suppose f is linear.*
1) $\delta^2 f(x) = 0$ for every $x \in X$.
2) If f is continuous, then it is twice Frechet differentiable for every $x \in X$.

Proof. 1)

$$\frac{f(x+\alpha h) - f(x) - \alpha f(h)}{\alpha^2} = 0$$

as $\alpha \to 0$.

2)

$$\frac{f(x+h) - f(x) - f(h)}{\|h\|^2} = \frac{f(x) + f(h) - f(x) - f(h)}{\|h\|^2} = 0.$$

■

Theorem 3.24 *Suppose K is bilinear and $f(x) = K(x,x)$ (so f is quadratic).*
1) $\delta^2 f(x)h = 2f(h)$ for every $x \in X$.
2) If K is continuous (on X^2), then f is twice Frechet differentiable.

Proof. 1)

$$\begin{aligned}\frac{f(x+\alpha h) - f(x) - \alpha \delta f(x)h}{\alpha^2} &= \frac{K(x+\alpha h, x+\alpha h) - K(x,x) - \alpha\left(K(x,h) + K(h,x)\right)}{\alpha^2} \\ &= \frac{\alpha^2 K(h,h)}{\alpha^2} \\ &= f(h).\end{aligned}$$

2) $\delta^2 f(x)$ is obviously quadratic. If K is continuous, then $\delta^2 f(x)$ is continuous for every x. The result follows from

$$\frac{f(x+h) - f(x) - \delta f(x)h - \frac{1}{2}\delta^2 f(x)h}{\|h\|^2} = \frac{K(h,h) - f(h)}{\|h\|^2} = 0.$$

■

In the finite-dimensional case, the second derivative reduces to the familiar quadratic form.

Theorem 3.25 *Let $X = \mathbb{R}^n$. If $J \in C^2$, then $J''(x)$ exists with*

$$J''(x)\,h = h^T \frac{\partial^2 J}{\partial x^2}\Big|_{x^*} h$$

for every $h \in \mathbb{R}^n$.

Proof. Since $C^2 \subset C^1$, Theorem 3.13 guarantees

$$\delta J(x^* + \alpha h)\,h = \frac{\partial J}{\partial x}\Big|_{x^*+\alpha h} h$$

for every $h \in \mathbb{R}^n$. Since $J \in C^2$, we may again differentiate to obtain

$$\begin{aligned}
\frac{d^2}{d\alpha^2} J(x^* + \alpha h)\Big|_{\alpha=0} &= \frac{d}{d\alpha}\left(\delta J(x^* + \alpha h)\,h\right)\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha}\left(\frac{\partial J}{\partial x}\Big|_{x^*+\alpha h} h\right)\Big|_{\alpha=0} \\
&= \frac{\partial}{\partial x}\left(\frac{\partial J}{\partial x}\Big|_{x^*} h\right) h \\
&= \frac{\partial}{\partial x}\left(h^T \frac{\partial J}{\partial x}^T\Big|_{x^*}\right) h \\
&= h^T \frac{\partial}{\partial x}\left(\frac{\partial J}{\partial x}^T\Big|_{x^*}\right) h \\
&= h^T \frac{\partial^2 J}{\partial x^2}\Big|_{x^*} h.
\end{aligned}$$

Existence of $J''(x)$ follows from Bartle, pp. 369-371. ■

3.2.9 Definite Quadratic Functionals

We say a quadratic functional $f : X \to \mathbb{R}$ is *positive semidefinite* (written $f \geq 0$) if $f(x) \geq 0$ for every $x \in X$ and *negative semidefinite* ($f \leq 0$) if $f(x) \leq 0$ for every $x \in X$.

Example 3.15 *For $X = \mathbb{R}^n$ and*

$$f(x) = x^T P x,$$

$f \geq 0$ iff $x^T P x \geq 0$ for every $x \in \mathbb{R}^n$. But this is just the definition of positive semidefiniteness of P. Similarly, $f \leq 0$ iff $P \leq 0$.

A quadratic functional f is *positive definite* (written $f > 0$) if there exists $\varepsilon > 0$ such that

$$f(x) > \varepsilon \|x\|^2$$

for every $x \neq 0$ and *negative definite* ($f < 0$) if there exists $\varepsilon > 0$ such that

$$f(x) < -\varepsilon \|x\|^2$$

for every $x \neq 0$. From the definitions, positive (negative) definiteness of f implies positive (negative) semidefiniteness of f. Note that definiteness (but not semidefiniteness) depends on the choice of norm. This fact can be seen explicitly in the following example.

Example 3.16 *Let $X =$ continuous functions $x : [0,1] \to \mathbb{R}$,*

$$\|x\|_\infty = \max_t |x(t)|,$$

$$\|x\|_2 = \sqrt{\int_0^1 x^2(t)\,dt},$$

$$f(x) = \|x\|_2^2.$$

Then

$$f(x) > \frac{1}{2}\|x\|_2^2,$$

so $f > 0$ relative to $\|\cdot\|_2$. Now let $\varepsilon > 0$ and set

$$x_\delta(t) = e^{-\frac{t}{\delta}}.$$

Then

$$f(x_\delta) = \int_0^1 x_\delta^2(t)\,dt = \frac{\delta}{2}\left(1 - e^{-\frac{2}{\delta}}\right) \to 0$$

as $\delta \to 0$ and

$$\|x_\delta\|_\infty = 1.$$

Hence,

$$f(x_\delta) \not> \varepsilon \|x_\delta\|_\infty^2$$

for sufficiently small δ, so $f \not> 0$ relative to $\|\cdot\|_\infty$.

Theorem 3.26 *Let $X = \mathbb{R}^n$ and*

$$f(x) = x^T P x.$$

Then f is positive (negative) definite iff P is positive (negative) definite.

Proof. The two norm theorem guarantees that definiteness is norm-independent. For convenience, choose

$$\|x\| = \sqrt{x^T x}.$$

Setting

$$Q = \frac{1}{2}\left(P + P^T\right),$$

we may write

$$f(x) = x^T Q x.$$

Note that

$$f(x) > \varepsilon x^T x$$

iff

$$x^T (Q - \varepsilon I)\, x > 0.$$

Hence, f is positive definite iff $Q - \varepsilon I > 0$ for some $\varepsilon > 0$. The eigenvalues of $Q - \varepsilon I$ are $\lambda - \varepsilon$, where λ ranges over the eigenvalues of Q. Since Q is symmetric, Theorem 2.8, part 3) states that $Q - \varepsilon I > 0$ iff each λ satisfies $\lambda > \varepsilon$. But we are free to choose ε, so positive definiteness of f is equivalent to $\lambda > 0$ for every eigenvalue of Q. This is the same as $Q > 0$, which is equivalent to $P > 0$. Negative definiteness is handled similarly. ∎

3.2.10 Second Derivative Conditions

Now we state conditions for local extrema in terms of second derivatives.

Theorem 3.27 *Suppose J has a second Gateaux derivative $\delta^2 J(x^*)$ at $x^* \in X$.*
1) If J achieves a local minimum at x^*, then $\delta^2 J(x^*) \geq 0$.
2) If J achieves a local maximum at x^*, then $\delta^2 J(x^*) \leq 0$.

Proof. 1) Let $h \in X$. If $J(x^*) \leq J(x)$ for $x \in B(x^*, \varepsilon)$, then

$$J(x^*) \leq J(x^* + \alpha h)$$

for

$$\alpha \in \left(-\frac{\varepsilon}{\|h\|}, \frac{\varepsilon}{\|h\|}\right).$$

Since J achieves an extremum at x^*, $\delta J(x^*) h = 0$. From (3.9),

$$0 \leq \frac{J(x^* + \alpha h) - J(x^*)}{\alpha^2} \to \frac{1}{2}\delta^2 J(x^*) h$$

as $\alpha \to 0$. Since h was arbitrary, $\delta^2 J(x^*) \geq 0$.
2) Similar to 1). ■

Theorem 3.28 *Suppose J has a critical point $x \in X$ and a second Frechet derivative $J''(x)$.*
1) If $J''(x) > 0$, then J achieves a strict local minimum at x.
2) If $J''(x) < 0$, then J achieves a strict local maximum at x.

Proof. 1) From the definition of positive definiteness, there exists $\varepsilon > 0$ such that

$$J''(x) h > \varepsilon \|h\|^2$$

for every $h \in X$. Then

$$\frac{J(x+h) - J(x) - \frac{1}{2}J''(x) h}{\|h\|^2} \to 0$$

as $h \to 0$, so there exists $\delta > 0$ such that

$$\frac{J(x+h) - J(x) - \frac{1}{2}J''(x) h}{\|h\|^2} > -\frac{\varepsilon}{2}$$

for $h \in B(0, \delta)$. Hence,

$$J(x+h) - J(x) > \frac{1}{2}\left(J''(x) h - \varepsilon \|h\|^2\right) > 0$$

for $h \in B(0, \delta)$.
2) Similar to 1). ■
Note that Theorems 2.11 and 2.12 follow as special cases of Theorems 3.27 and 3.28.

3.2.11 Legendre's Condition

Let us calculate the second Gateaux derivative of

$$J(x) = \int_0^1 F\left(x, \dot{x}, t\right) dt,$$

where $X = C^2$ functions $x : [0,1] \to \mathbb{R}$ and $F \in C^2$. The second directional derivative exists and equals

$$\begin{aligned}
\delta^2 J(x^*) h &= \frac{d^2}{d\alpha^2} \int_0^1 F\left(x^* + \alpha h, \dot{x}^* + \alpha \dot{h}, t\right) dt\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*+\alpha h} h + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*+\alpha h} \dot{h} \right) dt\Big|_{\alpha=0} \\
&= \int_0^1 \left(\frac{\partial^2 F}{\partial x^2}\Big|_{x^*} h^2 + 2\frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x^*} h\dot{h} + \frac{\partial^2 F}{\partial \dot{x}^2}\Big|_{x^*} \dot{h}^2 \right) dt.
\end{aligned}$$

Invoking the chain rule and integration by parts,

$$\begin{aligned}
\int_0^1 2\frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x^*} h\dot{h} dt &= \int_0^1 \frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x^*} \frac{d}{dt}\left(h^2\right) dt \\
&= -\int_0^1 \frac{d}{dt}\left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x^*} \right) h^2 dt + \frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x=x^*, t=1} h^2(1) - \frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x=x^*, t=0} h^2(0),
\end{aligned}$$

so

$$\delta^2 J(x^*) h = \int_0^1 \left(\left(\frac{\partial^2 F}{\partial x^2}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x^*} \right) \right) h^2 + \frac{\partial^2 F}{\partial \dot{x}^2}\Big|_{x^*} \dot{h}^2 \right) dt + \frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x=x^*, t=1} h^2(1) - \frac{\partial^2 F}{\partial x \partial \dot{x}}\Big|_{x=x^*, t=0} h^2(0). \tag{3.14}$$

Now we need a lemma reminiscent of the fundamental lemma, but for quadratic integrals.

Theorem 3.29 *Let $P, Q : [0,1] \to \mathbb{R}$ be continuous and*

$$f(h) = \int_0^1 \left(P h^2 + Q \dot{h}^2 \right) dt.$$

If $f(h) \geq 0$ for every $h \in X$ with

$$h(0) = h(1) = 0,$$

then $Q(t) \geq 0$ for every $t \in [0,1]$.

Proof. Suppose $Q(t) < 0$ for some $t \in [0,1]$ and let

$$M = \max_t P(t).$$

By continuity, there exists an interval $(a,b) \subset (0,1)$ and $\varepsilon > 0$ such that $Q(t) < -\varepsilon$ for all $t \in (0,1)$. In fact, there is no harm in assuming

$$b - a < 4\pi \sqrt{\frac{\varepsilon}{3M}}.$$

Setting

$$h(t) = \begin{cases} \sin^2\left(\frac{2\pi}{b-a}\left(t - \frac{a+b}{2}\right)\right), & \left|t - \frac{a+b}{2}\right| \le \frac{b-a}{2} \\ 0, & \text{else} \end{cases},$$

we obtain $h(0) = h(1) = 0$, so

$$\begin{aligned}
0 &\le f(h) \\
&\le \int_0^1 \left(M h^2 - \varepsilon \dot{h}^2 \right) dt \\
&= \int_a^b \left(M \sin^4\left(\frac{2\pi}{b-a}\left(t - \frac{a+b}{2}\right)\right) - \varepsilon \left(\frac{2\pi}{b-a}\right)^2 \sin^2\left(\frac{4\pi}{b-a}\left(t - \frac{a+b}{2}\right)\right)\right) dt \\
&= M \frac{b-a}{2\pi} \int_{-\pi}^{\pi} \sin^4 \tau d\tau - \varepsilon \frac{\pi}{b-a} \int_{-2\pi}^{2\pi} \sin^2 \tau d\tau \\
&= \frac{3}{8} M (b-a) - \frac{2\pi^2 \varepsilon}{b-a} \\
&< \frac{3}{8} M \left(4\pi \sqrt{\frac{\varepsilon}{3M}} \right) - 2\pi^2 \varepsilon \left(\frac{1}{4\pi} \sqrt{\frac{3M}{\varepsilon}} \right) \\
&= 0.
\end{aligned}$$

This is a contradiction, so $Q(t) \ge 0$ for every t. ■

In combination with Theorem 3.27, the following theorem establishes necessary conditions for an extremum based on the second derivative. Compare this result with Theorem 2.11.

Theorem 3.30 *(Legendre's Condition)*
1) If $\delta^2 J(x^*) \ge 0$, then $\left.\frac{\partial^2 F}{\partial \dot{x}^2}\right|_{x^*} \ge 0$ for every $t \in [0,1]$.
2) If $\delta^2 J(x^*) \le 0$, then $\left.\frac{\partial^2 F}{\partial \dot{x}^2}\right|_{x^*} \le 0$ for every $t \in [0,1]$.

Proof. 1) Let

$$P = \left.\frac{\partial^2 F}{\partial x^2}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial^2 F}{\partial x \partial \dot{x}}\right|_{x^*} \right),$$

$$Q = \left.\frac{\partial^2 F}{\partial \dot{x}^2}\right|_{x^*}$$

in (3.14). For any $h \in X$ with $h(0) = h(1) = 0$,

$$\delta^2 J(x^*) h = \int_0^1 \left(P h^2 + Q \dot{h}^2 \right) dt.$$

From Theorem 3.29, $Q(t) \ge 0$ for every t.

2) Obviously, $-\delta^2 J(x^*) \ge 0$. Negating (3.14) and applying the arguments in the proof of part 1) yields

$$-\left.\frac{\partial^2 F}{\partial \dot{x}^2}\right|_{x^*} \ge 0$$

for every t. ■

Example 3.17 *Recall Example 3.10.*

$$F\left(x,\dot{x}\right)=x^2+x\dot{x}+\dot{x}^2,$$

$$x^*\equiv 0.$$

Since

$$\frac{\partial^2 F}{\partial \dot{x}^2}=\frac{\partial}{\partial \dot{x}}\left(x+2\dot{x}\right)=2,$$

Theorems 3.27 and 3.30 indicate that J and x^* violate the necessary condition for a local maximum.

Example 3.18 *Recall Example 3.11.*

$$F\left(x,\dot{x}\right)=\sqrt{1+\dot{x}^2},$$

$$x^*\equiv a.$$

Then

$$\frac{\partial^2 F}{\partial \dot{x}^2}=\frac{\partial}{\partial \dot{x}}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right)=\frac{1}{\left(1+\dot{x}^2\right)^{\frac{3}{2}}}>0 \quad \forall x\in X, \quad t\in[0,1],$$

so a local maximum is ruled out for every a.

Example 3.19 *Recall Example 3.12.*

$$F\left(x,\dot{x}\right)=(x-y(t))^2+\left(\dot{x}-z(t)\right)^2,$$

$$\begin{aligned}x^*(t)=&\frac{z(1)-(\cosh 1)z(0)-\int_0^1(\cosh(1-\tau))\left(\dot{z}(\tau)-y(\tau)\right)d\tau}{\sinh 1}\cosh t+z(0)\sinh t\\&+\int_0^t(\sinh(t-\tau))\left(\dot{z}(\tau)-y(\tau)\right)d\tau.\end{aligned}$$

Then

$$\frac{\partial^2 F}{\partial \dot{x}^2}=\frac{\partial}{\partial \dot{x}}\left(2\left(\dot{x}-z(t)\right)\right)=2,$$

so a local maximum is ruled out.

For calculus of variations problems formulated in C^2, the second derivative sufficient conditions of Theorem 3.28 are rarely applicable.

Example 3.20 *As in Examples 3.10 and 3.17,*

$$F\left(x,\dot{x}\right)=x^2+x\dot{x}+\dot{x}^2,$$

$$x^*\equiv 0,$$

$$\frac{\partial^2 F}{\partial \dot{x}^2}\Big|_{x^*} = 2.$$

One can show that $J''(x^*)$ exists, so

$$J''(x^*)\,h = \int_0^1 \frac{\partial^2 F}{\partial \dot{x}^2}\Big|_{x^*} h^2 dt = 2\int_0^1 h^2 dt.$$

Let

$$h_\delta(t) = e^{-\frac{t}{\delta}}.$$

Then

$$\|h_\delta\|_{C^2} = \max_t |h_\delta(t)| + \max_t \left|\dot{h}_\delta(t)\right| + \max_t \left|\ddot{h}_\delta(t)\right| = 1 + \frac{1}{\delta} + \frac{1}{\delta^2} \to \infty,$$

$$J''(x^*)\,h_\delta = 2\int_0^1 h_\delta^2 dt = \delta\left(1 - e^{-\frac{2}{\delta}}\right) \to 0$$

as $\delta \to 0^+$. For any $\varepsilon > 0$ one can choose δ such that

$$0 < J''(x^*)\,h_\delta < \varepsilon \|h_\delta\|_{C^2}^2$$

for sufficiently small δ. Hence, $J''(x^*) \not< 0$ and $J''(x^*) \not> 0$.

3.3 Constrained Optimization in X

3.3.1 Introduction

Let X be a normed linear space, $J : X \to \mathbb{R}$, and $g : X \to \mathbb{R}^m$. Writing

$$\begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} = g,$$

we say g is *Frechet differentiable at x* if each g_i is Frechet differentiable at x. This is the same as

$$\frac{\|g(x+h) - g(x) - g'(x)\,h\|_\infty}{\|h\|} \to 0. \tag{3.15}$$

Here, the norm $\|\cdot\|_\infty$ on $\mathbb{R}^m$ is applied to the numerator, while $\|\cdot\|$ in the denominator is any norm on X. By the two norm theorem, the choice of norm for the numerator is actually irrelevant, so we may simply write

$$\frac{g(x+h) - g(x) - g'(x)\,h}{\|h\|} \to 0.$$

In the infinite-dimensional context, the notion of a constrained extremum carries over verbatim from the finite-dimensional case: We say J achieves a *constrained global minimum (maximum) at x^* subject to $\Omega \subset X$* if $x^* \in \Omega$ and $J(x^*) \le J(x)$ $(J(x^*) \ge J(x))$ for every $x \in \Omega$. Furthermore, J achieves a *constrained local minimum (maximum)* if there exists $\varepsilon > 0$ such that $J(x^*) \le J(x)$ $(J(x^*) \ge J(x))$ for every $x \in \Omega \cap B(x^*, \varepsilon)$. The extrema are *strict* if the inequalities are strict.

Theorem 2.13 generalizes without any changes.

Theorem 3.31 *If J achieves an unconstrained extremum at $x^* \in \Omega$, then J achieves a constrained extremum (of the same type) at x^* subject to Ω.*

3.3.2 Open Constraint Sets

Open sets are defined as before: A set Ω is *open* if for every $x \in \Omega$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \Omega$. Unfortunately, for infinite-dimensional spaces X, the open sets depend on the choice of norm. Hence, there is no infinite-dimensional version of Theorem 2.14. Nevertheless, Theorem 2.17 carries over verbatim.

Theorem 3.32 *If $\Omega \subset \mathbb{R}^n$ is open, $x^* \in \Omega$, and J achieves a constrained local extremum at x^* subject to Ω, then J achieves an unconstrained local extremum (of the same type) at x^*.*

Theorem 2.16 generalizes without change. Hence, strict inequalities with continuous functions $g(x) > 0$ determine open constraint sets. As in finite-dimensional problems, J need only be defined on an open set $\Omega \subset X$. A useful application arises in certain calculus of variations problems.

Theorem 3.33 *If $U \subset \mathbb{R}^3$ is open, then*

$$\Omega = \left\{ x \in X \;\middle|\; \left(x(t), \dot{x}(t), t \right) \in U \quad \forall t \in [0,1] \right\}$$

is open.

In view of Theorem 3.33, the integrand F need only be defined on an open set $U \subset \mathbb{R}^3$. This fact is useful in problems where $|F| \to \infty$ as $\left(x(t), \dot{x}(t), t \right)$ tends to the boundary of U. If F is time-invariant and $U \subset \mathbb{R}^2$ is open, then Theorem 3.33 states that

$$\begin{aligned} \Omega &= \left\{ x \in X \;\middle|\; \left(x(t), \dot{x}(t) \right) \in U \quad \forall t \in [0,1] \right\} \\ &= \left\{ x \in X \;\middle|\; \left(x(t), \dot{x}(t), t \right) \in U \times \mathbb{R} \quad \forall t \in [0,1] \right\} \end{aligned}$$

is open.

Example 3.21 *Let*

$$F\left(x, \dot{x} \right) = \frac{1}{\sqrt{1 - x^2}}$$

and $U = (-1, 1) \times \mathbb{R}$. Euler's equation is

$$-\frac{x}{\sqrt{1 - x^2}} = 0,$$

so the only extremal is $x^* \equiv 0$. The boundary conditions are trivial, so x^* is a critical point. Since

$$\frac{\partial F}{\partial \dot{x}} = 0,$$

Legendre's condition provides no information.

3.3.3 Affine Constraint Sets

We say g is *affine* if $\widetilde{g} = g - g(0)$ is linear. An optimization problem with affine g can be reduced to an unconstrained problem by recasting it in

$$\widetilde{X} = \left\{ x \in X \,\middle|\, \widetilde{g}(x) = 0 \right\}.$$

It is easy to check that $\widetilde{X}$ satisfies the axioms of a vector space. Let

$$\Omega = \left\{ x \in X \,\middle|\, g(x) = 0 \right\}.$$

Theorem 3.34 *Let $x_0 \in \Omega$. Then $\widetilde{X} = -x_0 + \Omega$.*

Proof. By linearity of $\widetilde{g}$,

$$\begin{aligned}
\widetilde{g}(x) &= \widetilde{g}(x) + g(x_0) \\
&= \widetilde{g}(x) + \widetilde{g}(x_0) + g(0) \\
&= \widetilde{g}(x + x_0) + g(0) \\
&= g(x + x_0).
\end{aligned}$$

Hence, $x \in \widetilde{X}$ iff $x + x_0 \in \Omega$ or, equivalently, $x \in -x_0 + \Omega$. ■

Define the *shift operator* $\Sigma : \widetilde{X} \to \Omega$ according to

$$\Sigma(x) = x + x_0.$$

Setting

$$\widetilde{J} : \widetilde{X} \to \mathbb{R},$$
$$\widetilde{J}(x) = J(\Sigma(x)) = J(x + x_0) \tag{3.16}$$

preserves cost under the transformation Σ and, hence, preserves global extrema. Since

$$\Sigma(B(x, \varepsilon)) = B(x + x_0, \varepsilon),$$

local extrema are also preserved. Hence, an optimization problem with an affine constraint set may be replaced by an unconstrained problem with on a reduced vector space $\widetilde{X}$.

Theorem 3.35 *1) If J is Gateaux differentiable at x, then $\widetilde{J}$ is Gateaux differentiable at $x - x_0$ and*

$$\delta\widetilde{J}(x - x_0)\,h = \delta J(x)\,h$$

for $h \in \widetilde{X}$.
2) If J is Frechet differentiable at x, then $\widetilde{J}$ is Frechet differentiable at $x - x_0$ and

$$\widetilde{J}'(x - x_0)\,h = J'(x)\,h$$

for $h \in \widetilde{X}$.
3) If J is twice Gateaux differentiable at x, then $\widetilde{J}$ is twice Gateaux differentiable at $x - x_0$ and

$$\delta^2\widetilde{J}(x - x_0)\,h = \delta^2 J(x)\,h$$

for $h \in \widetilde{X}$.
4) If J is twice Frechet differentiable at x, then $\widetilde{J}$ is twice Frechet differentiable at $x - x_0$ and $\widetilde{J}''(x - x_0)\,h = J''(x)\,h$ for $h \in \widetilde{X}$.

Proof. 1) From (3.16),

$$\frac{\widetilde{J}(x-x_0+\alpha h)-\widetilde{J}(x-x_0)}{\alpha}=\frac{J(x+\alpha h)-J(x)}{\alpha}\rightarrow \delta J(x)h$$

for any $h\in\widetilde{X}$ as $\alpha\rightarrow 0$.

2)

$$\frac{\widetilde{J}(x-x_0+h)-\widetilde{J}(x-x_0)-J'(x)h}{\|h\|}=\frac{J(x+h)-J(x)-J'(x)h}{\|h\|}\rightarrow 0$$

as $h\rightarrow 0$.

3)

$$\frac{\widetilde{J}(x-x_0+\alpha h)-\widetilde{J}(x-x_0)-\alpha\delta J(x)h}{\alpha^2}=\frac{J(x+\alpha h)-J(x)-\delta J(x)}{\alpha^2}\rightarrow\frac{1}{2}\delta^2 J(x)$$

as $\alpha\rightarrow 0$.

4)

$$\frac{\widetilde{J}(x-x_0+h)-\widetilde{J}(x-x_0)-J'(x)h-\frac{1}{2}J''(x)h}{\|h\|^2}=\frac{J(x+h)-J(x)-J'(x)h-\frac{1}{2}J''(x)h}{\|h\|^2}\rightarrow 0$$

as $h\rightarrow 0$. ■

In view of Theorem 3.35, optimization problems with affine constraints can be solved by restricting the first and second derivatives of J to $\widetilde{X}$ and applying the theorems of unconstrained optimization, discarding critical points $x^*\notin\Omega$.

3.3.4 Fixed End Points

Let $X=C^2$ functions $x:[0,1]\rightarrow\mathbb{R}$, $\|\cdot\|=\|\cdot\|_{C^2}$, and

$$J(x)=\int_0^1 F\left(x,\dot{x},t\right)dt,$$

where $F:\mathbb{R}^3\rightarrow\mathbb{R}$ lies in C^2. We impose the fixed end point condition $x(0)=a$ or, equivalently,

$$g(x)=x(0)-a=0.$$

Since g is affine, we may transform the problem into an unconstrained one. Compare the following result with Theorem 3.16.

Theorem 3.36 *x^* is a critical point of L iff*

$$\left.\frac{\partial F}{\partial x}\right|_{x^*}=\frac{d}{dt}\left(\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right) \tag{3.17}$$

for every $t\in[0,1]$,

$$x^*(0)=a,$$

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1}=0. \tag{3.18}$$

Proof. Let $x_0 \equiv a$. Then

$$\widetilde{X} = -x_0 + \Omega = \left\{ x \in X \,\middle|\, x(0) = 0 \right\}.$$

For $h \in \widetilde{X}$, the Gateaux derivative of J at x^* is

$$\delta J(x^*) h = \int_0^1 \left(\frac{\partial F}{\partial x}\bigg|_{x^*} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}}\bigg|_{x^*} \right) h dt + \frac{\partial F}{\partial \dot{x}}\bigg|_{x=x^*, t=1} h(1).$$

Applying the fundamental lemma and the constraint yields (3.17)-(3.18). ■

3.3.5 Extensions and Examples

Using arguments similar to those in the proof of Theorem 3.36, one can also derive necessary conditions for the following cases.

1) Right end point constraint $x(1) = b$:

$$g(x) = x(1) - b = 0$$

$$\widetilde{X} = \left\{ x \in X \,\middle|\, x(1) = 0 \right\}$$

$$\frac{\partial F}{\partial x}\bigg|_{x^*} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\bigg|_{x^*} \right)$$

$$\frac{\partial F}{\partial \dot{x}}\bigg|_{x=x^*, t=0} = 0$$

2) Left and right end point constraints $x(0) = a, \quad x(1) = b$:

$$g(x) = \begin{bmatrix} x(0) - a \\ x(1) - b \end{bmatrix} = 0$$

$$\widetilde{X} = \left\{ x \in X \,\middle|\, x(0) = x(1) = 0 \right\}$$

$$\frac{\partial F}{\partial x}\bigg|_{x^*} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\bigg|_{x^*} \right)$$

Example 3.22 *We wish to find the C^2 function x with $x(0) = 0$ and $x(1) = 1$ having minimum arc length. From Example 3.11,*

$$F\left(x, \dot{x}\right) = \sqrt{1 + \dot{x}^2},$$

which yields Euler's equation

$$\ddot{x} \equiv 0$$

and the extremals

$$x(t) = at + b$$

for arbitrary $a, b \in \mathbb{R}$. Since this is a two end point problem, we need only apply the constraints

$$b = x(0) = 0,$$

$$a + b = x(1) = 1.$$

Hence, $a = 1$ and the only possible solution is

$$x^* = t.$$

Example 3.23 *(Surface of Rotation) The problem is to find the C^2 function x with*

$$x(0) = x(1) = 1$$

and $x(t) > 0$ for every $t \in [0,1]$, which generates the surface of revolution with minimum area. Let

$$\Omega_3 = \left\{ x \in X \,\middle|\, x(t) > 0 \right\}$$

and $x^* \in \Omega_3$. Since

$$U = [0, \infty) \times \mathbb{R}$$

is open in $\mathbb{R}^2$, Theorem 3.33 guarantees that Ω_3 is open. The end point conditions determine the equality constraint

$$g_1(x) = \begin{bmatrix} x(0) - 1 \\ x(1) - 1 \end{bmatrix} = 0.$$

Although we have a problem with mixed constraints, the open constraint can be dealt with by simply imposing $x(t) > 0$ on the set of extremals. The cost functional is obtained by finding the incremental surface of revolution:

$$ds = \sqrt{1 + \dot{x}^2}\, dt$$

$$dA = 2\pi x ds = 2\pi x \sqrt{1 + \dot{x}^2}\, dt$$

$$J(x) = 2\pi \int_0^1 x \sqrt{1 + \dot{x}^2}\, dt$$

This is the same cost functional we analyzed in Example 3.13. The extremals are the catenaries

$$x(t) = a \cosh\left(\frac{t + b}{a}\right),$$

where a and b are arbitrary constants. For $a > 0$, the solutions satisfy $x(t) > 0$. The end point conditions dictate

$$x(0) = a \cosh \frac{b}{a} = 1,$$

$$x(1) = a \cosh\left(\frac{1 + b}{a}\right) = 1.$$

Solving the two equations yields

$$(a, b) = (0.255, -0.5), \quad (0.848, -0.5).$$

The corresponding values of J are

$$J(x_1) = 5.99, \quad J(x_2) = 5.96$$

leaving the second solution as the only candidate for a global minimum.

Example 3.24 *(Brachistochrone) The brachistochrone problem was originally posed by Johann Bernoulli in 1696. Solutions were provided by several mathematicians, including Newton, Jacob Bernoulli, Leibniz, and l'Hospital. It is generally acknowledged that this was the origin of the calculus of variations.*

The problem is to find the C^2 function $x : [0,1] \to \mathbb{R}$ with $x(0) = 1$ and $x(1) = 0$ that minimizes the time required by a point-mass sliding without friction along the curve to move from the left end point to the right, starting with tangential velocity $v_0 \geq 0$. We will begin by assuming $v_0 > 0$ and write z, rather than t, for the independent variable, since time has a different physical meaning in this problem. The incremental arc length is

$$ds = \sqrt{1 + \left(\frac{dx}{dz}\right)^2} dz,$$

so the tangential velocity is

$$v = \frac{ds}{dt} = \sqrt{1 + \left(\frac{dx}{dz}\right)^2} \frac{dz}{dt}.$$

Hence, the mass moves a horizontal distance dz in time

$$dt = \frac{1}{v} \sqrt{1 + \left(\frac{dx}{dz}\right)^2} dz.$$

Set

$$a = \frac{v_0^2}{2g} + 1.$$

From physics,

$$\frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 + mg(1 - x),$$

$$v = \sqrt{v_0^2 + 2g(1 - x)} = \sqrt{2g}\sqrt{a - x},$$

which must be real, so $x(z) \leq a$ for every $z \in [0,1]$. In fact, $x(z) = a$ makes no sense either, since then $v(z) = 0$, which would result in the mass stopping at position z, taking infinite time to reach the end point. Setting

$$x' = \frac{dx}{dz},$$

the cost functional is

$$J(x) = \int_0^1 \frac{1}{v} \sqrt{1 + x'^2} dz = \frac{1}{\sqrt{2g}} \int_0^1 \sqrt{\frac{1 + x'^2}{a - x}} dz.$$

Although F is not continuous (and therefore not C^2) on $\mathbb{R}^2$, F is C^2 on the open half-plane

$$U = \left\{ \left(x, \dot{x}\right) \in \mathbb{R}^2 \,\middle|\, x < a \right\}.$$

Hence, we may proceed, keeping in mind that critical points must lie in the constraint set

$$\Omega = \left\{ x \in X \,\middle|\, x(t) < a \quad \forall t \in [0,1] \right\}.$$

Ω is open by Theorem 3.33. The partial derivatives are

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x} \sqrt{\frac{1 + x'^2}{a - x}} = \frac{1}{2} \sqrt{\frac{1 + x'^2}{(a - x)^3}},$$

$$\frac{\partial F}{\partial x'} = \frac{\partial}{\partial x'}\sqrt{\frac{1+x'^2}{a-x}} = \frac{x'}{\sqrt{(1+x'^2)(a-x)}},$$

yielding Euler's equation

$$\frac{1}{2}\sqrt{\frac{1+x'^2}{(a-x)^3}} = \frac{d}{dz}\frac{x'}{\sqrt{(1+x'^2)(a-x)}} = \frac{2(a-x)x'' + x'^4 + x'^2}{2\sqrt{(1+x'^2)^3(a-x)^3}},$$

$$2(x-a)x'' + x'^2 + 1 = 0.$$

This is a hard differential equation, so we apply the Beltrami identity to obtain

$$\sqrt{\frac{1+x'^2}{a-x}} - \frac{x'^2}{\sqrt{(1+x'^2)(a-x)}} = b,$$

$$b\sqrt{(1+x'^2)(a-x)} = 1.$$

Obviously, $b \neq 0$, *so*

$$\left(1+x'^2\right)(a-x) = \frac{1}{b^2}.$$

Since the slope of x *must be negative for some* z*,*

$$x' = -\sqrt{\frac{1-b^2(a-x)}{b^2(a-x)}}.$$

Separating variables yields

$$-\int\sqrt{\frac{b^2(a-x)}{1-b^2(a-x)}}dx = z + c,$$

$$\frac{\arccos\left(1-2b^2(a-x)\right) - \sqrt{1-\left(1-2b^2(a-x)\right)^2}}{2b^2} = z + c. \tag{3.19}$$

Applying the boundary conditions,

$$\arccos\left(1-2b^2(a-1)\right) - \sqrt{1-\left(1-2b^2(a-1)\right)^2} - 2b^2c = 0, \tag{3.20}$$

$$\arccos\left(1-2b^2a\right) - \sqrt{1-\left(1-2b^2a\right)^2} - 2b^2(1+c) = 0. \tag{3.21}$$

Recalling that a *is a constant determined by* v_0 *and* g*, (3.20)-(3.21) can be solved simultaneously for* b *and* c*. Setting*

$$\theta = \arccos\left(1-2b^2(a-x^*)\right),$$

and invoking (3.19), we obtain x^* *in parametric form:*

$$\sin\theta = \sin\left(\arccos\left(1-2b^2(a-x^*)\right)\right) = \sqrt{1-\left(1-2b^2(a-x^*)\right)^2}$$

$$x^* = a - \frac{1-\cos\theta}{2b^2}$$

$$z^* = \frac{\theta - \sin\theta}{2b^2} - c$$

This function is called a ***cycloid*** *– the curve traced by a point on the perimeter of a rolling wheel. The parameter θ varies over the interval $[\theta_0, \theta_1]$, where*

$$\theta_0 = \arccos\left(1 - 2b^2(a-1)\right),$$

$$\theta_1 = \arccos\left(1 - 2b^2 a\right).$$

The case $v_0 = 0$ is mathematically more difficult, since $x^*(0) = 1$ forces x^* to lie outside Ω. However, physical considerations dictate that letting $v_0 \to 0$ in the solution above yields the optimal curve given by

$$b^* = 0.934, \quad c^* = 0,$$

$$\theta_0 = 0, \quad \theta_1 = 2.41,$$

$$x^* = 1 - \frac{1-\cos\theta}{2b^{*2}} = 0.427 + 0.573\cos\theta,$$

$$z^* = \frac{\theta - \sin\theta}{2b^{*2}} = 0.573(\theta - \sin\theta).$$

3.3.6 Banach Spaces

For constraint sets Ω that are neither open nor affine, the situation becomes harder. Such problems can be handled with an infinite-dimensional version of Lagrange multipliers. In order to prove such a result, we need to make an assumption on the normed linear space X. This requires some additional background.

A *sequence* in X is any infinite list of vectors $x_1, x_2, x_3, \ldots \in X$. The sequence is often denoted x_k. Let $\|\cdot\|$ be any norm on X. A sequence in X is *Cauchy* if for every $\varepsilon > 0$ there exists $N < \infty$ such that $\|x_k - x_l\| < \varepsilon$ whenever $k, l > N$. In other words, x_k being Cauchy means $x_k - x_l \to 0$ as $k, l \to \infty$.

Theorem 3.37 *Every convergent sequence is Cauchy.*

Proof. If $x_k \to x$, then

$$\begin{aligned}\|x_k - x_l\| &= \|(x_k - x) - (x_l - x)\| \\ &\leq \|x_k - x\| + \|x_l - x\| \\ &\to 0.\end{aligned}$$

■

In certain spaces, the converse of Theorem 3.37 is false.

Example 3.25 *Let $X =$ continuous functions $x : [0,1] \to \mathbb{R}$ with norm $\|\cdot\|_2$. Consider the sequence*

$$x_k(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2} \\ k\left(t - \frac{1}{2}\right), & \frac{1}{2} \leq t < \frac{1}{2} + \frac{1}{k} \\ 1, & \frac{1}{2} + \frac{1}{k} \leq t \leq 1 \end{cases}.$$

For $k > l$,

$$x_k(t) - x_l(t) = \begin{cases} (k-l)\left(t - \frac{1}{2}\right), & \frac{1}{2} \leq t < \frac{1}{2} + \frac{1}{k} \\ 1 - l\left(t - \frac{1}{2}\right), & \frac{1}{2} + \frac{1}{k} \leq t < \frac{1}{2} + \frac{1}{l} \\ 0, & \text{else} \end{cases}.$$

Then

$$
\begin{aligned}
\|x_k - x_l\|^2 &= \int_0^1 (x_k - x_l)^2 \, dt \\
&= (k-l)^2 \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{k}} \left(t - \frac{1}{2}\right)^2 dt + \int_{\frac{1}{2}+\frac{1}{k}}^{\frac{1}{2}+\frac{1}{l}} \left(1 - l\left(t - \frac{1}{2}\right)\right)^2 dt \\
&= \frac{1}{3l}\left(\frac{k-l}{k}\right)^2 \\
&< \frac{1}{3l} \\
&\to 0
\end{aligned}
$$

as $l \to \infty$. For $k < l$, simply interchange k and l to yield a similar result. The case $k = l$ trivially yields $\|x_k - x_l\|^2 = 0$. Hence, $\|x_k - x_l\| \to 0$, so x_k is Cauchy. But x_k converges to the step function

$$
x(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2} \\ 0, & \frac{1}{2} \leq t \leq 1 \end{cases},
$$

which does not belong to X.

A normed linear space X is *complete* if every Cauchy sequence in X converges (to a point in X.) Example 3.25 describes a normed linear space that is not complete. Actually, the notion of completeness applies to more general spaces that normed linear spaces (e.g. metric spaces). If a normed linear space is complete, it is a *Banach space.*

Example 3.26 *The following are Banach spaces:*
1) $\mathbb{R}^n$ with any norm.
2) $X =$ continuous functions $x : [0,1] \to \mathbb{R}$ with norm $\|\cdot\|_\infty$.
3) $X = C^k$ functions $x : [0,1] \to \mathbb{R}$ with norm

$$
\|\cdot\|_{C^k} = \sum_{i=0}^{k} \max_t \left| x^{(i)}(t) \right|.
$$

4) $X = L^p$ functions $x : [0,1] \to \mathbb{R}$ with norm

$$
\|\cdot\|_p = \left(\int_0^1 |x(t)|^p \, dt \right)^{\frac{1}{p}}.
$$

We will see that completeness plays a role in the infinite-dimensional version of Lagrange multipliers.

3.3.7 Strict Frechet Differentiability

We say $J : X \to \mathbb{R}$ is *strictly Frechet differentiable* at $x^* \in X$ if

$$
\frac{J(x+h) - J(x) - J'(x^*)\,h}{\|h\|} \to 0
$$

as $h \to 0$ and $x \to x^*$. If J is strictly Frechet differentiable, then it is Frechet differentiable, since we may take the limit with $x = x^*$. In this case, $J'(x^*)$ is the Frechet derivative (and the Gateaux derivative).

Strict Frechet differentiability specializes naturally to $X = \mathbb{R}^n$.

Theorem 3.38 *If $X = \mathbb{R}^n$ and $J \in C^1$, then J is strictly Frechet differentiable.*

Proof. Choose any $R > 0$. By Taylor's theorem, there exists $M_1 < \infty$ such that

$$\frac{\left| J(x+h) - J(x) - \frac{\partial J(x)}{\partial x} h \right|}{\|h\|} \leq M_1 \|h\|$$

for every $x \in B(x^*, R)$ and $h \in B(0, R)$. From Theorem 2.4, there exists $M_2 < \infty$ such that

$$\left| \left(\frac{\partial J(x)}{\partial x} - \left. \frac{\partial J}{\partial x} \right|_{x^*} \right) h \right| \leq M_2 \left\| \frac{\partial J(x)}{\partial x} - \left. \frac{\partial J}{\partial x} \right|_{x^*} \right\| \|h\|.$$

Hence,

$$\begin{aligned}
\frac{\left| J(x+h) - J(x) - \left. \frac{\partial J}{\partial x} \right|_{x^*} h \right|}{\|h\|} &\leq \frac{\left| J(x+h) - J(x) - \frac{\partial J(x)}{\partial x} h \right|}{\|h\|} + \frac{\left| \left(\frac{\partial J(x)}{\partial x} - \left. \frac{\partial J}{\partial x} \right|_{x^*} \right) h \right|}{\|h\|} \\
&\leq M_1 \|h\| + M_2 \left\| \frac{\partial J(x)}{\partial x} - \left. \frac{\partial J}{\partial x} \right|_{x^*} \right\| \\
&\to 0
\end{aligned}$$

as $x \to x^*$ and $h \to 0$. ■

We can also prove strict Frechet differentiability for functionals of the form

$$J(x) = \int_0^1 F\left(x, \dot{x}, t\right) dt,$$

where $F \in C^2$. To do so, we need a different version of Taylor's theorem.

Theorem 3.39 *(Taylor's Theorem) Let $f : \mathbb{R}^n \times [0, 1] \to \mathbb{R}$ lie in C^2, $x^* \in \mathbb{R}^n$, and $R > 0$. Then there exists $M < \infty$ such that*

$$\left| f(x+h, t) - f(x, t) - \frac{\partial f(x, t)}{\partial x} h \right| \leq M \|h\|^2$$

for every $x \in B(x^*, R)$, $h \in B(0, R)$, and $t \in [0, 1]$.

In the following result, we let $X = C^2$ functions $x : [0, 1] \to \mathbb{R}$.

Theorem 3.40 *If $F \in C^2$, then J is strictly Frechet differentiable at every $x^* \in X$.*

Proof. Let $\varepsilon > 0$ and impose the norm $\|\cdot\|_1$ on R^2. If $x \in B_{C^2}(x^*, \varepsilon)$ and $h \in B_{C^2}(0, \varepsilon)$, then

$$\left\| \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} - \begin{bmatrix} x^*(t) \\ \dot{x}^*(t) \end{bmatrix} \right\|_1 = |x(t) - x^*(t)| + \left| \dot{x}(t) - \dot{x}^*(t) \right| \leq \|x - x^*\|_{C^2} < \varepsilon,$$

$$\left\| \begin{bmatrix} h(t) \\ \dot{h}(t) \end{bmatrix} \right\|_1 = |h(t)| + \left| \dot{h}(t) \right| \leq \|h\|_{C^2} < \varepsilon,$$

So

$$\left[\begin{array}{c} x(t) \\ \dot{x}(t) \end{array}\right] \in B\left(\left[\begin{array}{c} x^{*}(t) \\ \dot{x}^{*}(t) \end{array}\right], R\right),$$

$$\left[\begin{array}{c} h(t) \\ \dot{h}(t) \end{array}\right] \in B(0, R).$$

Setting $n = 2$, $f = F$, and $R = \varepsilon$ in Taylor's theorem, we obtain

$$\begin{aligned}
&\left|F\left(x(t)+h(t), \dot{x}(t)+\dot{h}(t), t\right)-F\left(x(t), \dot{x}(t), t\right)-\left[\begin{array}{cc} \left.\frac{\partial F}{\partial x}\right|_{x(t)} & \left.\frac{\partial F}{\partial \dot{x}}\right|_{x(t)} \end{array}\right]\left[\begin{array}{c} h(t) \\ \dot{h}(t) \end{array}\right]\right| \\
&\leq M_1\left\|\left[\begin{array}{c} h(t) \\ \dot{h}(t) \end{array}\right]\right\|_1^2 \\
&\leq M_1\|h\|_{C^2}^2 .
\end{aligned}$$

From Theorem 2.4,

$$\begin{aligned}
&\left|\left[\begin{array}{cc} \left.\frac{\partial F}{\partial x}\right|_{x(t)}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}(t)} & \left.\frac{\partial F}{\partial \dot{x}}\right|_{x(t)}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}(t)} \end{array}\right]\left[\begin{array}{c} h(t) \\ \dot{h}(t) \end{array}\right]\right| \\
&\leq M_2\left\|\left[\begin{array}{cc} \left.\frac{\partial F}{\partial x}\right|_{x(t)}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}(t)} & \left.\frac{\partial F}{\partial \dot{x}}\right|_{x(t)}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}(t)} \end{array}\right]\right\|_1\left\|\left[\begin{array}{c} h(t) \\ \dot{h}(t) \end{array}\right]\right\|_1 \\
&\leq M_2\left\|\frac{\partial F}{\partial x}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}}\right\|_{C^2}\|h\|_{C^2} .
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\left|J(x+h)-J(x)-\delta J\left(x^{*}\right) h\right|}{\|h\|_{C^2}} &= \frac{\left|\int_0^1\left(\begin{array}{c} F\left(x+h, \dot{x}+\dot{h}, t\right)-F\left(x, \dot{x}, t\right)-\frac{\partial F}{\partial x} h-\frac{\partial F}{\partial \dot{x}} \dot{h} \\ +\left(\frac{\partial F}{\partial x}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}}\right) h-\left(\frac{\partial F}{\partial \dot{x}}-\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^{*}}\right) \dot{h} \end{array}\right) dt\right|}{\|h\|_{C^2}} \\
&\leq \frac{\int_0^1\left|F\left(x+h, \dot{x}+\dot{h}, t\right)-F\left(x, \dot{x}, t\right)-\frac{\partial F}{\partial x} h-\frac{\partial F}{\partial \dot{x}} \dot{h}\right| dt}{\|h\|_{C^2}} \\
&+\frac{\int_0^1\left|\left(\frac{\partial F}{\partial x}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}}\right) h-\left(\frac{\partial F}{\partial \dot{x}}-\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^{*}}\right) \dot{h}\right| dt}{\|h\|_{C^2}} \\
&\leq \frac{\int_0^1 M_1\|h\|_{C^2}^2 \, dt+\int_0^1 M_2\left\|\frac{\partial F}{\partial x}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}}\right\|_{C^2}\|h\|_{C^2} \, dt}{\|h\|_{C^2}} \\
&= M_1\|h\|_{C^2}+M_2\left\|\frac{\partial F}{\partial x}-\left.\frac{\partial F}{\partial x}\right|_{x^{*}}\right\|_{C^2} \\
&\rightarrow 0
\end{aligned}$$

as $x \rightarrow x^{*}$ and $h \rightarrow 0$. ■

Note that F in Examples 3.7 and 3.8 lies in C^2. Since ε may be chosen arbitrarily small in the proof of Theorem 3.40, the result applies equally well to functions F which are defined only on an open subset $U \subset \mathbb{R}^3$.

The definition of strict differentiability may also be applied to $g : X \to \mathbb{R}^m$ verbatim. Strict differentiability will be required in order to obtain an appropriate generalization of Theorem 2.18, which in turn will play a role in developing the Lagrange multiplier method.

3.3.8 Equality Constraints and Lagrange Multipliers

Reference: Luenberger, Chapter 9

Consider the constraint set

$$\Omega = \left\{ x \in X \,\middle|\, g(x) = 0 \right\}$$

and define the *Lagrangian*

$$L : X \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R},$$

according to

$$L(x, \mu, \lambda) = \mu J(x) - \lambda^T g(x).$$

For each $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^m$, denote

$$\delta L(x^*, \mu, \lambda) = \mu \delta J(x^*) - \lambda^T \delta g(x^*),$$

$$L'(x^*, \mu, \lambda) = \mu J'(x^*) - \lambda^T g'(x^*).$$

For the finite-dimensional case, recall that a constrained extremum of J at x^* may lead to a saddle point of $L(\cdot, \lambda)$ at x^*. The same can occur in the infinite-dimensional setting.

A function $f : X \to Y$ is *onto* or *surjective* if $f(X) = Y$. A vector $x \in \Omega$ is *regular* if $\delta g(x)$ is onto. We can prove an alternative characterization of regularity.

Theorem 3.41 *$x \in \Omega$ is regular iff there exist $h_1, \ldots, h_m \in X$ such that the $m \times m$ matrix*

$$\begin{bmatrix} \delta g(x) h_1 & \cdots & \delta g(x) h_m \end{bmatrix} \tag{3.22}$$

is nonsingular.

Proof. (Sufficient) Choose any $y \in \mathbb{R}^m$. Since (3.22) is nonsingular, we may define

$$v = \begin{bmatrix} \delta g(x) h_1 & \cdots & \delta g(x) h_m \end{bmatrix}^{-1} y.$$

Let

$$H = \begin{bmatrix} h_1 & \cdots & h_m \end{bmatrix}$$

and $h = Hv$. Then

$$\delta g(x) h = \delta g(x)(Hv) = \begin{bmatrix} \delta g(x) h_1 & \cdots & \delta g(x) h_m \end{bmatrix} v = y.$$

Since y was arbitrary, $\delta g(x)$ is onto.

(Sufficient) Since $\delta g(x)$ is onto, for each unit vector e_i there exists $h_i \in X$ such that $\delta g(x) h_i = e_i$. Then

$$\begin{bmatrix} \delta g(x) h_1 & \cdots & \delta g(x) h_m \end{bmatrix} = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix} = I,$$

which is nonsingular. ■

The next result shows that finite and infinite-dimensional regularity coincide as long as $g \in C^1$.

Theorem 3.42 *Let $X = \mathbb{R}^n$ and $g \in C^1$. Then x^* is regular iff $\left.\frac{\partial g}{\partial x}\right|_{x^*}$ has rank m.*

Proof. Applying Theorem 3.13 to each g_i yields

$$\delta g\left(x^*\right) h = \left.\frac{\partial g}{\partial x}\right|_{x^*} h.$$

From matrix theory, $\delta g\left(x^*\right)$ is onto iff it has rank m. ■

Now we can generalize Theorem 2.18 to infinite-dimensional problems.

Theorem 3.43 *If X is a Banach space, $x^* \in \Omega$ is regular, and g is strictly Frechet differentiable at x^*, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$B\left(g\left(x^*\right), \delta\right) \subset g\left(B\left(x^*, \varepsilon\right)\right).$$

Proof. L. M. Graves, "Some Mapping Theorems", *Duke Mathematical Journal*, 17 (1950), 111-114. ■

Theorem 3.44 *(Lagrange Multipliers) Let X be a Banach space and J and g be strictly Frechet differentiable at $x^* \in X$. If J achieves a constrained local extremum at x^* subject to $g(x) = 0$, then there exist $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^m$, not both zero, such that*

$$L'\left(x^*, \mu, \lambda\right) = 0.$$

If x^* is regular, then we may set $\mu = 1$.

Proof. Let

$$f(x) = \begin{bmatrix} J(x) \\ g(x) \end{bmatrix}.$$

Then f is strictly Frechet differentiable at x^*, and

$$f'\left(x^*\right) = \begin{bmatrix} J'\left(x^*\right) \\ g'\left(x^*\right) \end{bmatrix}.$$

Suppose x^* is a regular point of f and let $\varepsilon > 0$. From Theorem 3.43, there exists $\delta > 0$ such that

$$B\left(f\left(x^*\right), \delta\right) \subset f\left(B\left(x^*, \varepsilon\right)\right).$$

But

$$\begin{bmatrix} J\left(x^*\right) + \alpha \\ 0 \end{bmatrix} = f\left(x^*\right) + \alpha e_1 \in B\left(f\left(x^*\right), \delta\right)$$

for

$$\alpha \in \left(-\frac{\delta}{\|e_1\|}, \frac{\delta}{\|e_1\|}\right),$$

so

$$\left(J\left(x^*\right) - \frac{\delta}{\|e_1\|}, J\left(x^*\right) + \frac{\delta}{\|e_1\|}\right) \subset J\left(B\left(x^*, \varepsilon\right)\right).$$

This contradicts the assumption that J achieves a constrained local extremum at x^*. Hence, x^* is not a regular point of f. From Theorem 3.41,

$$\det \begin{bmatrix} f'\left(x^*\right) e_1 & \cdots & f'\left(x^*\right) e_m \end{bmatrix} = 0,$$

so there exist μ and λ, not both zero, such that

$$\begin{bmatrix} \mu & \lambda^T \end{bmatrix} \begin{bmatrix} f'(x^*)\, e_1 & \cdots & f'(x^*)\, e_m \end{bmatrix} = 0.$$

For any

$$h = \sum_{i=1}^{m+1} h_i e_i \in X,$$

we obtain

$$\begin{aligned} L'(x^*, \mu, \lambda)\, h &= \begin{bmatrix} \mu & \lambda^T \end{bmatrix} \begin{bmatrix} J'(x^*)\, h \\ g'(x^*)\, h \end{bmatrix} \\ &= \begin{bmatrix} \mu & \lambda^T \end{bmatrix} f'(x^*)\, h \\ &= \begin{bmatrix} \mu & \lambda^T \end{bmatrix} \sum_{i=1}^{m+1} h_i f'(x^*)\, e_i \\ &= \begin{bmatrix} \mu & \lambda^T \end{bmatrix} \begin{bmatrix} f'(x^*)\, e_1 & \cdots & f'(x^*)\, e_m \end{bmatrix} h \\ &= 0. \end{aligned} \tag{3.23}$$

Now suppose x^* is a regular point of g. If $\mu = 0$, then $\lambda \neq 0$ and

$$\lambda^T g'(x^*) = 0.$$

Since $g'(x^*)$ is onto, for every $y \in \mathbb{R}^m$ there exists $h \in X$ such that

$$g'(x^*)\, h = y.$$

Hence,

$$\lambda^T y = \lambda^T g'(x^*)\, h = 0,$$

yielding $\lambda = 0$, a contradiction. Hence, $\mu \neq 0$. Dividing μ and λ by μ yields another solution with $\mu = 1$. is a contradiction. In the latter case, $2x_1 = 0$, which is a also contradiction. ■

We say $x^* \in \Omega$ is a *critical point of* L if x^* is regular and there exists $\lambda \in \mathbb{R}^m$ such that $L'(x^*, \lambda) = 0$.

3.3.9 Terminal Manifolds

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function. In the calculus of variations framework, rather than imposing a fixed end point constraint, suppose we merely require $\phi(x(0)) = 0$. The set

$$T = \left\{ w \in \mathbb{R}^n \,\middle|\, \phi(w) = 0 \right\}$$

is called a *terminal manifold*. Note that this is the same as saying $g(x) = 0$, where g is the functional $g(x) = \phi(x(0))$. The Gateaux derivative of g is

$$\delta g(x^*)\, h = \frac{d}{d\alpha} \phi(x^*(0) + \alpha h(0)) \Big|_{\alpha=0} = \frac{\partial \phi}{\partial x(0)} \Big|_{x^*(0)} h(0).$$

x^* is regular iff

$$\frac{\partial \phi}{\partial x(0)} \Big|_{x^*(0)} \neq 0.$$

Theorem 3.45 *g is strictly Frechet differentiable.*

Proof. For any norm $\|\cdot\|$ on $\mathbb{R}^n$, $x \in B_{C^2}(x^*, \varepsilon)$ and $h \in B_{C^2}(0, \varepsilon)$ implies

$$\|x(0) - x^*(0)\| \leq \|x - x^*\|_{C^2} < \varepsilon,$$

$$\|h(0)\| \leq \|h\|_{C^2} < \varepsilon.$$

From Taylor's theorem (Theorem 3.39),

$$\left|\phi(x(0) + h(0)) - \phi(x(0)) - \frac{\partial \phi}{\partial x(0)} h(0)\right| \leq M_1 \|h(0)\|^2.$$

From Theorem 2.4,

$$\left|\left(\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right) h(0)\right| \leq M_2 \left\|\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right\| \|h(0)\|.$$

Hence,

$$\begin{aligned}
\frac{|g(x+h) - g(x) - \delta g(x^*) h|}{\|h\|_{C^2}} &= \frac{\left|\phi(x(0) + h(0)) - \phi(x(0)) - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)} h(0)\right|}{\|h\|_{C^2}} \\
&\leq \frac{\left|\phi(x(0) + h(0)) - \phi(x(0)) - \frac{\partial \phi}{\partial x(0)} h(0)\right|}{\|h\|_{C^2}} \\
&+ \frac{\left|\left(\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right) h(0)\right|}{\|h\|_{C^2}} \\
&\leq \frac{M_1 \|h(0)\|^2 + M_2 \left\|\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right\| \|h(0)\|}{\|h\|_{C^2}} \\
&\leq \frac{M_1 \|h\|_{C^2}^2 + M_2 \left\|\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right\| \|h\|_{C^2}}{\|h\|_{C^2}} \\
&\leq M_1 \|h\|_{C^2} + M_2 \left\|\frac{\partial \phi}{\partial x(0)} - \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)}\right\| \\
&\to 0
\end{aligned}$$

as $x \to x^*$ and $h \to 0$. ■

Theorem 3.46 *A regular point x^* is a critical point of L iff there exists $\lambda \in \mathbb{R}$ such that*

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right) = 0 \tag{3.24}$$

for every $t \in [0, 1]$ and

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*, t=0} + \lambda \left.\frac{\partial \phi}{\partial x(0)}\right|_{x^*(0)} = 0, \tag{3.25}$$

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*, t=1} = 0. \tag{3.26}$$

Proof. The critical points are the solutions of

$$L'(x^*,\lambda) = \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} h + \frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \dot{h} \right) dt - \lambda \frac{\partial \phi}{\partial x(0)}\Big|_{x^*(0)} h(0) = 0.$$

Applying integration by parts,

$$\int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \right) \right) h dt + \frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} h(1) - \left(\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} + \lambda \frac{\partial \phi}{\partial x(0)}\Big|_{x^*(0)} \right) h(0) = 0 \quad (3.27)$$

for every $h \in X$. Restricting to h with $h(0) = h(1)$ and invoking the fundamental lemma (Theorem 3.15), we obtain Euler's equation (3.24). Setting $h(0)$ and $h(1)$ arbitrarily, we obtain

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} + \lambda \frac{\partial \phi}{\partial x(0)}\Big|_{x^*(0)} = 0, \quad (3.28)$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} = 0. \quad (3.29)$$

Conversely, substituting (3.24), (3.28), and (3.29) into (3.27) yields equality. ■
Equation (3.25) is called a *transversality condition*.

If we impose a terminal manifold at $t = 1$ rather than $t = 0$, we obtain the critical point conditions

$$\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \right) = 0,$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} = 0,$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} - \lambda \frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)} = 0.$$

x^* is regular iff

$$\frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)} \neq 0.$$

We may also impose terminal manifolds at both endpoints, determined by functions ϕ_0 and ϕ_1. In this case,

$$g(x) = \begin{bmatrix} \phi_0(x(0)) \\ \phi_1(x(0)) \end{bmatrix}$$

and the strict Frechet derivative of g is

$$g'(x^*)h = \frac{d}{d\alpha} \begin{bmatrix} \phi_0(x^*(0) + \alpha h(0)) \\ \phi_1(x^*(1) + \alpha h(1)) \end{bmatrix}\Bigg|_{\alpha=0} = \begin{bmatrix} \frac{\partial \phi_0}{\partial x(0)}\Big|_{x^*(0)} & 0 \\ 0 & \frac{\partial \phi_1}{\partial x(1)}\Big|_{x^*(1)} \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix}. \quad (3.30)$$

An analysis similar to Theorem 3.46 yields the conditions

$$\frac{\partial F}{\partial x}\Big|_{x^*} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\Big|_{x^*} \right) = 0,$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=0} + \lambda_1 \frac{\partial \phi_0}{\partial x(0)}\Big|_{x^*(0)} = 0,$$

$$\frac{\partial F}{\partial \dot{x}}\Big|_{x=x^*,t=1} - \lambda_2 \frac{\partial \phi_1}{\partial x(1)}\Big|_{x^*(1)} = 0.$$

x^* is regular iff the 2×2 matrix in (3.30) is nonsingular or, equivalently,

$$\frac{\partial \phi_0}{\partial x(0)}\Big|_{x^*(0)} \neq 0, \quad \frac{\partial \phi_1}{\partial x(1)}\Big|_{x^*(1)} \neq 0.$$

Example 3.27 *Find the curve $x^* : [0,1] \to \mathbb{R}^2$ that minimized the distance between the line*

$$\phi_0(x) = x_2 - x_1 - 1 = 0$$

at $t = 0$ and the parabola

$$\phi_1(x) = x_1 - x_2^2 = 0$$

at $t = 1$. From

$$ds = \sqrt{dt^2 + dx_1^2 + dx_2^2} = \sqrt{1 + \dot{x}_1^2 + \dot{x}_2^2}\, dt = \sqrt{1 + \dot{x}^T \dot{x}}\, dt,$$

we obtain the integrand

$$F\left(x, \dot{x}\right) = \sqrt{1 + \dot{x}^T \dot{x}}.$$

Every point is regular, since

$$\frac{\partial \phi_0}{\partial x(0)}\Big|_{x^*(0)} = \begin{bmatrix} -1 & 1 \end{bmatrix} \neq 0,$$

$$\frac{\partial \phi_1}{\partial x(1)}\Big|_{x^*(1)} = \begin{bmatrix} 1 & -2x_2(1) \end{bmatrix} \neq 0.$$

The critical point conditions are

$$0 = \frac{d}{dt}\left(\frac{\dot{x}^T}{\sqrt{1 + \dot{x}^T \dot{x}}}\right) = \frac{\ddot{x}^T}{\left(1 + \dot{x}^T \dot{x}\right)^{\frac{3}{2}}}, \tag{3.31}$$

$$\frac{\dot{x}^T(0)}{\sqrt{1 + \dot{x}^T(0)\,\dot{x}(0)}} + \lambda_1 \begin{bmatrix} -1 & 1 \end{bmatrix} = 0,$$

$$\frac{\dot{x}^T(1)}{\sqrt{1 + \dot{x}^T(1)\,\dot{x}(1)}} - \lambda_2 \begin{bmatrix} 1 & -2x_2(1) \end{bmatrix} = 0.$$

From (3.31),

$$x(t) = a + bt$$

for some $a, b \in \mathbb{R}^2$. Then

$$\frac{\begin{bmatrix} b_1 & b_2 \end{bmatrix}}{\sqrt{1 + b_1^2 + b_2^2}} + \lambda_1 \begin{bmatrix} -1 & 1 \end{bmatrix} = 0,$$

$$\frac{\begin{bmatrix} b_1 & b_2 \end{bmatrix}}{\sqrt{1 + b_1^2 + b_2^2}} - \lambda_2 \begin{bmatrix} 1 & -2(a_2 + b_2) \end{bmatrix} = 0,$$

$$a_2 - a_1 - 1 = 0,$$

$$a_1 + b_1 - (a_2 + b_2)^2 = 0,$$

which consists of six equations in six variables. Solving simultaneously yields

$$a = \begin{bmatrix} -\frac{1}{8} \\ \frac{7}{8} \end{bmatrix}, \quad b = \begin{bmatrix} \frac{3}{8} \\ -\frac{3}{8} \end{bmatrix},$$

$$\lambda_{1,2} = \frac{3}{\sqrt{82}}.$$

The optimal end points are

$$x^*(0) = a = \begin{bmatrix} -\frac{1}{8} \\ \frac{7}{8} \end{bmatrix},$$

$$x^*(1) = a + b = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

3.3.10 Integral Constraints

In calculus of variations problems, we may also encounter an equality constraint of the form

$$g(x) = \int_0^1 G\left(x, \dot{x}, t\right) dt = 0.$$

If we assume $G \in C^2$, then Theorem 3.40 guarantees that g is strictly Frechet differentiable. The derivative is given by

$$\begin{aligned} \delta g(x^*) h &= \int_0^1 \left(\left.\frac{\partial G}{\partial x}\right|_{x^*} h + \left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*} \dot{h} \right) dt \\ &= \int_0^1 \left(\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*} \right) \right) h dt + \left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=1} h(1) - \left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=0} h(0). \end{aligned}$$

Theorem 3.47 *$x^* \in \Omega$ is regular iff at least one of the following conditions holds:*
1) $\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*} \right) \not\equiv 0$
2) $\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=0} \neq 0$
3) $\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=1} \neq 0$

Proof. $\delta g(x^*) : X \to \mathbb{R}$ is clearly linear, so $\delta g(x^*)$ is onto iff there exists h such that $\delta g(x^*) h \neq 0$. If 1)-3) all fail, then $\delta g(x^*) = 0$ and x^* is not regular. Conversely, if 1) holds, then the fundamental lemma (Theorem 3.15) guarantees that there exists h such that $h(0) = h(1) = 0$ and $\delta g(x^*) h \neq 0$. If 1) fails but 2) holds, choose any h such that

$$h(0) = -\frac{1}{\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=0}}$$

and $h(1) = 0$. If 1) and 2) fail, but 3) holds, choose h such that

$$h(1) = \frac{1}{\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=1}}.$$

In either case, $\delta g(x^*)\,h = 1$. ■

For an integral constraint, the Lagrangian is

$$L(x,\lambda) = \int_0^1 F\left(x,\dot{x},t\right)dt - \lambda\int_0^1 G\left(x,\dot{x},t\right)dt.$$

Applying integration by parts and the fundamental lemma in the usual way leads to the critical point conditions

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right) = \lambda\left(\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*}\right)\right), \tag{3.32}$$

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=0} = \lambda\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*,t=0},$$

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*,t=1} = \lambda\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*,t=1}. \tag{3.33}$$

Example 3.28 *Find the curve x^* maximizing*

$$J(x) = \int_0^1 x(t)\,dt$$

subject to

$$\int_0^1 x^2(t)\,dt = 1.$$

Setting

$$G\left(x,\dot{x},t\right) = x^2 - 1,$$

we obtain

$$\frac{\partial G}{\partial x} = 2x.$$

Since $x \equiv 0$ does not lie in Ω, every x is regular by Theorem 3.47, part 1). Equation (3.32) yields

$$1 = 2\lambda x.$$

The boundary conditions are trivial, so we need only compare the extremals

$$x^* \equiv \frac{1}{2\lambda}$$

to the constraint

$$g(x^*) = \int_0^1 \frac{1}{4\lambda^2}dt = \frac{1}{4\lambda^2} = 1,$$

yielding $\lambda = \pm\frac{1}{2}$ and $x^* \equiv \pm 1$. Obviously, the maximum must be nonnegative.

3.3.11 Non-strict Inequality Constraints

Consider a non-strict inequality constraint

$$\Omega = \left\{ x \in X \middle| \, g(x) \geq 0 \right\},$$

where $g : X \to \mathbb{R}^m$ is strictly Frechet differentiable. As in the finite-dimensional case, write

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}.$$

Partitioning the constraint set

$$\Omega = \bigcup_{\pi} \left(\Omega_\pi \cap \Omega_{\pi_+} \right),$$

allows us to apply Lagrange multipliers to each piece separately. Theorem 2.21 (and its proof) applies to the infinite-dimensional setting verbatim:

Theorem 3.48 *If g is continuous, $x^* \in \Omega_\pi \cap \Omega_{\pi_+}$, and J achieves a constrained local extremum at x^* subject to Ω, then J achieves a constrained local extremum (of the same type) at x^* subject to Ω_π.*

Setting

$$L_\pi(x, \mu, \lambda) = \mu J(x) - \lambda^T g_\pi,$$

we say $x^* \in \Omega_\pi$ is a *critical point of* L if x^* is a critical point of L_π.

Theorem 3.49 *(Kuhn-Tucker Theorem) Let X be a Banach space and J and g be strictly Frechet differentiable at $x^* \in \Omega_\pi$. If J achieves a constrained local extremum at x^* subject to $g(x) \geq 0$, then x^* is a critical point of L. If the extremum is a constrained local minimum (maximum) at x^*, then $\lambda \geq 0$ ($\lambda \leq 0$).*

Proof. Apply Theorems 3.44 and 3.48. The sign of λ is proven in Luenberger, Section 9.4, Theorem 1. ■

In this setting, $x^* \in \Omega_\pi \cap \Omega_{\pi_+}$ is *regular* if $\delta g_\pi(x^*)$ is onto.

3.3.12 Integral Constraint with Inequality

Let

$$J(x) = \int_0^1 F\left(x, \dot{x}, t\right) dt$$

and consider the constraint

$$g(x) = \int_0^1 G\left(x, \dot{x}, t\right) dt \geq 0.$$

Ω consists of two components:

$$\Omega = \{g(x) > 0\} \cup \{g(x) = 0\}.$$

A point $x^* \in \Omega$ is regular if either $g(x^*) > 0$ or one of the conditions in Theorem 3.47 holds. According to the Kuhn-Tucker theorem, we need to check for critical points of J with $g(x) > 0$ and then apply Lagrange multipliers to

$$L_{\{1\}}(x, \mu, \lambda) = \mu J(x) - \lambda g(x).$$

Example 3.29 *Find the curve of minimum length connecting $t = 0$ and $t = 1$ subject to*

$$x(1) \geq x(0) + 1. \tag{3.34}$$

$$F\left(x, \dot{x}\right) = \sqrt{1 + \dot{x}^2}$$

$$G\left(x, \dot{x}, t\right) = \dot{x} - 1$$

For $g(x) > 0$, Theorem 3.16 states that the critical points of J are the solutions of

$$0 = \frac{\ddot{x}}{\left(1 + \dot{x}^2\right)^{\frac{3}{2}}},$$

$$\frac{\dot{x}(0)}{\sqrt{1 + \dot{x}^2(0)}} = \frac{\dot{x}(1)}{\sqrt{1 + \dot{x}^2(1)}} = 0.$$

Hence,

$$x(t) = a + bt$$

with $b = 0$. But this contradicts $g(x) > 0$.
Under $g(x) = 0$, Theorem 3.47, parts 2) and 3) guarantee that every point is regular, since $\frac{\partial G}{\partial \dot{x}} = 1$. From (3.32)-(3.33), the critical points are the solutions of

$$-\frac{\ddot{x}}{\left(1 + \dot{x}^2\right)^{\frac{3}{2}}} = 0,$$

$$\frac{\dot{x}(0)}{\sqrt{1 + \dot{x}^2(0)}} = \lambda,$$

$$\frac{\dot{x}(1)}{\sqrt{1 + \dot{x}^2(1)}} = \lambda,$$

and the constraint. These lead to

$$x = a + bt,$$

$$\lambda = \frac{b}{\sqrt{1 + b^2}},$$

$$b - 1 = 0,$$

$$x^*(t) = a + t,$$

$$\lambda = \frac{1}{\sqrt{2}}.$$

Since $\lambda \geq 0$, J does not achieve a maximum at x^*.

3.3.13 Mixed Constraints

The methods above can also be used to handle combinations of the various kinds of constraints. For example, suppose we have a combination of end point and integral constraints:

$$g(x) = \begin{bmatrix} \int_0^1 G\left(x, \dot{x}, t\right) dt \\ x(0) - a \\ x(1) - b \end{bmatrix}.$$

If G is linear, then g is affine and the problem can be reduced to an unconstrained one. In fact, even for nonlinear G, the end point constraints can be eliminated by shifting and redefining the underlying vector space. However, it is perhaps simpler to just apply Lagrange multiplier techniques to the given problem.

Theorem 3.50 *$x^* \in \Omega$ is regular iff*

$$\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*}\right) \not\equiv 0.$$

Proof. Let

$$g_1(x) = \int_0^1 G\left(x, \dot{x}, t\right) dt.$$

The derivative of g is

$$\delta g(x^*) h = \begin{bmatrix} \delta g_1(x^*) h \\ h(0) \\ h(1) \end{bmatrix},$$

where

$$\delta g_1(x^*) h = \int_0^1 \left(\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*}\right)\right) h dt + \left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=1} h(1) - \left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=0} h(0).$$

According to Theorem 3.41, x^* is regular iff the 3×3 matrix

$$\begin{bmatrix} \delta g(x^*) h_1 & \delta g(x^*) h_2 & \delta g(x^*) h_3 \end{bmatrix} = \begin{bmatrix} \delta g_1(x^*) h_1 & \delta g_1(x^*) h_2 & \delta g_1(x^*) h_3 \\ h_1(0) & h_2(0) & h_3(0) \\ h_1(1) & h_2(1) & h_3(1) \end{bmatrix}$$

is nonsingular for some h_i. If

$$\left.\frac{\partial G}{\partial x}\right|_{x^*} - \frac{d}{dt}\left(\left.\frac{\partial G}{\partial \dot{x}}\right|_{x^*}\right) \not\equiv 0, \tag{3.35}$$

then the fundamental lemma guarantees that there exists h_1 such that $h_1(0) = h_1(1) = 0$ and $\delta g_1(x^*) h_1 \neq 0$. Then we need only choose h_2 and h_3 to make

$$\begin{bmatrix} h_2(0) & h_3(0) \\ h_2(1) & h_3(1) \end{bmatrix}$$

nonsingular. Conversely, if (3.35) fails, then

$$\begin{bmatrix} \delta g(x^*) h_1 & \delta g(x^*) h_2 & \delta g(x^*) h_3 \end{bmatrix} = \begin{bmatrix} 1 & -\left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=0} & \left.\frac{\partial G}{\partial \dot{x}}\right|_{x=x^*, t=1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ h_1(0) & h_2(0) & h_3(0) \\ h_1(1) & h_2(1) & h_3(1) \end{bmatrix}$$

is singular. ■

Differentiation of the Lagrangian yields

$$\delta L\left(x^{*},\mu,\lambda\right)h=\mu\int_{0}^{1}\left(\left.\frac{\partial F}{\partial x}\right|_{x^{*}}-\frac{d}{dt}\left(\left.\frac{\partial F}{\partial\dot{x}}\right|_{x^{*}}\right)\right)hdt-\lambda_{1}\int_{0}^{1}\left(\left.\frac{\partial G}{\partial x}\right|_{x^{*}}-\frac{d}{dt}\left(\left.\frac{\partial G}{\partial\dot{x}}\right|_{x^{*}}\right)\right)hdt$$
$$+\mu\left(\left.\frac{\partial F}{\partial\dot{x}}\right|_{x=x^{*},t=1}h\left(1\right)-\left.\frac{\partial F}{\partial\dot{x}}\right|_{x=x^{*},t=0}h\left(0\right)\right)-\lambda_{1}\left(\left.\frac{\partial G}{\partial\dot{x}}\right|_{x=x^{*},t=1}h\left(1\right)-\left.\frac{\partial G}{\partial\dot{x}}\right|_{x=x^{*},t=0}h\left(0\right)\right)$$
$$-\lambda_{2}h\left(0\right)-\lambda_{3}h\left(1\right).$$

Hence, the critical point conditions are determined by

$$\left(\left.\frac{\partial F}{\partial x}\right|_{x^{*}}-\frac{d}{dt}\left(\left.\frac{\partial F}{\partial\dot{x}}\right|_{x^{*}}\right)\right)-\lambda_{1}\left(\left.\frac{\partial G}{\partial x}\right|_{x^{*}}-\frac{d}{dt}\left(\left.\frac{\partial G}{\partial\dot{x}}\right|_{x^{*}}\right)\right)=0, \tag{3.36}$$

$$\lambda_{2}=-\left.\frac{\partial F}{\partial\dot{x}}\right|_{x=x^{*},t=0}+\lambda_{1}\left.\frac{\partial G}{\partial\dot{x}}\right|_{x=x^{*},t=0},$$

$$\lambda_{3}=\left.\frac{\partial F}{\partial\dot{x}}\right|_{x=x^{*},t=1}-\lambda_{1}\left.\frac{\partial G}{\partial\dot{x}}\right|_{x=x^{*},t=1}.$$

We may treat λ_2 and λ_3 as extraneous variables, making (3.36) the only relevant condition.

Example 3.30 *Design a trough spanning* $(x,t)=(1,0)$ *to* $(1,1)$ *with unit breadth and surface area* $\frac{\pi}{2}$ *such that it holds maximum water. The problem corresponds to the functionals*

$$J\left(x\right)=\int_{0}^{1}\left(1-x\right)dt,$$

$$g\left(x\right)=\begin{bmatrix}\int_{0}^{1}\left(\sqrt{1+\dot{x}^{2}}-\frac{\pi}{2}\right)dt\\ x\left(0\right)-1\\ x\left(1\right)-1\end{bmatrix}.$$

The regularity condition is

$$\frac{\partial G}{\partial x}-\frac{d}{dt}\left(\frac{\partial G}{\partial\dot{x}}\right)=-\frac{d}{dt}\frac{\dot{x}}{\sqrt{1+\dot{x}^{2}}}=-\frac{\ddot{x}}{\left(1+\dot{x}^{2}\right)^{\frac{3}{2}}}\not\equiv0.$$

Hence, the only possible nonregular points have the form

$$x\left(t\right)=a+bt.$$

But then

$$g\left(x\right)=\begin{bmatrix}\int_{0}^{1}\left(\sqrt{1+b^{2}}-\frac{\pi}{2}\right)dt\\ a-1\\ a+b-1\end{bmatrix}=0,$$

which implies $a=1,\ b=0,$ *and*

$$\int_{0}^{1}\left(1-\frac{\pi}{2}\right)dt=0,$$

which is a contradiction. Consequently, every point in the constraint set is regular. The critical point condition is

$$-1 = -\lambda_1 \frac{d}{dt} \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}.$$

Obviously, $\lambda_1 \neq 0$, *so*

$$\frac{d}{dt} \frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = \frac{1}{\lambda_1},$$

$$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = \frac{t}{\lambda_1} + c,$$

$$\dot{x}^2 = \frac{(t+c\lambda_1)^2}{\lambda_1^2 - (t+c\lambda_1)^2}.$$

$$\dot{x} = \pm \frac{t+c\lambda_1}{\sqrt{\lambda_1^2 - (t+c\lambda_1)^2}},$$

$$x = \mp \sqrt{\lambda_1^2 - (t+c\lambda_1)^2} + d,$$

$$(x-d)^2 + (t+c\lambda_1)^2 = \lambda_1^2,$$

which describes a circle. The second and third constraints state that the circle passes through $(0,1)$ *and* $(1,1)$:

$$(1-d)^2 + (c\lambda_1)^2 = \lambda_1^2 \tag{3.37}$$

$$(1-d)^2 + (1+c\lambda_1)^2 = \lambda_1^2 \tag{3.38}$$

Subtracting (3.37) from (3.38) yields

$$c\lambda_1 = -\frac{1}{2}.$$

In terms of the angle θ *subtended by the circular arc, the length of the curve is*

$$|\lambda_1|\,\theta = \int_0^1 \sqrt{1+\dot{x}^2}\,dt = \frac{\pi}{2}.$$

From trigonometry,

$$\sin\frac{\theta}{2} = \frac{\frac{1}{2}}{|\lambda_1|} = \frac{\theta}{\pi},$$

so

$$\theta = \pi, \quad \lambda_1 = \pm\frac{1}{2}.$$

From (3.37),

$$d = 1 \pm \sqrt{\lambda_1^2 - (c\lambda_1)^2} = 1,$$

$$(x-1)^2 + \left(t - \frac{1}{2}\right)^2 = \frac{1}{4},$$

$$x^*(t) = 1 - \sqrt{1 - \left(t - \frac{1}{2}\right)^2}.$$

3.3.14 Variable Initial and Final Time

Another extension of the theory is to allow the end points of the time interval on which x is defined to be variable. Let X be the set of C^2 functions $x : [t_0, 1] \to \mathbb{R}$, where $t_0 < 1$ is a variable (i.e. dependent on x). Unfortunately, addition is not defined for two functions with different t_0, so X is not a vector space. Hence, there is no way to define a norm on X, and local extrema have no meaning. We are restricted to considering only global extrema. Although our previous techniques do not apply directly here, we can avoid these difficulties through "state augmentation" and reparametrization.

Consider the Banach space Y of C^2 functions $y : [0,1] \to \mathbb{R}^2$ and write

$$y(\theta) = \begin{bmatrix} y_1(\theta) \\ y_2(\theta) \end{bmatrix}.$$

We restrict attention to the set

$$\Omega_1 = \left\{ y \in Y \,\middle|\, y_2(0) < 1, \quad y_2'(\theta) > 0 \quad \forall \theta \in [0,1] \right\}.$$

By Theorem 3.33, Ω_1 is open. y_2 may be viewed as a reparametrization of time:

$$t = y_2(\theta). \tag{3.39}$$

Theorem 3.51 *If $f : [0,1] \to \mathbb{R}$ lies in C^2 and $f'(\theta) > 0$ for every θ, then f is one-to-one, f^{-1} lies in C^2, and*

$$\frac{d}{dt}\left(f^{-1}(t)\right) = \frac{1}{f'\left(f^{-1}(t)\right)}.$$

Proof. Suppose f is not one-to-one. Then there exist $a, b \in [0,1]$ with $a < b$ such that $f(a) = f(b)$. From the mean value theorem (Bartle, Theorem 27.6), there exists $\theta \in (a,b)$ such that

$$(b-a) f'(\theta) = f(b) - f(a) = 0.$$

Hence, $f'(\theta) = 0$, which is a contradiction. Hence, f is one-to-one and

$$f^{-1} : [f(0), f(1)] \to [0,1]$$

exists. From the inverse function theorem (Bartle, Theorem 41.8), $f^{-1} \in C^1$ with

$$\frac{d}{dt}\left(f^{-1}(t)\right) = \frac{1}{f'\left(f^{-1}(t)\right)}$$

for every $t \in [f(0), f(1)]$. But $f' \in C^1$, so

$$\frac{d^2}{dt^2}\left(f^{-1}(t)\right) = -\frac{f''\left(f^{-1}(t)\right) \frac{d}{dt}\left(f^{-1}(t)\right)}{\left(f'\left(f^{-1}(t)\right)\right)^2} = -\frac{f''\left(f^{-1}(t)\right)}{\left(f'\left(f^{-1}(t)\right)\right)^3}.$$

Since f^{-1}, f', and f'' are continuous, $f^{-1} \in C^2$. ■

In view of Theorem 3.51, $y \in \Omega_1$ implies $y_2^{-1} \in C^2$. Thus we can define a map

$$\Pi : \Omega_1 \to X$$

according to

$$\Pi(y) = y_1 \circ y_2^{-1}. \tag{3.40}$$

In other words, $y \in \Omega_1$ determines a unique function $x : [t_0, 1] \to \mathbb{R}$ given by

$$x(t) = y_1\left(y_2^{-1}(t)\right), \tag{3.41}$$

where $t_0 = y_2(0)$. Conversely, for any C^2 function $x : [t_0, 1] \to \mathbb{R}$ with $t_0 < 1$, we may set

$$y_2(\theta) = t_0 + (1 - t_0)\theta \tag{3.42}$$

and

$$y(\theta) = \begin{bmatrix} x(y_2(\theta)) \\ y_2(\theta) \end{bmatrix}.$$

Then

$$y_2(0) = t_0 < 1$$

and

$$y_2'(\theta) = 1 - t_0 > 0$$

for every $\theta \in [0, 1]$, so $y \in \Omega_1$ and

$$\Pi(y) = x \circ y_2 \circ y_2^{-1} = x.$$

This argument shows that Π maps Ω_1 **onto** X. The map is not one-to-one, since many different choices of y_1 and y_2 yield the same x in (3.40). For example, we may replace (3.42) with

$$y_2(\theta) = t_0 + \left(\frac{1 - t_0}{2}\right)\theta(1 + \theta).$$

Now consider the effect of the map Π on cost functionals

$$J(x) = \int_{t_0}^{1} F\left(x, \dot{x}, t\right) dt.$$

The change of variable (3.39) yields

$$dt = y_2'(\theta)\, d\theta, \tag{3.43}$$
$$x(t) = y_1\left(y_2^{-1}(t)\right) = y_1(\theta).$$

From theorem 3.51,

$$\dot{x}(t) = \frac{dx}{dt} = \frac{y_1'(\theta)\, d\theta}{y_2'(\theta)\, d\theta} = \frac{y_1'(\theta)}{y_2'(\theta)}. \tag{3.44}$$

Define

$$\widetilde{J}(y) = J(x) = \int_0^1 F\left(y_1(\theta), \frac{y_1'(\theta)}{y_2'(\theta)}, y_2(\theta)\right) y_2'(\theta)\, d\theta.$$

Then each $y \in \Omega_1$ maps into an $x \in X$ with the same cost. Hence, every global extremum $x^* \in X$ corresponds to a global extremum in $y^* \in Y$ with $\Pi(y^*) = x^*$. Since Π is not one-to-one, y^* is not unique. However, extrema need not be strict in order to apply Lagrange multipliers.

Let

$$\widetilde{F}(y, y') = F\left(y_1, \frac{y_1'}{y_2'}, y_2\right) y_2'.$$

Then

$$\frac{\partial \widetilde{F}}{\partial y} = \begin{bmatrix} \frac{\partial \widetilde{F}}{\partial y_1} & \frac{\partial \widetilde{F}}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial x} y_2' & \frac{\partial F}{\partial t} y_2' \end{bmatrix},$$

$$\frac{\partial \widetilde{F}}{\partial y'} = \begin{bmatrix} \frac{\partial \widetilde{F}}{\partial y_1'} & \frac{\partial \widetilde{F}}{\partial y_2'} \end{bmatrix} = \begin{bmatrix} \frac{\partial F}{\partial \dot{x}} & F - \frac{\partial F}{\partial \dot{x}} \frac{y_1'}{y_2'} \end{bmatrix},$$

so Euler's equation becomes

$$\frac{\partial F}{\partial x} y_2' = \frac{d}{d\theta} \left(\frac{\partial F}{\partial \dot{x}} \right), \tag{3.45}$$

$$\frac{\partial F}{\partial t} y_2' = \frac{d}{d\theta} \left(F - \frac{\partial F}{\partial \dot{x}} \frac{y_1'}{y_2'} \right). \tag{3.46}$$

In view of (3.43), we may divide (3.45) by y_2' to obtain the original form of Euler's equation

$$\frac{\partial F}{\partial x} = \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right).$$

Dividing (3.46) by y_2' and applying (3.43) and (3.44) yields

$$\begin{aligned}
\frac{\partial F}{\partial t} &= \frac{d}{dt} \left(F - \frac{\partial F}{\partial \dot{x}} \dot{x} \right) \\
&= \frac{\partial F}{\partial x} \dot{x} + \frac{\partial F}{\partial \dot{x}} \ddot{x} + \frac{\partial F}{\partial t} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) \dot{x} - \frac{\partial F}{\partial \dot{x}} \ddot{x} \\
&= \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{x}} \right) \right) \dot{x} \\
&= \frac{\partial F}{\partial t},
\end{aligned}$$

which provides no information.

Without any additional constraints at $\theta = 0$, we have the boundary condition

$$\left. \frac{\partial \widetilde{F}}{\partial y'} \right|_{\theta=0} = \begin{bmatrix} \left. \frac{\partial F}{\partial \dot{x}} \right|_{t=t_0} & F(x(t_0), \dot{x}(t_0), t_0) - \left. \frac{\partial F}{\partial \dot{x}} \right|_{t=t_0} \dot{x}(t_0) \end{bmatrix} = 0$$

or, equivalently,

$$\left. \frac{\partial F}{\partial \dot{x}} \right|_{t=t_0} = 0, \tag{3.47}$$

$$F(x(t_0), \dot{x}(t_0), t_0) = \left. \frac{\partial F}{\partial \dot{x}} \right|_{t=t_0} \dot{x}(t_0). \tag{3.48}$$

At $\theta = 1$, (3.41) implies $y_1(1) = x(1)$ so we must apply the end point constraint $y_2(1) = 1$. This defines a terminal manifold corresponding to

$$\phi_1(y(1)) = y_2(1) - 1.$$

Every $y \in Y$ is regular, since

$$\frac{\partial \phi_1}{\partial y(1)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \neq 0.$$

The transversality condition is

$$\left[\begin{array}{cc} \frac{\partial \widetilde{F}}{\partial y_1'} & \frac{\partial \widetilde{F}}{\partial y_2'} \end{array}\right]\Big|_{\theta=1} + \lambda \frac{\partial \phi_1}{\partial y(1)} = 0$$

(cf. equation (3.25)). Equivalently,

$$\left[\begin{array}{cc} \frac{\partial F}{\partial \dot{x}}\Big|_{t=1} & F(x(1), \dot{x}(1), 1) - \frac{\partial F}{\partial \dot{x}}\Big|_{t=1} \frac{y_1'(1)}{y_2'(1)} + \lambda \end{array}\right] = 0.$$

The second entry merely provides the value of λ, leaving us with

$$\frac{\partial F}{\partial \dot{x}}\Big|_{t=1} = 0. \tag{3.49}$$

Equations (3.47) and (3.49) are just the usual boundary conditions encountered when there are no end point constraints. Condition (3.48) is unique to the variable initial time problem.

If $t_0 = 0$ and t_1 is variable, a similar analysis yields Euler's equation and the boundary conditions

$$\frac{\partial F}{\partial \dot{x}}\Big|_{t=0} = \frac{\partial F}{\partial \dot{x}}\Big|_{t=t_1} = 0$$

along with the extra equation

$$F(x(t_1), \dot{x}(t_1), t_1) = \frac{\partial F}{\partial \dot{x}}\Big|_{t=t_1} \dot{x}(t_1). \tag{3.50}$$

If both t_0 and t_1 are variable, then both (3.48) and (3.50) must be applied.

Various constraints may be imposed on this general framework. For example, end point conditions

$$x(t_0) = a, \tag{3.51}$$

$$x(1) = b \tag{3.52}$$

map to the manifold constraint

$$\phi_0(y(0)) = y_1(0) - a$$

and the final end point condition

$$y(1) = \left[\begin{array}{c} b \\ 1 \end{array}\right].$$

Every $y \in Y$ is regular, since

$$\frac{\partial \phi_0}{\partial y(0)} = \left[\begin{array}{cc} 1 & 0 \end{array}\right] \neq 0.$$

The transversality condition is

$$\left[\begin{array}{cc} \frac{\partial \widetilde{F}}{\partial y_1'} & \frac{\partial \widetilde{F}}{\partial y_2'} \end{array}\right]\Big|_{\theta=0} + \lambda \frac{\partial \phi_0}{\partial y(0)} = \left[\begin{array}{cc} \frac{\partial F}{\partial \dot{x}}\Big|_{t=t_0} + \lambda & F(x(t_0), \dot{x}(t_0), t_0) - \frac{\partial F}{\partial \dot{x}}\Big|_{t=t_0} \dot{x}(t_0) \end{array}\right] = 0$$

or

$$F(x(t_0), \dot{x}(t_0), t_0) = \frac{\partial F}{\partial \dot{x}}\Big|_{t=t_0} \dot{x}(t_0).$$

Hence, we retain (3.48) and replace (3.47) by (3.51). The final end point constraint replaces the terminal manifold ϕ_1, so (3.49) is replaced by (3.52).

Alternatively, we may require that the initial value $x(t_0)$ lie on a manifold

$$\phi_0(x(t_0), t_0) = 0. \tag{3.53}$$

Note that the manifold is time-varying, since ϕ_0 depends explicitly on t_0. This constraint can be equivalently imposed on $y(0)$ by defining $\widetilde{\phi}_0 : \mathbb{R}^2 \to \mathbb{R}$ according to

$$\widetilde{\phi}_0(y(0)) = \phi_0(y_1(0), y_2(0)).$$

The Jacobian of $\widetilde{\phi}_0$ is

$$\frac{\partial \widetilde{\phi}_0}{\partial y(0)} = \begin{bmatrix} \frac{\partial \phi_0}{\partial x(t_0)} & \frac{\partial \phi_0}{\partial t_0} \end{bmatrix}.$$

A point $y \in \Omega$ is regular iff either

$$\frac{\partial \phi_0}{\partial x(t_0)} \neq 0 \tag{3.54}$$

or

$$\frac{\partial \phi_0}{\partial t_0} \neq 0. \tag{3.55}$$

The transversality condition is

$$\left.\frac{\partial \widetilde{F}}{\partial y'}\right|_{\theta=0} + \lambda \frac{\partial \widetilde{\phi}_0}{\partial y(0)} = 0.$$

In terms of x,

$$\begin{bmatrix} \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0} & F\left(x(t_0), \dot{x}(t_0), t_0\right) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0} \dot{x}(t_0) \end{bmatrix} + \lambda \begin{bmatrix} \frac{\partial \phi_0}{\partial x(t_0)} & \frac{\partial \phi_0}{\partial t_0} \end{bmatrix} = 0. \tag{3.56}$$

If (3.54) holds, then we may solve for λ from the first entry in (3.56) and substitute it into the second to obtain

$$\left(F(x(t_0), \dot{x}(t_0), t_0) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0} \dot{x}(t_0)\right) \frac{\partial \phi_0}{\partial x(t_0)} = \frac{\partial \phi_0}{\partial t_0} \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0}. \tag{3.57}$$

Similarly, if (3.55) holds, we may eliminate λ from the second entry and substitute it into the first to obtain (3.57). If t_1 is variable, we also have the condition

$$\left(F(x(t_1), \dot{x}(t_1), t_1) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_1} \dot{x}(t_1)\right) \frac{\partial \phi_1}{\partial x(t_1)} = \frac{\partial \phi_1}{\partial t_1} \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_1}. \tag{3.58}$$

Example 3.31 *Find the curve of minimum length joining the manifolds*

$$x(t_0) = t_0^2,$$

$$x(t_1) = t_1 - 1.$$

In this problem $n = 1$, the integrand is

$$F\left(x, \dot{x}\right) = \sqrt{1 + \dot{x}^2},$$

and the manifolds are described by

$$\phi_0(x(t_0), t_0) = x(t_0) - t_0^2,$$

$$\phi_1 (x (t_1), t_1) = x (t_1) - t_1 + 1.$$

Euler's equation is

$$0 = \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{1 + \dot{x}^2}} \right) = \frac{\ddot{x}}{\left(1 + \dot{x}^2\right)^{\frac{3}{2}}},$$

yielding

$$x = c + dt.$$

The transversality conditions are

$$\left(F (x (t_0), \dot{x} (t_0), t_0) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0} \dot{x} (t_0) \right) \frac{\partial \phi_0}{\partial x (t_0)} = \frac{\partial \phi_0}{\partial t_0} \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0},$$

$$\left(F (x (t_1), \dot{x} (t_1), t_1) - \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_0} \dot{x} (t_1) \right) \frac{\partial \phi_1}{\partial x (t_1)} = \frac{\partial \phi_1}{\partial t_1} \left.\frac{\partial F}{\partial \dot{x}}\right|_{t=t_1},$$

from which we obtain

$$\left(\sqrt{1 + d^2} - \frac{d^2}{\sqrt{1 + d^2}} \right) = -2t_0 \frac{d}{\sqrt{1 + d^2}},$$

$$\left(\sqrt{1 + d^2} - \frac{d^2}{\sqrt{1 + d^2}} \right) = -\frac{d}{\sqrt{1 + d^2}}$$

to yield the solution

$$d = -1, \quad t_0 = \frac{1}{2},$$

$$c - \frac{1}{2} = x (t_0) = t_0^2 = \frac{1}{4},$$

$$c = \frac{3}{4},$$

$$\frac{3}{4} - t_1 = x (t_1) = t_1 - 1,$$

$$t_1 = \frac{7}{8}.$$

3.3.15 Second Derivative Conditions

Recalling Example 3.20, the second derivative sufficient conditions (Theorem 3.28) are not applicable to calculus of variations problems framed in C^2. Furthermore, as in the finite-dimensional case, J having a constrained extremum at x^* does not guarantee that $L (\cdot, \lambda)$ has a local extremum at x^* for any λ. Hence, Legendre's condition (Theorem 3.30) is the only appropriate use of second derivatives for variational calculus in C^2.

Applicability of Second Derivative Conditions

	Unconstrained	Constrained
Necessary	Yes	No
Sufficient	No	No

3.4 L^2 Theory

3.4.1 Functionals on L^2

Besides C^2, another important space in which calculus of variations problems can be posed is L^2 :

$$X = \left\{ x : [0,1] \to \mathbb{R} \,\middle|\, \int_0^1 x^2(t)\,dt < \infty \right\},$$

$$\|x\| = \sqrt{\int_0^1 x^2(t)\,dt}.$$

It can be shown that X is a Banach space. One disadvantage here is that many L^2 functions are not differentiable, so the cost functional J cannot depend on $\dot{x}$:

$$J(x) = \int_0^1 F(x,t)\,dt \tag{3.59}$$

(We will see later that problems where F depends on $\dot{x}$ can be handled in the "optimal control" framework.) Another disadvantage is that varying $x(t)$ at a single value of t does not affect J, so imposing constraints on $x(0)$ and $x(1)$ has no meaning. L^2 analysis works best for problems with an integral constraint.

Suppose $F \in C^2$ and that there exists $M < \infty$ such that

$$|F(x,t)| \leq Mx^2, \tag{3.60}$$

$$\left| \frac{\partial F(x,t)}{\partial x} \right| \leq M|x|, \tag{3.61}$$

$$\left| \frac{\partial^2 F(x,t)}{\partial x^2} \right| \leq M \tag{3.62}$$

for every $x \in \mathbb{R}$ with $|x| > M$ and every $t \in [0,1]$. These assumptions ensure that the cost functional (3.59) and the integrals

$$J_1(x)\,h = \int_0^1 \frac{\partial F}{\partial x} h\,dt,$$

$$J_2(x)\,h = \int_0^1 \frac{\partial^2 F}{\partial x^2} h^2\,dt$$

all exist. J_1 and J_2 are the obvious candidates for the derivatives of J.

Theorem 3.52 *J is strictly Frechet differentiable with $J'(x) = J_1(x)$ for every $x \in X$.*

3.4.2 Second Derivatives

Unfortunately, assuming $F \in C^2$ and (3.60)-(3.62) is not enough to guarantee that J is twice Frechet differentiable.

Example 3.32 *Let*

$$F(x) = x^2 \arctan x.$$

Then

$$F'(x) = 2x \arctan x + \frac{x^2}{1+x^2},$$

$$F''(x) = 2 \arctan x + \frac{2x(2+x^2)}{(1+x^2)^2}.$$

Since

$$|\arctan x| < \pi,$$

$$|F'(x)| \leq \frac{\pi}{2} x^2,$$

$$|F'(x)| \leq 2|x| |\arctan x| + \left|\frac{x^2}{1+x^2}\right| < \pi |x| + 1 < 4|x|; \quad |x| \geq 4,$$

$$|F''(x)| \leq 2|\arctan x| + 2\frac{|x| |2+x^2|}{|1+x^2|^2} < 4.72.$$

For $x^* = 0$,

$$\begin{aligned}
&\frac{J(x^*+h) - J(x^*) - \int_0^1 F'(x^*) h dt - \frac{1}{2}\int_0^1 F''(x^*) h^2 dt}{\|h\|^2} \\
&= \frac{\int_0^1 \left(F(x^*+h) - F(x^*) - F'(x^*) h - \frac{1}{2} F''(x^*) h^2\right) dt}{\|h\|^2} \\
&= \frac{\int_0^1 h^2 \arctan h dt}{\|h\|^2}.
\end{aligned}$$

Let

$$h_\delta(t) = \begin{cases} \delta^{-\frac{1}{4}}, & 0 \leq t \leq \delta \\ 0, & \delta < t \leq 1 \end{cases}.$$

Then

$$\|h_\delta\|^2 = \int_0^1 h^2 dt = \int_0^\delta \delta^{-\frac{1}{2}} dt = \delta^{\frac{1}{2}} \to 0$$

as $\delta \to 0$, *but*

$$\frac{\int_0^1 h_\delta^2 \arctan h_\delta dt}{\|h_\delta\|^2} = \frac{\int_0^\delta \delta^{-\frac{1}{2}} \arctan \delta^{-\frac{1}{2}} dt}{\delta^{\frac{1}{2}}} = \arctan \delta^{-\frac{1}{2}} \to \frac{\pi}{2},$$

so J *is not twice Frechet differentiable.*

We can prove a weaker result which will lead to necessary conditions on the second derivative.

Theorem 3.53 *J is twice Gateaux differentiable with $\delta^2 J(x) = J_2(x)$ for every $x \in X$.*

Theorem 3.54 *1) If $\delta^2 J(x^*) \geq 0$, then $\left.\frac{\partial^2 F}{\partial x^2}\right|_{x^*} \geq 0$ for every $t \in [0,1]$.*
2) If $\delta^2 J(x^*) \geq 0$, then $\left.\frac{\partial^2 F}{\partial x^2}\right|_{x^*} \geq 0$ for every $t \in [0,1]$.

Proof. 1) Suppose

$$\frac{\partial^2 F}{\partial x^2}\Big|_{x^*} < 0 \tag{3.63}$$

for some $t_0 \in [0,1]$. Since $F \in C^2$, $\frac{\partial^2 F}{\partial x^2}$ is continuous, so there exists $a, b \in \mathbb{R}$ such that $a < t_0 < b$ and (3.63) holds on (a,b). Let

$$h(t) = \begin{cases} 1, & a < t < b \\ 0, & \text{else} \end{cases}.$$

Then $h \in X$ and

$$\delta^2 J(x^*)\, h = \int_a^b \frac{\partial^2 F}{\partial x^2}\Big|_{x^*} dt < 0,$$

contradicting $\delta^2 J(x^*) \geq 0$.

2) Similar to part 1). ■

The applicability of second derivative necessary and sufficient conditions is the same as in the case $X = C^2$.

3.4.3 Integral Constraints

Now consider an integral constraint

$$g(x) = \int_0^1 G(x,t)\, dt,$$

where $G \in C^2$,

$$|G(x,t)| \leq M x^2,$$

$$\left|\frac{\partial G(x,t)}{\partial x}\right| \leq M|x|,$$

$$\left|\frac{\partial^2 G(x,t)}{\partial x^2}\right| \leq M.$$

By Theorem 3.52, g is strictly Frechet differentiable with

$$g'(x^*)\, h = \int_0^1 \frac{\partial G}{\partial x}\Big|_{x^*} h dt.$$

Applying the Lagrange multiplier theorem (Theorem 3.44) yields

$$J'(x^*)\, h - \lambda g'(x^*)\, h = \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{x^*} - \lambda \frac{\partial G}{\partial x}\Big|_{x^*}\right) h dt = 0 \tag{3.64}$$

for every $h \in X$. We need a fundamental lemma applicable to problems in L^2.

Theorem 3.55 *(Fundamental Lemma in L^2) If $x \in X$ and*

$$\int_0^1 x^T h dt = 0$$

for every $h \in X$, then $x = 0$.

Proof. Setting $h = x$ yields

$$\|x\|^2 = \int_0^1 x^T x dt = 0.$$

Since the norm is positive definite, $x = 0$. ■

Applying Theorem 3.55 to (3.64) yields

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} = \lambda \left.\frac{\partial G}{\partial x}\right|_{x^*}. \tag{3.65}$$

(Compare equations (3.32) and (3.65).)

3.4.4 Quadratic Cost

Let

$$X = \left\{ x : [0,1] \to \mathbb{R}^n \;\middle|\; \int_0^1 x^T x dt < \infty \right\},$$

$$\|x\| = \sqrt{\int_0^1 x^T x dt}.$$

Recall that a functional J is quadratic if there exists a bilinear functional K such that $J(x) = K(x, x)$ for every $x \in X$. In particular, let $Q : [0,1] \to \mathbb{R}^{n \times n}$ be continuous and

$$K(x, y) = \int_0^1 x^T P(t) y dt.$$

Then

$$K(x+z, y) = \int_0^1 (x+z)^T P(t) y dt = \int_0^1 x^T P(t) y dt + \int_0^1 z^T P(t) y dt = K(x, y) + K(z, y),$$

$$K(x, y+z) = \int_0^1 x^T P(t)(y+z) dt = \int_0^1 x^T P(t) y dt + \int_0^1 x^T P(t) z dt = K(x, y) + K(x, z),$$

so K is bilinear. Setting

$$F(x, t) = x^T P(t) x$$

yields a quadratic cost functional

$$J(x) = \int_0^1 x^T P(t) x dt. \tag{3.66}$$

Note that, since P is continuous, there exists $M < \infty$ such that

$$\|P(t)\| < M$$

for every $t \in [0,1]$.

Theorem 3.56 *J is continuous.*

Proof. The result follows by observing that

$$\begin{aligned}
|J(x) - J(x^*)| &= \left| \int_0^1 \left(x^T P(t) x - x^{*T} P(t) x^* \right) dt \right| \\
&= \left| \int_0^1 \left((x - x^*)^T P(t) x + x^{*T} P(t) (x - x^*) \right) dt \right| \\
&\leq \int_0^1 \left| (x - x^*)^T P(t) x \right| dt + \int_0^1 \left| x^{*T} P(t) (x - x^*) \right| dt \\
&\leq \int_0^1 \|P(t)\| \|x - x^*\| \|x\| dt + \int_0^1 \|P(t)\| \|x^*\| \|x - x^*\| dt \\
&\leq M \|x - x^*\| (\|x\| + \|x^*\|) \\
&\to 0
\end{aligned}$$

as $x \to x^*$. ■

Functionals of the form (3.66) are particularly suitable for L^2 analysis.

In view of Theorem 3.10, J is twice Frechet differentiable with

$$J'(x) h = K(x, h) + K(h, x) = \int_0^1 x^T P(t) h dt + \int_0^1 h^T P(t) x dt = \int_0^1 x^T \left(P(t) + P^T(t) \right) h dt,$$

$$J''(x) h = 2J(h) = 2 \int_0^1 h^T P(t) h dt.$$

Theorem 3.57 *Let $x \in X$.*
1) $J''(x) \geq 0$ iff $P(t) \geq 0$ for every $t \in [0, 1]$.
2) $J''(x) \leq 0$ iff $P(t) \leq 0$ for every $t \in [0, 1]$.
3) $J''(x) > 0$ iff $P(t) > 0$ for every $t \in [0, 1]$.
4) $J''(x) < 0$ iff $P(t) < 0$ for every $t \in [0, 1]$.

Proof. 1) If $P(t) \geq 0$ for every t, then

$$h^T(t) P(t) h(t) \geq 0$$

for every $t \in [0, 1]$. Hence,

$$J''(x) h = 2 \int_0^1 h^T P(t) h dt \geq 0.$$

Conversely, if $P(t) \ngeq 0$ for every t, then there exists $v \in \mathbb{R}^n$ and $t_1 \in [0, 1]$ such that $v^T P(t_1) v < 0$. Since P is continuous, there exists an interval $(a, b) \subset [0, 1]$ such that $v^T P(t_1) v < 0$ for all $t \in (a, b)$. Setting

$$h(t) = \begin{cases} v, & a < t < b \\ 0, & \text{else} \end{cases}$$

yields $h \in X$ and

$$J''(x) h = 2 \int_a^b v^T P(t) v dt < 0.$$

2) Apply part 1) to $-J$ and $-P$.

3) From Theorem 2.8, $P(t) > 0$ implies that every eigenvalue of

$$Q(t) = P(t) + P^T(t)$$

satisfies $\lambda_i(t) > 0$. Since Q is continuous, the eigenvalues are continuous, so Theorem 3.1 guarantees that there exists $\varepsilon > 0$ such that $\lambda_i(t) > \varepsilon$ for every i and t. Hence, the eigenvalues of $Q(t) - \varepsilon I$ satisfy $\lambda_i(t) - \varepsilon > 0$, making $Q(t) - \varepsilon I > 0$ for every t and

$$v^T Q(t) v > \varepsilon v^T v$$

for every $v \in \mathbb{R}^n - \{0\}$ and $t \in [0,1]$. Thus

$$J''(x) h = 2 \int_0^1 h^T P(t) h dt = \int_0^1 h^T Q(t) h dt > \varepsilon \int_0^1 h^T h dt = \varepsilon \|h\|^2$$

for all $h \neq 0$, so $J''(x) > 0$.

Conversely, if $P(t) \not> 0$ for every t, then there exists $v \neq 0$ and $t_1 \in [0,1]$ such that $v^T Q(t_1) v \leq 0$. If $t_1 > 0$, let

$$h_\delta(t) = \begin{cases} v, & t_1 - \delta < t < t_1 \\ 0, & \text{else} \end{cases}.$$

Then $h_\delta \in X$ for every $\delta > 0$. Since P is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $t \in [t_1 - \delta, t_1]$ implies $v^T P(t) v < \frac{\varepsilon}{2} v^T v$. Hence,

$$J''(x) h_\delta = 2 \int_{t_1 - \delta}^{t_1} v^T P(t) v dt < \varepsilon \delta v^T v,$$

$$\|h_\delta\|^2 = \int_0^1 h_\delta^T h_\delta dt = \int_{t_1 - \delta}^{t_1} v^T v dt = \delta v^T v,$$

so

$$J''(x) h_\delta < \varepsilon \|h_\delta\|^2,$$

violating positive definiteness of $J''(x)$. If $t_1 = 0$, let

$$h_\delta(t) = \begin{cases} v, & 0 < t < \delta \\ 0, & \text{else} \end{cases}$$

and apply similar arguments.

4) Apply part 3) to $-J$ and $-P$. ■

Example 3.33 *Let*

$$P(t) = \begin{bmatrix} 1 & 2t \\ 0 & t^2 \end{bmatrix}.$$

Setting

$$Q = P + P^T,$$

Extrema must satisfy

$$Q(t) x = \begin{bmatrix} 2 & 2t \\ 2t & 2t^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2(x_1 + t x_2) \\ 2t(x_1 + t x_2) \end{bmatrix} = 0,$$

yielding

$$x^*(t) = \alpha \begin{bmatrix} -t \\ 1 \end{bmatrix}$$

for any $\alpha \in \mathbb{R}$. $Q(t)$ is positive semidefinite for every t, so $J''(x^*)$ is positive semidefinite for every α. This means that the x^* cannot be maxima. $Q(t)$ is not positive definite, so $J''(x^*)$ is not positive definite. Thus the sufficient condition for a strict local minimum fails.

3.4.5 Quadratic Cost and Affine Constraint

One of the most common scenarios in optimization theory involves quadratic cost and affine constraint. Let

$$J(x) = \int_0^1 x^T P(t)\, x dt,$$

$$g(x) = a + \int_0^1 y^T(t)\, x dt$$

for some $a \in \mathbb{R}$ and $y \in X - \{0\}$. From Theorem 2.3,

$$\int_0^1 \left| y^T(t)\, x \right| dt < \infty,$$

so $g(x)$ is well defined. Setting

$$x_0 = -\frac{a}{\|y\|^2} y$$

yields

$$g(x_0) = a - \frac{a}{\|y\|^2} \int_0^1 y^T(t)\, y(t)\, dt = 0.$$

Hence, we may reformulate the problem on the vector space

$$\widetilde{X} = -x_0 + \Omega = \left\{ x \in X \;\middle|\; \int_0^1 y^T(t)\, x dt = 0 \right\}$$

with cost

$$\begin{aligned}
\widetilde{J}(x) &= J(x + x_0) \\
&= \int_0^1 \left(x + x_0^T(t) \right) P(t) \left(x + x_0(t) \right) dt \\
&= \int_0^1 x_0^T(t)\, P(t)\, x_0(t)\, dt + \int_0^1 x_0^T(t) \left(P(t) + P^T(t) \right) x dt + \int_0^1 x^T P(t)\, x dt \\
&= c + \int_0^1 l^T(t)\, x dt + \int_0^1 x^T P(t)\, x dt,
\end{aligned}$$

where

$$c = \int_0^1 x_0^T(t)\, P(t)\, x_0(t)\, dt,$$

$$l = \left(P^T + P \right) x_0.$$

From Theorems 3.22, 3.23, and 3.24, $\widetilde{J}(x)$ is twice Frechet differentiable with

$$\widetilde{J}'(x)\, h = \int_0^1 l^T(t)\, h dt + \int_0^1 x(t)^T \left(P(t) + P^T(t) \right) h dt, \tag{3.67}$$

$$\widetilde{J}''(x)\, h = 2 \int_0^1 h^T P(t)\, h dt.$$

According to Theorem 3.57, if $P(t) \geq 0$ for every t, then J does not achieve a constrained maximum at any critical point. If $P(t) \leq 0$, then J does not achieve a constrained maximum. If $P(t) > 0$

for every t, then J achieves a strict constrained local minimum at every critical point. If $P(t) < 0$, then J achieves a strict constrained local maximum.

Unfortunately, Theorem 3.55 is not applicable here, since $\widetilde{X} \neq L^2$. A better approach is to return to the original formulation in terms of J and g and apply Lagrange multipliers. From Theorem 3.40, J is strictly Frechet differentiable. Since g is affine, Theorem 3.8 implies that g has Gateaux derivative

$$\delta g(x) h = g(h) - a = \int_0^1 y^T(t) h dt.$$

We can establish strict Frechet differentiability of g with the aid of a stronger version of the Cauchy-Schwarz Inequality.

Theorem 3.58 *(Cauchy-Schwarz Inequality) If $x, y \in X$, then*

$$\left| \int_0^1 y^T x dt \right| \leq \|x\| \|y\|.$$

Proof. Luenberger, Section 3.2, Lemma 1 ■

Theorem 3.59 *1) g is continuous.*
2) g is strictly Frechet differentiable.

Proof. 1) By the Cauchy-Schwarz inequality,

$$\begin{aligned} |g(x) - g(x^*)| &= \left| \int_0^1 y^T(t)(x - x^*) dt \right| \\ &\leq \|y(t)\| \|x - x^*\| \\ &\to 0 \end{aligned}$$

as $x \to x^*$.

2) For any $x, h \in X$ with $h \neq 0$,

$$\frac{g(x+h) - g(x) - \delta g(x) h}{\|h\|} = \frac{\int_0^1 \left(y^T(t)(x+h) - y^T(t) x - y^T(t) h \right) dt}{\|h\|} = 0.$$

■

Applying Lagrange multipliers (Theorem 3.44), the critical points x^* must satisfy

$$J'(x^*) h - \lambda g'(x^*) h = \int_0^1 x^{*T} \left(P(t) + P^T(t) \right) h dt - \lambda \int_0^1 y^T(t) h dt = 0.$$

From the fundamental lemma (Theorem 3.55),

$$\left(P^T + P \right) x^* - \lambda y = 0. \tag{3.68}$$

If P is definite, then $P^T(t) + P(t)$ is nonsingular for every t by Theorem 2.8. In this case,

$$x^* = \lambda \left(P^T + P \right)^{-1} y,$$

$$g(x^*) = a + \int_0^1 y^T(t) x^* dt = a + \lambda \int_0^1 y^T(t) \left(P^T(t) + P(t) \right)^{-1} y(t) dt = 0.$$

Assuming $y \neq 0$, we obtain

$$\lambda = -\frac{a}{\int_0^1 y^T \left(P^T + P\right)^{-1} y dt},$$

$$x^* = -\frac{a}{\int_0^1 y^T \left(P^T + P\right)^{-1} y dt} \left(P^T + P\right)^{-1} y,$$

which must be a strict constrained local extremum. If P is merely semidefinite, then $P^T + P$ is singular and the critical points must be found by solving (3.68) and $g(x) = 0$ simultaneously.

4 Optimal Control

4.1 L^2 Theory

4.1.1 Lagrange Multipliers

To handle problems in optimal control, we must generalize the Lagrange multiplier framework. Let X and Y be Banach spaces and consider cost and constraint functions

$$J : X \to \mathbb{R},$$

$$g : X \to Y.$$

Since g is a mapping between infinite-dimensional spaces, we refer to it as an *operator*. The theory of differentiation of operators is much the same as for functionals:

$$\frac{g(x + \alpha h) - g(x)}{\alpha} \to \delta g(x) h,$$

$$\frac{g(x + \alpha h) - g(x) - \delta g(x) h}{\alpha^2} \to \frac{1}{2} \delta^2 g(x) h$$

as $\alpha \to 0$;

$$\frac{g(x + h) - g(x) - g'(x) h}{\|h\|} \to 0,$$

$$\frac{g(x + h) - g(x) - g'(x) h - \frac{1}{2} g''(x) h}{\|h\|^2} \to 0$$

as $h \to 0$. The first Frechet derivative is *strict* if

$$\frac{g(x + h) - g(x) - g'(x^*) h}{\|h\|} \to 0$$

as $x \to x^*$ and $h \to 0$. All preceding results on derivatives of functionals (Theorems (3.4)-(3.10) and (3.20)-(3.24)) carry over verbatim to operators. The notion of regularity is also the same: x^* is *regular* if the linear operator

$$g'(x^*) : X \to Y$$

is onto.

We will perform our analysis in the space of L^2 functions

$$X = \left\{ x : [0, 1] \to \mathbb{R}^n \;\middle|\; \int_0^1 x^T x dt < \infty \right\},$$

$$Y = \left\{ y : [0,1] \rightarrow \mathbb{R}^k \;\middle|\; \int_0^1 y^T y dt < \infty \right\}.$$

In order to define the Lagrangian, we need the *inner product* of any two vectors $w, x \in X$:

$$\langle w, x \rangle = \int_0^1 x^T w dt.$$

Note that

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Similar statements apply to Y. In this context, the *Lagrangian* is

$$L : X \times \mathbb{R} \times Y \rightarrow \mathbb{R},$$

$$L(x, \mu, \lambda) = \mu J(x) - \langle g(x), \lambda \rangle .$$

We denote

$$L'(x, \mu, \lambda) = \mu J'(x) - \langle g'(x), \lambda \rangle .$$

Theorem 4.1 *(Lagrange Multipliers) Let J and g be strictly Frechet differentiable at $x^* \in X$. If J achieves a constrained local extremum at x^* subject to $g(x) = 0$, then*

$$L'(x^*, \mu, \lambda) = 0.$$

If x^* is regular, then we may set $\mu = 1$.

The proof is essentially the same as for $Y = \mathbb{R}^m$. (See Theorem 3.44.)

4.1.2 Differential Equations

The advantage of allowing g to map into an infinite-dimensional space is that now we can handle differential equations. Let

$$U = \left\{ u : [0,1] \rightarrow \mathbb{R}^m \;\middle|\; \int_0^1 u^T u dt < \infty \right\}$$

and consider the constraints

$$\dot{x}(t) = f(x(t), u(t)), \tag{4.1}$$

$$x(0) = x_0, \tag{4.2}$$

where $x \in X$ and $u \in U$. In control theory, the differential equation represents the dynamic system or "plant" that we wish to control.

The first issue we face is differentiability of x. It is an unfortunate fact that many L^2 functions are not differentiable. (For example, consider the unit step.) For this reason we must rewrite (4.1) in an L^2-friendly form.

Theorem 4.2 *Suppose f is continuous and there exists $M < \infty$ such that*

$$\|f(\xi, \nu)\| \leq M \left(\|\xi\|^2 + \|\nu\|^2 \right) \tag{4.3}$$

for every $\xi \in \mathbb{R}^n$, $\nu \in \mathbb{R}^m$ *with* $\|\xi\|^2 + \|\nu\|^2 > M$.
1)

$$\int_0^1 \|f(x(t), u(t))\| \, dt < \infty$$

for every $x \in X$ *and* $u \in U$.
2) The function $y : [0,1] \to \mathbb{R}^n$ *given by*

$$y(t) = \int_0^t f(x(t), u(t)) \, d\tau$$

belongs to X *and is differentiable for "almost every"* t.

Theorem 4.2 tells us that, if f satisfies the quadratic bound (4.3), then we may integrate through the differential equation (4.1) with (4.2) to obtain

$$x(t) - x_0 = \int_0^t f(x(\tau), u(\tau)) \, d\tau. \tag{4.4}$$

The integral equation (4.4) is equivalent to (4.1)-(4.2). Theorem 4.2 guarantees that the right side of (4.4) belongs to L^2 whenever $x \in L^2$.

4.1.3 A Maximum Principle

Perhaps the simplest optimal control problem may be stated as follows. Let

$$\begin{aligned} J &: X \times U \to \mathbb{R}, \\ J(x,u) &= \int_0^1 F(x,u) \, dt, \\ g &: X \times U \to X, \\ g(x,u)(t) &= x(t) - x_0 - \int_0^t f(x(\tau), u(\tau)) \, d\tau, \end{aligned} \tag{4.5}$$

where $x_0 \in \mathbb{R}^n$. Typically, we assume $F \in C^2$ and $f \in C^1$ so that we can take the Jacobian and Hessian of J and the Jacobian of f. It is easy to show that $X \times U$ is a Banach space. We further adopt the assumption (4.3) to ensure that g is a well-defined operator on $X \times U$.

We wish to find the extrema (x^*, u^*) of J subject to the equality constraint

$$(x^*, u^*) \in \Omega = \left\{ (x,u) \in X \times U \,\middle|\, g(x,u) = 0 \right\}.$$

Applying Lagrange multipliers,

$$\begin{aligned} J'(x^*, u^*)(h,k) &= \frac{d}{d\alpha} \int_0^1 F(x^* + \alpha h, u^* + \alpha k) \, dt \Big|_{\alpha=0} \\ &= \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{(x^*,u^*)} h + \frac{\partial F}{\partial u}\Big|_{(x^*,u^*)} k \right) dt, \end{aligned}$$

$$\begin{aligned} g'(x^*, u^*)(h,k) &= \frac{d}{d\alpha} \left(x^* + \alpha h - x_0 - \int f(x^* + \alpha h, u^* + \alpha k) \, d\tau \right) \Big|_{\alpha=0} \\ &= h - \int \left(\frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} h + \frac{\partial f}{\partial u}\Big|_{(x^*,u^*)} k \right) d\tau. \end{aligned}$$

(An integral written without limits denotes an indefinite integral.)

Theorem 4.3 *Let $A : [0,1] \to \mathbb{R}^{n\times n}$ be continuous and $\xi \in X$. Then there exists $x \in X$ such that*

$$x(t) - \int_0^t A(\tau)\, x(\tau)\, d\tau = \xi(t)$$

for every $t \in [0,1]$.

Proof. *Let Φ be the state-transition matrix of the time-varying linear state-space system corresponding to A – i.e.*

$$\frac{\partial \Phi(t,\tau)}{\partial t} = A(t)\,\Phi(t,\tau), \quad \Phi(\tau,\tau) = I$$

with $\Phi \in C^1$. Let

$$x(t) = \xi(t) + \int_0^t \Phi(t,\tau)\, A(\tau)\,\xi(\tau)\, d\tau.$$

Then

$$\int_\eta^t A(\tau)\,\Phi(\tau,\eta)\, d\tau = \int_\eta^t \frac{\partial \Phi(\tau,\eta)}{\partial \tau} d\tau = \Phi(t,\eta) - I,$$

$$\begin{aligned}\int_0^t A(\tau)\left(\int_0^\tau \Phi(\tau,\eta)\, A(\eta)\,\xi(\eta)\, d\eta\right) d\tau &= \int_0^t \left(\int_\eta^t A(\tau)\,\Phi(\tau,\eta)\, d\tau\right) A(\eta)\,\xi(\eta)\, d\eta \\ &= \int_0^t (\Phi(t,\eta) - I)\, A(\eta)\,\xi(\eta)\, d\eta,\end{aligned}$$

$$\begin{aligned}x(t) - \int_0^t A(\tau)\, x(\tau)\, d\tau &= \xi(t) + \int_0^t \Phi(t,\tau)\, A(\tau)\,\xi(\tau)\, d\tau \\ &\quad - \int_0^t A(\tau)\left(\xi(\tau) + \int_0^\tau \Phi(\tau,\eta)\, A(\eta)\,\xi(\eta)\, d\eta\right) d\tau \\ &= \xi(t) + \int_0^t \Phi(t,\tau)\, A(\tau)\,\xi(\tau)\, d\tau \\ &\quad - \int_0^t A(\tau)\,\xi(\tau)\, d\tau - \int_0^t (\Phi(t,\eta) - I)\,(A(\eta)\,\xi(\eta))\, d\eta \\ &= \xi(t).\end{aligned}$$

■

Theorem 4.4 *Every $(x^*, u^*) \in \Omega$ is regular.*

Proof. Setting

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}$$

in Theorem 4.3 shows that for every $\xi \in X$ there exists $h \in X$ such that

$$g'(x^*, u^*)(h, 0) = \xi.$$

Hence, $g'(x^*, u^*)$ is onto. ■

The critical points are the solutions $(x^*, u^*) \in \Omega$ of the equation

$$\begin{aligned} L'(x^*, u^*, \lambda) &= J'(x^*, u^*) - \langle g'(x^*, u^*), \lambda \rangle \qquad (4.6) \\ &= \int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{(x^*,u^*)} h + \frac{\partial F}{\partial u}\Big|_{(x^*,u^*)} k \right) dt - \int_0^1 \lambda^T \left(h - \int \left(\frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} h + \frac{\partial f}{\partial u}\Big|_{(x^*,u^*)} k \right) d\tau \right) dt \\ &= 0 \end{aligned}$$

for every $h \in X$ and $k \in U$.

Setting $k = 0$ yields

$$\int_0^1 \frac{\partial F}{\partial x}\Big|_{(x^*,u^*)} h dt - \int_0^1 \lambda^T \left(h - \int \frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} h d\tau \right) dt = 0 \qquad (4.7)$$

for every $h \in X$. Define the *costate*

$$p(t) = \int_t^1 \lambda(\tau)\, d\tau. \qquad (4.8)$$

Since

$$p(t) = \int_0^1 \lambda(\tau)\, d\tau - \int_0^t \lambda(\tau)\, d\tau,$$

Theorem 4.2 guarantees that $p \in X$ and p is differentiable. From (4.8),

$$\dot{p} = -\lambda, \quad p(1) = 0.$$

For any function ψ, integration by parts yields

$$\begin{aligned} \int_0^1 \lambda^T(t) \left(\int_0^t \psi(\tau)\, d\tau \right) dt &= \left(\int_0^1 \lambda^T(\tau)\, d\tau \right) \left(\int_0^1 \psi(t)\, dt \right) - \int_0^1 \left(\int_0^t \lambda^T(\tau)\, d\tau \right) \psi(t)\, dt \qquad (4.9) \\ &= \int_0^1 \left(\int_t^1 \lambda^T(\tau)\, d\tau \right) \psi(t)\, dt \\ &= \int_0^1 p(t)\, \psi(t)\, dt. \end{aligned}$$

Setting

$$\psi = \frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} h$$

and substituting p for λ, (4.7) becomes

$$\int_0^1 \left(\frac{\partial F}{\partial x}\Big|_{(x^*,u^*)} + \dot{p}^T + p^T \frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} \right) h dt = 0.$$

By the fundamental lemma (Theorem 3.55),

$$\dot{p} = -\left(\frac{\partial f}{\partial x}\Big|_{(x^*,u^*)} \right)^T p - \left(\frac{\partial F}{\partial x}\Big|_{(x^*,u^*)} \right)^T.$$

Similarly, set $h = 0$ in (4.6) to obtain

$$\int_0^1 \left(\frac{\partial F}{\partial u}\Big|_{(x^*,u^*)} k + \lambda^T \left(\int \frac{\partial f}{\partial u}\Big|_{(x^*,u^*)} k d\tau \right) \right) dt = 0$$

for every $k \in U$. Setting

$$\psi = \frac{\partial f}{\partial u}\bigg|_{(x^*,u^*)} k$$

in (4.9) yields

$$\int_0^1 \left(\frac{\partial F}{\partial u}\bigg|_{(x^*,u^*)} + p^T \frac{\partial f}{\partial u}\bigg|_{(x^*,u^*)} \right) k dt = 0.$$

From the fundamental lemma,

$$\frac{\partial F}{\partial u}\bigg|_{(x^*,u^*)} + p^T \frac{\partial f}{\partial u}\bigg|_{(x^*,u^*)} = 0. \tag{4.10}$$

Define the *Hamiltonian*

$$H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R},$$

$$H(x, u, p) = F(x, u) + p^T f(x, u).$$

Then the differential equation (4.1) may be written

$$\dot{x} = \left(\frac{\partial H}{\partial p} \right)^T.$$

We have proven the following result.

Theorem 4.5 *(Maximum Principle) If J achieves a local extremum at (x^*, u^*) subject to (4.1) and (4.2), then there exists a differentiable $p \in X$ such that*

1) $\dot{p} = -\left(\frac{\partial H}{\partial x}\Big|_{(x^*,u^*)} \right)^T,$

2) $p(1) = 0,$

3) $\frac{\partial H}{\partial u}\Big|_{(x^*,u^*)} = 0.$

4.1.4 Time-Varying Problems

Suppose F or f depends explicitly on t :

$$J(x, u) = \int_0^1 F(x, u, t)\, dt,$$

$$\dot{x} = f(x, u, t).$$

The corresponding Hamiltonian is defined as

$$H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R},$$

$$H(x, u, p, t) = F(x, u, t) + p^T f(x, u, t).$$

The problem can be transformed into the time-invariant framework described in the previous section by defining an $(n + 1)$th state

$$\dot{y} = 1, \quad y(0) = 0.$$

In other words, $y(t) = t$. Then we set

$$\widetilde{x} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\widetilde{F}(\widetilde{x},u) = F(x,u,y),$$

$$\widetilde{f}(\widetilde{x},u) = \begin{bmatrix} f(x,u,y) \\ 1 \end{bmatrix}.$$

The Hamiltonian requires an augmented costate

$$\widetilde{p} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

Then

$$\begin{aligned}\widetilde{H}(\widetilde{x},u,\widetilde{p}) &= \widetilde{F}(\widetilde{x},u) + \widetilde{p}^T \widetilde{f}(\widetilde{x},u) \\ &= F(x,u,y) + p^T f(x,u) + q \\ &= H(x,u,p,y) + q.\end{aligned}$$

The necessary conditions become

$$\begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} = \dot{\widetilde{p}} = -\left(\left.\frac{\partial \widetilde{H}}{\partial \widetilde{x}}\right|_{(\widetilde{x}^*,u^*)} \right)^T = -\begin{bmatrix} \left(\left.\frac{\partial H}{\partial x}\right|_{(x^*,u^*)} \right)^T \\ \left.\frac{\partial H}{\partial t}\right|_{(x^*,u^*)} \end{bmatrix},$$

$$\begin{bmatrix} p(1) \\ q(1) \end{bmatrix} = 0,$$

$$\left.\frac{\partial H}{\partial u}\right|_{(x^*,u^*)} = \left.\frac{\partial \widetilde{H}}{\partial u}\right|_{(\widetilde{x}^*,u^*)} = 0.$$

q is an extraneous variable, which may be ignored. Hence, we are left with the same necessary conditions 1)-3) as in Theorem 4.5.

4.1.5 Calculus of Variations

A general L^2-based calculus of variations can be achieved using the maximum principle. Let

$$J(x) = \int_0^1 F\left(x, \dot{x}, t\right) dt$$

with end point constraint

$$x(0) = x_0.$$

In general, $x \in L^2$ is not differentiable. However, we may introduce an additional variable u and impose the differential equation

$$\dot{x} = u.$$

The cost function may then be rewritten

$$J(x,u) = \int_0^1 F(x,u,t)\, dt.$$

The Hamiltonian is

$$H(x,u,p,t) = F(x,u,t) + p^T u,$$

so the necessary conditions are

$$*\dot{p} = -\left(\frac{\partial F}{\partial x}\right)^T,$$

$$p(1) = 0,$$

$$\frac{\partial F}{\partial \dot{x}} + p^T = 0.$$

Solving for p, we obtain Euler's equation

$$\left.\frac{\partial F}{\partial x}\right|_{x^*} = \frac{d}{dt}\left(\left.\frac{\partial F}{\partial \dot{x}}\right|_{x^*}\right)$$

and boundary condition

$$\left.\frac{\partial F}{\partial \dot{x}}\right|_{x=x^*, t=1} = 0.$$

The appropriate conditions for L^2 problems with additional constraints can be derived similarly.

4.1.6 State Regulation

Consider the optimal control problem with

$$F(x,u,t) = \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q(t) & N(t) \\ N^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{4.11}$$

$$f(x,u,t) = A(t)x + B(t)u, \tag{4.12}$$

In view of the structure of F and f, the problem is referred to as *linear-quadratic regulation*. In other words, the cost is quadratic and the plant is linear; minimizing J requires that we drive x close to 0 while keeping u small. We assume Q and R are symmetric.

Theorem 4.6 *Suppose* $R > 0$*. Then*

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geq 0 \quad (>0)$$

iff

$$Q - NR^{-1}N^T \geq 0 \quad (>0).$$

Proof. Let

$$P = \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix}, \quad M = \begin{bmatrix} I & -NR^{-1} \\ 0 & I \end{bmatrix},$$

and note that M is nonsingular and

$$MPM^T = \begin{bmatrix} Q - NR^{-1}N^T & 0 \\ 0 & R \end{bmatrix}.$$

(Sufficient) Let $z \in \mathbb{R}^{n+m}$ and define

$$\begin{bmatrix} v \\ w \end{bmatrix} = \left(M^T\right)^{-1} z.$$

Then

$$\begin{aligned} z^T P z &= \begin{bmatrix} v^T & w^T \end{bmatrix} M P M^T \begin{bmatrix} v \\ w \end{bmatrix} \\ &= v^T \left(Q - N R^{-1} N^T\right) v + w^T R w \\ &\geq 0. \end{aligned}$$

If $Q - NR^{-1}N^T$ is nonsingular, then so is MPM^T and, hence, P.

(Necessary) For any $v \in \mathbb{R}^n$,

$$v^T \left(Q - N R^{-1} N^T\right) v = \begin{bmatrix} v^T & 0 \end{bmatrix} M P M^T \begin{bmatrix} v \\ 0 \end{bmatrix} \geq 0.$$

If P is nonsingular, then so is MPM^T and, hence, $Q - NR^{-1}N^T$. ■

In view of Theorem 4.6, we assume that

$$R(t) > 0,$$

$$Q(t) - N(t) R^{-1}(t) N^T(t) \geq 0$$

for every $t \in [0, 1]$.

The maximum principle provides the necessary conditions:

$$H(x, u, p, t) = x^T Q(t) x + 2x^T N(t) u + u^T R(t) u + p^T (A(t) x + B(t) u)$$

$$\dot{p} = -\left(2x^T Q + 2u^T N^T + p^T A\right)^T = -A^T p - 2Qx - 2Nu$$

$$p(1) = 0$$

$$2N^T x + 2Ru + B^T p = 0$$

Since R is nonsingular,

$$u = -R^{-1}\left(N^T x + \frac{1}{2} B^T p\right),$$

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}N^T & -\frac{1}{2} BR^{-1}B^T \\ -2\left(Q - NR^{-1}N^T\right) & -\left(A - BR^{-1}N\right)^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \tag{4.13}$$

$$x(0) = x_0, \quad p(1) = 0. \tag{4.14}$$

Note that equations (4.13)-(4.14) require solving a differential equation from boundary conditions partially specified at both end points. This is referred to as a *two-point boundary value problem.*

Example 4.1 *Let*

$$A = 0, \quad B = 1,$$

$$Q = R = 1, \quad N = 0.$$

We obtain

$$\begin{bmatrix} A - BR^{-1}N^T & -\frac{1}{2} BR^{-1}B^T \\ -2Q & -A^T \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ -2 & 0 \end{bmatrix},$$

which has state-transition matrix

$$\Phi(t, \tau) = \exp\left((t - \tau)\begin{bmatrix} 0 & -\frac{1}{2} \\ -2 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cosh(t - \tau) & -\frac{1}{2}\sinh(t - \tau) \\ -2\sinh(t - \tau) & \cosh(t - \tau) \end{bmatrix}.$$

Then

$$\begin{bmatrix} x(1) \\ 0 \end{bmatrix} = \Phi(1,0) \begin{bmatrix} x_0 \\ p(0) \end{bmatrix} = \begin{bmatrix} \cosh 1 & -\frac{1}{2}\sinh 1 \\ -2\sinh 1 & \cosh 1 \end{bmatrix} \begin{bmatrix} x_0 \\ p(0) \end{bmatrix},$$

$$p(0) = 2\frac{\sinh 1}{\cosh 1} x_0,$$

$$\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Phi(t,0) \begin{bmatrix} x_0 \\ p(0) \end{bmatrix} = \begin{bmatrix} \cosh t & -\frac{1}{2}\sinh t \\ -2\sinh t & \cosh t \end{bmatrix} \begin{bmatrix} 1 \\ 2\frac{\sinh 1}{\cosh 1} \end{bmatrix} x_0,$$

$$x^*(t) = \frac{\cosh 1 \cosh t - \sinh 1 \sinh t}{\cosh 1} x_0 = \frac{\cosh(1-t)}{\cosh 1} x_0,$$

$$u^*(t) = \dot{x}^*(t) = -\frac{\sinh(1-t)}{\cosh 1} x_0,$$

$$J(x^*, u^*) = \int_0^1 \left(x^{*2} + u^{*2}\right) dt = \left(\frac{x_0}{\cosh 1}\right)^2 \int_0^1 \cosh(2(1-t))\, dt = \frac{e^4 - 1}{4e^2} \left(\frac{x_0}{\cosh 1}\right)^2.$$

4.1.7 Final End Point Constraint

Another way to extend the basic optimal control problem is to append a final end point constraint

$$x(1) = x_1.$$

In order to treat such a condition in L^2, we exploit equation (4.5) to write

$$x_1 = x_0 + \int_0^1 f(x,u)\, dt.$$

Then

$$g : X \times U \to X \times \mathbb{R}^n,$$

$$g(x,u) = \begin{bmatrix} x - x_0 - \int f(x,u)\, d\tau \\ x_1 - x_0 - \int_0^1 f(x,u)\, dt \end{bmatrix},$$

$$g'(x^*,u^*)(h,k) = \begin{bmatrix} h - \int \left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} h + \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} k \right) d\tau \\ -\int_0^1 \left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} h + \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} k \right) dt \end{bmatrix}.$$

We need to determine when (x,u) is regular.

Let $A : [0,1] \to \mathbb{R}^{n\times n}$ and $B : [0,1] \to \mathbb{R}^{n\times m}$ be continuous. We say that (A,B) is *controllable on* $[0,1]$ if for every $x_0, x_1 \in \mathbb{R}^n$ there exists an input function u that drives the system

$$\dot{x} = Ax + Bu$$

from $x(0) = x_0$ to $x(1) = x_1$. Let $\Phi(t,\tau)$ be the state-transition matrix corresponding to A, and define the *controllability Gramian*

$$W(t) = \int_0^t \Phi(0,\tau) B(\tau) B^T(\tau) \Phi^T(0,\tau)\, d\tau.$$

Note that $W(t)$ is symmetric, positive semidefinite for every $t \in [0,1]$.

Theorem 4.7 *The following are equivalent:*
1) (A,B) *is controllable on* $[0,1]$.
2) The rows of $\Phi(0,\cdot)B(\cdot)$ *are linearly independent on* $[0,1]$.
3) $W(1)$ *is nonsingular.*

Theorem 4.8 *Let* $A:[0,1]\to\mathbb{R}^{n\times n}$ *and* $B:[0,1]\to\mathbb{R}^{n\times m}$ *be continuous,* $\xi\in X$, *and* $w\in\mathbb{R}^n$. *Then there exists* $x\in X$ *and* $u\in U$ *such that*

$$x(t)-\int_0^t (A(\tau)x(\tau)+B(\tau)u(\tau))\,d\tau=\xi(t) \tag{4.15}$$

for every $t\in[0,1]$ *and*

$$\int_0^1 (A(t)x(t)+B(t)u(t))\,dt=w \tag{4.16}$$

iff (A,B) *is controllable on* $[0,1]$.

Proof. *(Sufficient) Let* Φ *be the state-transition matrix of the time-varying linear state-space system corresponding to* A *– i.e.*

$$\frac{\partial\Phi(t,\tau)}{\partial t}=A(t)\Phi(t,\tau),\quad \Phi(\tau,\tau)=I$$

with $\Phi\in C^1$. *Then*

$$\int_\eta^t A(\tau)\Phi(\tau,\eta)\,d\tau=\int_\eta^t\frac{\partial\Phi(\tau,\eta)}{\partial\eta}d\tau=\Phi(t,\eta)-I.$$

From controllability, there exists $u\in U$ *such that*

$$\int_0^1\Phi(1,\tau)B(\tau)u(\tau)\,d\tau=w-\int_0^1\Phi(1,\tau)A(\tau)\xi(\tau)\,d\tau.$$

Let

$$x(t)=\xi(t)+\int_0^t\Phi(t,\tau)(A(\tau)\xi(\tau)+B(\tau)u(\tau))\,d\tau.$$

From the double integral

$$\int_0^t A(\tau)\left(\int_0^\tau\Phi(\tau,\eta)(A(\eta)\xi(\eta)+B(\eta)u(\eta))\,d\eta\right)d\tau=\int_0^t\left(\int_\eta^t A(\tau)\Phi(\tau,\eta)\,d\tau\right)\begin{pmatrix}A(\eta)\xi(\eta)\\+B(\eta)u(\eta)\end{pmatrix}d\eta$$
$$=\int_0^t(\Phi(t,\eta)-I)(A(\eta)\xi(\eta)+B(\eta)u(\eta))\,d\eta,$$

we obtain

$$\begin{aligned}x(t)-\int_0^t(A(\tau)x(\tau)+B(\tau)u(\tau))\,d\tau&=\xi(t)+\int_0^t\Phi(t,\tau)(A(\tau)\xi(\tau)+B(\tau)u(\tau))\,d\tau\\&\quad-\int_0^t\begin{pmatrix}A(\tau)\left(\xi(\tau)+\int_0^\tau\Phi(\tau,\eta)(A(\eta)\xi(\eta)+B(\eta)u(\eta))\,d\eta\right)\\+B(\tau)u(\tau)\end{pmatrix}d\tau\\&=\xi(t)+\int_0^t\Phi(t,\tau)(A(\tau)\xi(\tau)+B(\tau)u(\tau))\,d\tau\\&\quad-\int_0^t(A(\tau)\xi(\tau)+B(\tau)u(\tau))\,d\tau\\&\quad-\int_0^t(\Phi(t,\eta)-I)(A(\eta)\xi(\eta)+B(\eta)u(\eta))\,d\eta\\&=\xi(t),\end{aligned}$$

$$\int_0^1 (A(t)x(t) + B(t)u(t))\,dt = \int_0^1 \left(\begin{array}{c} A(t)\left(\xi(t) + \int_0^t \Phi(t,\tau)(A(\tau)\xi(\tau) + B(\tau)u(\tau))\,d\tau\right) \\ +B(t)u(t) \end{array} \right) dt$$

$$= \int_0^1 (A(t)\xi(t) + B(t)u(t))\,dt$$

$$+ \int_0^1 (\Phi(1,\tau) - I)(A(\tau)\xi(\tau) + B(\tau)u(\tau))\,d\tau$$

$$= \int_0^1 \Phi(1,\tau)(A(\tau)\xi(\tau) + B(\tau)u(\tau))\,d\tau$$

$$= w.$$

(Necessary) Set $\xi = 0$ and choose any w. From (4.15), x is differentiable with

$$\dot{x} = Ax + Bu.$$

From (4.15) and (4.16),

$$x(0) = 0,$$

$$x(1) = \int_0^1 (A(t)x(t) + B(t)u(t))\,dt = w.$$

Hence, every state w is reachable from the origin at $t = 1$, making (A,B) controllable on $[0,1]$. ■

Theorem 4.9 *$(x^*, u^*) \in \Omega$ is regular iff*

$$\left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}, \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} \right)$$

is controllable on $[0,1]$.

Proof. Setting

$$(A,B) = \left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}, \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} \right)$$

in Theorem 4.8 shows that $g'(x^*,u^*)$ is onto iff (A,B) is controllable on $[0,1]$. ■

In this context, we define the inner product on $X \times \mathbb{R}^n$ to be

$$\left\langle \begin{bmatrix} x \\ v \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix} \right\rangle = \langle x,y\rangle + \langle v,w\rangle = \int_0^1 \left(y^T x\right) dt + w^T v.$$

The critical points are those $(x^*, u^*) \in \Omega$ satisfying

$$L'(x^*,u^*,\lambda)(h,k) = J'(x^*,u^*) - \langle g'(x^*,u^*), \lambda\rangle$$

$$= \int_0^1 \left(\frac{\partial F}{\partial x}h + \frac{\partial F}{\partial u}k \right) dt - \int_0^1 \lambda_1^T \left(h - \int \left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} h + \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} k \right) d\tau \right) dt$$

$$+ \lambda_2^T \int_0^1 \left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)} h + \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)} k \right) dt$$

$$= 0$$

for every $h \in X$ and $k \in U$. Here we define the costate

$$p(t) = \int_t^1 \lambda_1(\tau)\,d\tau + \lambda_2.$$

By an analysis similar to (4.7)-(4.10), we obtain the following form of the maximum principle.

Theorem 4.10 *(Maximum Principle with Final End Point) If J achieves a local extremum at a regular point (x^*, u^*) subject to*

$$\dot{x} = f(x,u),$$

$$x(0) = x_0, \quad x(1) = x_1,$$

then there exists a differentiable $p \in X$ such that

1) $\dot{p} = -\left(\left.\frac{\partial H}{\partial x}\right|_{(x^*,u^*)}\right)^T$,

2) $\left.\frac{\partial H}{\partial u}\right|_{(x^*,u^*)} = 0.$

4.1.8 Minimum Control Energy

We wish to drive a linear time-invariant plant from the initial state $x(0) = x_0$ to the final state $x(1) = x_1$ while minimizing the "control energy"

$$J(x,u) = \int_0^1 u^T u dt.$$

Thus

$$f(x,u) = Ax + Bu,$$

$$F(x,u) = u^T u,$$

In this case,

$$\frac{\partial f}{\partial x} = A, \quad \frac{\partial f}{\partial u} = B$$

are independent of the choice of (x,u), so by Theorem 4.9 either every (x,u) is regular or none is. Regularity is equivalent to controllability of (A,B), which we assume.

The Hamiltonian is

$$H(x,u,p) = u^T u + p^T (Ax + Bu).$$

From Theorem 4.10, the necessary conditions are

$$\dot{p} = -A^T p, \tag{4.17}$$

$$2u^T + p^T B = 0. \tag{4.18}$$

Solving (4.18) for u yields

$$u = -\frac{1}{2} B^T p, \tag{4.19}$$

which leads to the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -\frac{1}{2}BB^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix},$$

$$x(t_0) = x_0, \quad x(t_1) = x_1.$$

In this context, the controllability Gramian is

$$W(t) = \int_0^t \exp(-\tau A) BB^T \exp\left(-\tau A^T\right) d\tau.$$

From (4.17),

$$p(t) = \exp\left(-tA^T\right) p(0).$$

By (4.19),

$$\begin{aligned} x_1 &= \exp(A)\, x_0 + \int_0^1 \exp((1-\tau)A)\, Bu(\tau)\, d\tau \qquad (4.20) \\ &= \exp(A)\, x_0 + \int_0^1 \exp((1-\tau)A)\, B\left(-\frac{1}{2}B^T p(\tau)\right) d\tau \\ &= \exp(A)\, x_0 + \int_0^1 \exp((1-\tau)A)\, B\left(-\frac{1}{2}B^T \exp\left(-\tau A^T\right) p(0)\right) d\tau \\ &= \exp(A)\left(x_0 - \frac{1}{2}W(1)\, p(0)\right). \end{aligned}$$

From controllability of (A, B) and Theorem 4.7, we may solve (4.20) to obtain

$$p(0) = 2W^{-1}(1)\,(x_0 - \exp(-A)\, x_1),$$

yielding

$$u^*(t) = -B^T \exp\left(-tA^T\right) W^{-1}(1)\,(x_0 - \exp(-A)\, x_1), \qquad (4.21)$$

$$\begin{aligned} x^*(t) &= \exp(tA)\, x_0 - \int_0^t \exp((t-\tau)A)\, BB^T \exp\left(-\tau A^T\right) W^{-1}(1)\,(x_0 - \exp(-A)\, x_1)\, d\tau \qquad (4.22) \\ &= \exp(tA)\left(I - W(t)\, W^{-1}(1)\right) x_0 + \exp(tA)\, W(t)\, W^{-1}(1) \exp(-A)\, x_1. \end{aligned}$$

Example 4.2 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $x_1 = 0$. *Then*

$$\exp(tA) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

$$\begin{aligned} W(t) &= \int_0^t \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\tau & 1 \end{bmatrix} d\tau \\ &= \int_0^t \begin{bmatrix} \tau^2 & -\tau \\ -\tau & 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix}, \end{aligned}$$

$$W^{-1}(1) = \begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix},$$

$$\begin{aligned} u^*(t) &= -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix} x_0 \\ &= \begin{bmatrix} 6(2t-1) & 2(3t-2) \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} x^*(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 6 & 4 \end{bmatrix} \right) x_0 \\ &= \begin{bmatrix} 1 - 3t^2 + 2t^3 & t - 2t^2 + t^3 \\ t - 2t^2 + t^3 & 1 - 4t + 3t^2 \end{bmatrix} x_0. \end{aligned}$$

4.1.9 Terminal Manifolds

In addition to the differential equation

$$\dot{x} = f(x,u)$$

and initial condition

$$x(0) = x_0,$$

consider a final state constraint

$$\phi(x(1)) = 0,$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^l$ belongs to C^2. The problem is to drive the system from x_0 at time $t = 0$ to the terminal manifold

$$T = \left\{ w \in \mathbb{R}^n \,\middle|\, \phi(w) = 0 \right\}.$$

The constraint function is

$$g(x,u) = \begin{bmatrix} x - x_0 - \int f(x,u)\,d\tau \\ \phi\left(x_0 + \int_0^1 f(x,u)\,dt\right) \end{bmatrix}$$

with strict Frechet derivative

$$g'(x,u)(h,k) = \begin{bmatrix} h - \int \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)d\tau \\ \frac{\partial \phi}{\partial x(1)} \int_0^1 \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)dt \end{bmatrix}.$$

Theorem 4.11 *(x^*,u^*) is a regular point of Ω if $\left(\left.\frac{\partial f}{\partial x}\right|_{(x^*,u^*)}, \left.\frac{\partial f}{\partial u}\right|_{(x^*,u^*)}\right)$ is controllable on $[0,1]$ and $\left.\frac{\partial \phi}{\partial x(1)}\right|_{x^*(1)}$ has rank l.*

Proof. Let $\xi \in X$ and $z \in \mathbb{R}^l$. Since $\frac{\partial \phi}{\partial x(1)}$ has rank l, there exists $w \in \mathbb{R}^n$ such that

$$\frac{\partial \phi}{\partial x(1)} w = z.$$

From Theorem 4.8, there exist $h, k \in X$ such that

$$h - \int \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)d\tau = \xi,$$

$$\int_0^1 \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)dt = w,$$

making $g'(x^*,u^*)$ onto. ■

The Frechet derivative of the Lagrangian is

$$\begin{aligned} L'(x,u,\lambda)(h,k) = & \int_0^1 \left(\frac{\partial F}{\partial x}h + \frac{\partial F}{\partial u}k\right)dt - \int_0^1 \lambda_1^T \left(h - \int \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)d\tau\right)dt \\ & - \lambda_2^T \frac{\partial \phi}{\partial x(1)} \int_0^1 \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right)dt \\ & = 0. \end{aligned}$$

Setting

$$p(t) = \int_t^1 \lambda_1(\tau)\, d\tau - \left(\frac{\partial \phi}{\partial x(1)}\right)^T \lambda_2,$$

integrating by parts, and applying the fundamental lemma yields

$$\dot{p} = -\left(\frac{\partial H}{\partial x}\Big|_{(x^*,u^*)}\right)^T, \tag{4.23}$$

$$p(1) = -\left(\frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2, \tag{4.24}$$

$$\frac{\partial H}{\partial u}\Big|_{(x^*,u^*)} = 0. \tag{4.25}$$

Equation (4.24) is the *transversality condition*, which replaces $p(1) = 0$ in the maximum principle.

4.1.10 Minimum Control Energy with a Terminal Manifold

Consider the problem of driving an LTI state-space system from x_0 at $t = 0$ to a terminal manifold at $t = 1$ with minimum control energy.

$$F(x,u) = u^T u$$

$$f(x,u) = Ax + Bu$$

$$x(0) = x_0, \quad \phi(x(1)) = 0$$

The necessary conditions are

$$\dot{p} = -A^T p,$$

$$p(1) = \left(\frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2,$$

$$2u^T + p^T B = 0.$$

The solution is obtained from

$$p(t) = \exp\left((1-t)A^T\right) p(1),$$

$$\begin{aligned} u^*(t) &= -\frac{1}{2} B^T p(t) \\ &= -\frac{1}{2} B^T \exp\left((1-t)A^T\right) p(1) \\ &= -\frac{1}{2} B^T \exp\left((1-t)A^T\right) \left(\frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2, \end{aligned} \tag{4.26}$$

$$\begin{aligned} x^*(t) &= \exp(tA)\, x_0 + \int_0^t \exp((t-\tau)A)\, B \left(-\frac{1}{2} B^T \exp\left((1-\tau)A^T\right) p(1)\right) d\tau \\ &= \exp(tA) \left(x_0 - \frac{1}{2} W(t) \exp\left(A^T\right) \left(\frac{\partial \phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2\right), \end{aligned} \tag{4.27}$$

$$x(1) = \exp(A)\left(x_0 - \frac{1}{2}W(1)\exp\left(A^T\right)\left(\frac{\partial\phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2\right).$$

x^* and u^* are obtained by solving

$$\phi\left(\exp(A)\left(x_0 - \frac{1}{2}W(1)\exp\left(A^T\right)\left(\frac{\partial\phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2\right)\right) = 0$$

for λ_2 and substituting the result into (4.26) and (4.27). Then

Example 4.3 *As in Example 4.2, let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Instead of a final state constraint, consider the terminal manifold

$$\phi(x(1)) = x_1(1).$$

In other words, we wish to drive the system from $x(0) = x_0$ to the x_2-axis with minimum control energy.

$$\exp(tA) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

$$W(t) = \begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix},$$

$$\begin{aligned} x(1) &= \exp(A)\left(x_0 - \frac{1}{2}W(1)\exp\left(A^T\right)\left(\frac{\partial\phi}{\partial x(1)}\Big|_{x^*(1)}\right)^T \lambda_2\right) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\left(x_0 - \frac{1}{2}\begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda_2\right) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_0 + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{4} \end{bmatrix}\lambda_2, \end{aligned}$$

$$\lambda_2 = 6\begin{bmatrix} 1 & 1 \end{bmatrix} x_0,$$

$$\begin{aligned} x^*(t) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\left(x_0 - 3\begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} x_0\right) \\ &= \begin{bmatrix} 1 - \frac{3}{2}t^2 + \frac{1}{2}t^3 & t - \frac{3}{2}t^2 + \frac{1}{2}t^3 \\ -3t + \frac{3}{2}t^2 & 1 - 3t + \frac{3}{2}t^2 \end{bmatrix} x_0 \end{aligned}$$

$$u^*(t) = -3\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1-t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} x_0 = 3(t-1)\begin{bmatrix} 1 & 1 \end{bmatrix} x_0.$$

4.1.11 Terminal Cost

Up to this point, we have imposed a "hard constraint" on the final state $x(1)$, consisting of a point or terminal manifold where we require $x(1)$ to reside. An alternative is to consider a "soft constraint", where final distance from a manifold is penalized but not required to be 0. This is accomplished by adding a *terminal cost* to the cost functional:

$$J(x,u) = T(x(1)) + \int_0^1 F(x,u)\,dt$$

We assume that $T \in C^2$. The constraint function is simply

$$g(x,u) = x - x_0 - \int f(x,u)\,d\tau.$$

From Theorem 4.4, every $(x,u) \in \Omega$ is regular. Noting that

$$x(1) = x_0 + \int_0^1 f(x,u)\,dt,$$

the Lagrangian is

$$\begin{aligned} L(x,u,\lambda)(h,k) &= T(x(1)) + \int_0^1 F(x,u)\,dt - \lambda^T g(x,u) \\ &= T\left(x_0 + \int_0^1 f(x,u)\,dt\right) + \int_0^1 \left(F(x,u) - \lambda^T\left(x - x_0 - \int f(x,u)\,d\tau\right)\right) dt \end{aligned}$$

with derivative

$$L'(x,u,\lambda)(h,k) = \frac{\partial T}{\partial x(1)} \int_0^1 \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right) dt + \int_0^1 \left(\frac{\partial F}{\partial x}h + \frac{\partial F}{\partial u}k\right) dt - \int_0^1 \lambda^T \left(h - \int \left(\frac{\partial f}{\partial x}h + \frac{\partial f}{\partial u}k\right) d\tau\right) d$$

Defining the costate

$$p(t) = \int_t^1 \lambda(\tau)\,d\tau + \left(\frac{\partial T}{\partial x(1)}\right)^T,$$

integrating by parts, and applying the fundamental lemma yields the conditions

$$\dot{p} = -\left(\frac{\partial H}{\partial x}\right)^T,$$

$$p(1) = \left(\frac{\partial T}{\partial x(1)}\right)^T$$

$$\frac{\partial H}{\partial u} = 0.$$

4.1.12 Minimum Control Energy with Terminal Cost

Let

$$F(x,u) = u^T u,$$
$$f(x,u) = Ax + Bu,$$
$$T(x(1)) = x^T(1)\, x(1).$$

The Hamiltonian is

$$H(x,u,p) = u^T u + p^T (Ax + Bu).$$

The necessary conditions are

$$\dot{x} = Ax + Bu$$
$$x(0) = x_0, \tag{4.28}$$
$$\dot{p} = -A^T p, \tag{4.29}$$
$$p(1) = 2x(1) \tag{4.30}$$
$$2u^T + p^T B = 0.$$

Then

$$u = -\frac{1}{2} B^T p,$$

from which we obtain the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & -\frac{1}{2}B^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

with boundary conditions (4.28) and (4.30).

Solving (4.29) yields

$$p(t) = \exp\left((1-t)A^T\right) p(1) = 2\exp\left((1-t)A^T\right) x(1). \tag{4.31}$$

As always,

$$\begin{aligned} x(t) &= \exp(tA)\, x_0 + \int_0^t \exp((t-\tau)A)\, Bu(\tau)\, d\tau \qquad (4.32) \\ &= \exp(tA)\, x_0 - \frac{1}{2}\int_0^t \exp((t-\tau)A)\, BB^T p(\tau)\, d\tau \\ &= \exp(tA)\, x_0 - \int_0^t \exp((t-\tau)A)\, BB^T \exp\left((1-\tau)A^T\right) x(1)\, d\tau \\ &= \exp(tA)\left(x_0 - W(t)\exp\left(A^T\right) x(1)\right). \end{aligned}$$

Hence,

$$x(1) = \exp(A)\left(x_0 - W(1)\exp\left(A^T\right) x(1)\right). \tag{4.33}$$

Since $W(1)$ is symmetric, positive definite, so is $I + \exp(A)\, W(1)\exp\left(A^T\right)$. From (4.33),

$$\begin{aligned} \exp\left(A^T\right) x(1) &= \exp\left(A^T\right)\left(I + \exp(A)\, W(1)\exp\left(A^T\right)\right)^{-1} \exp(A)\, x_0 \\ &= \left(\exp\left(-A^T\right)\exp(-A) + W(1)\right)^{-1} x_0 \end{aligned}$$

From (4.32),

$$x^*(t) = \exp(tA)\left(I - W(t)\left(\exp(-A)\exp\left(-A^T\right) + W(1)\right)^{-1}\right)x_0.$$

From (4.31),

$$\begin{aligned} u^*(t) &= -B^T \exp\left((1-t)A^T\right)x(1) \\ &= -B^T \exp\left(-tA^T\right)\left(\exp(-A)\exp\left(-A^T\right) + W(1)\right)^{-1} x_0. \end{aligned}$$

Example 4.4 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

As in Example 3.54,

$$\exp(tA) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

$$W(t) = \begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix}.$$

Hence,

$$\left(\exp(-A)\exp\left(-A^T\right) + W(1)\right)^{-1} = \left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\right)^{-1} = \frac{1}{29}\begin{bmatrix} 24 & 18 \\ 18 & 28 \end{bmatrix},$$

$$x^*(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\left(I - \frac{1}{29}\begin{bmatrix} \frac{t^3}{3} & -\frac{t^2}{2} \\ -\frac{t^2}{2} & t \end{bmatrix}\begin{bmatrix} 24 & 18 \\ 18 & 28 \end{bmatrix}\right)x_0 = \frac{1}{29}\begin{bmatrix} 29 - 9t^2 + 4t^3 & 29t - 14t^2 + 3t^3 \\ -18t + 12t^2 & 29 - 28t + 9t^2 \end{bmatrix}x_0,$$

$$u^*(t) = -\frac{1}{29}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}\begin{bmatrix} 24 & 18 \\ 18 & 28 \end{bmatrix}x_0 = \frac{1}{29}\begin{bmatrix} -18 + 24t & -28 + 18t \end{bmatrix}x_0.$$

4.1.13 Second Derivatives

For state regulation (4.11)-(4.12), we may perform second derivative analysis. From Theorems 3.23 and 3.24,

$$J''(x,u)(h,k) = 2\int_0^1 \begin{bmatrix} h^T & k^T \end{bmatrix}\begin{bmatrix} Q(t) & N(t) \\ N^T(t) & R(t) \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} dt,$$

$$g''(x,u) = 0.$$

From Theorems 3.57 and 4.6,

$$Q(t) - N(t)R^{-1}(t)N^T(t) > 0$$

for every $t \in [0,1]$ implies that J achieves a strict constrained local minimum at every solution of the necessary conditions.

For minimum control energy with a final end point,

$$J''(x,u)(h,k) = \int_0^1 k^T Rk dt,$$

$$g''(x,u) = 0,$$

$$L''(x,u,\lambda) = J''(x,u).$$

From Theorem 3.57, part 3), J has a strict constrained local minimum at (x^*, u^*) as given by (4.21)-(4.22).

For minimum control energy with a terminal manifold, $g''(x,u)$ can only be calculated if $l = 1$. In this case,

$$J''(x,u)(h,k) = \int_0^1 k^T Rk dt,$$

$$g''(x,u)(h,k) = \left(\int_0^1 (Ah + Bk)\, dt\right)^T \frac{\partial^2 \phi}{\partial x(1)^2} \int_0^1 (Ah + Bk)\, dt,$$

$$L''(x,u,\lambda)(h,k) = \int_0^1 k^T Rk dt - \lambda_2 \left(\int_0^1 (Ah + Bk)\, dt\right)^T \frac{\partial^2 \phi}{\partial x(1)^2} \int_0^1 (Ah + Bk)\, dt,$$

which may not be positive definite. In Example 4.3,

$$\int_0^1 (Ah + Bk)\, dt = \int_0^1 \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} h + \begin{bmatrix} 0 \\ 1 \end{bmatrix} k\right) dt = \int_0^1 \begin{bmatrix} h_2 \\ k \end{bmatrix} dt,$$

$$\begin{aligned} L''(x,u,\lambda)(h,k) &= \int_0^1 k^2 dt - \left(12 \begin{bmatrix} 1 & 1 \end{bmatrix} x_0\right) \left(\int_0^1 \begin{bmatrix} h_2 \\ k \end{bmatrix} dt\right)^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \int_0^1 \begin{bmatrix} h_2 \\ k \end{bmatrix} dt \\ &= \int_0^1 k^2 dt - 12 (x_{01} + x_{02}) \left(\int_0^1 h_2 dt\right)^2. \end{aligned}$$

Taking $k = h_2 = 0$ and $h_1 \neq 0$ yields $L''(x,u,\lambda) = 0$, so the second derivative is not positive definite for any x_0.

For minimum control energy with terminal cost,

$$J''(x,u)(h,k) = 2\left(\int_0^1 (Ah + Bk)\, dt\right)^T Q \int_0^1 (Ah + Bk)\, dt + \int_0^1 k^T Rk dt,$$

$$g''(x,u) = 0,$$

$$L''(x,u,\lambda) = J''(x,u).$$

Let

$$h(t) = \sin 2\pi t, \quad k = 0.$$

Then

$$\|(h,0)\|_2^2 = \int_0^1 h^2(t)\, dt = \frac{1}{2},$$

$$\int_0^1 h(t)\, dt = 0,$$

$$L''(x,u,\lambda)(h,0) = 2\left(\int_0^1 h dt\right)^T A^T QA \int_0^1 h dt = 0 \not> \varepsilon \|(h,0)\|_2^2$$

for any $\varepsilon > 0$. Hence, the second derivative is not positive definite.

4.1.14 Pointwise Inequality Constraints

Consider a constraint function

$$g : X \times U \rightarrow Y,$$

where $Y = L^2$ functions $y : [0,1] \rightarrow \mathbb{R}$. For $y \in Y$, we write $y > 0$ if $y(t) > 0$ for every t. We may wish to consider constraints of the form

$$g(x,u) > 0.$$

Equivalently,

$$g(x,u)(t) > 0$$

for every t. Since the inequality is applied at every t, it is said to be a *pointwise constraint.* Similarly, we write $y \geq 0$ if $y(t) \geq 0$ for every t. In this case, we obtain the pointwise constraint

$$g(x,u)(t) \geq 0.$$

Since the constraint consists of inequalities, an extension of the Kuhn-Tucker theorem is called for. Unfortunately, this approach is often not tractable using classical variational methods. One can at least glimpse the difficulties involved by considering the "first orthant"

$$Y^+ = \left\{ y \in Y \,\middle|\, y > 0 \right\}.$$

In finite dimensions, Y^+ is an open set. However, in L^2 this no longer the case.

Theorem 4.12 *Y^+ is not open.*

Proof. The point $y \equiv 1$ obviously belongs to Y^+. Let

$$h(t) = \begin{cases} -\frac{1}{\varepsilon}, & 0 \leq t \leq \varepsilon^4 \\ 0, & \text{else} \end{cases}.$$

Then

$$\|h\|^2 = \int_0^1 h^2 dt = \int_0^{\varepsilon^4} \frac{1}{\varepsilon^2} dt = \varepsilon^2,$$

so

$$y + h \in B(y, \varepsilon)$$

for $\varepsilon < 1$. But

$$y(t) + h(t) = 1 - \frac{1}{\varepsilon} < 0$$

for $0 \leq t \leq \varepsilon^4$. Hence,

$$B(y, \varepsilon) \not\subset Y^+$$

for any $\varepsilon < 1$. ■

It can be further proven that Y^+ contains no ball $B(y, \varepsilon)$ for any y and ε. For this and other technical reasons, inequality constraints require a non-classical theory. Such a theory was provided by the Russian mathematician Pontryagin in 1956.

4.2 The Pontryagin Maximum Principle

Reference: Pontryagin, Chapter 1

4.2.1 Background

The Pontryagin Maximum Principle (PMP) extends the optimal control theory obtained through Lagrange multipliers, allowing for pointwise inequality constraints. The approach maintains the basic structure of the necessary conditions as previously stated, but with additional generality. Pontryagin formulated his theory to handle optimal control problems with variable terminal time. Hence, the theory is not based on normed linear spaces, and so does not address local extrema.

We say a function $u : [0, t_1] \to \mathbb{R}^m$ is *piecewise continuous* if there exist finitely many points $0 = \tau_1 < \ldots < \tau_k = t_1$ such that u is continuous on each interval $[\tau_i, \tau_{i+1}]$ and both limits $\lim_{t \to \tau_i^+} u(t)$ and $\lim_{t \to \tau_{i+1}^-} u(t)$ exist. That is, u has a *jump discontinuity* at each τ_i. We denote the set of all such functions (with t_1 variable) as PC^0. We say a function $x : [0, t_1] \to \mathbb{R}^n$ is *piecewise* C^1 if x is continuous on $[0, t_1]$ and there exist finitely many points $0 = \tau_1 < \ldots < \tau_k = t_1$ such that x is C^1 on each $[\tau_i, \tau_{i+1}]$. We denote the set of all such functions as PC^1. Since t_1 is variable, neither PC^1 nor PC^0 is a vector space.

4.2.2 Differential Equations

We must examine the nature of differential equations

$$\dot{x} = f(x, u) \tag{4.34}$$

for $x \in PC^1$ and $u \in PC^0$.

Theorem 4.13 *If $f \in C^1$, then for every $x_0 \in \mathbb{R}^n$ and $u \in PC^0$ with $u : [0, t_1] \to \mathbb{R}^n$ there exists $t_2 \in (0, t_1]$ and $x \in PC^1$ with $x : [0, t_2] \to \mathbb{R}^n$ such that*
1) $x(0) = x_0$,
2) the points of non-differentiability of x are points of discontinuity of u,
3) $\dot{x}(t) = f(x(t), u(t))$ for all t where x is differentiable.
If $y : [0, t_3] \to \mathbb{R}^n$ is another such function, then $y(t) = x(t)$ for every $t \in [0, \min\{t_2, t_3\}]$.

Proof. Athans and Falb, Theorem 3-14 ■

Unfortunately, it may happen that $t_2 < t_1$.

Example 4.5 *Let $u : [0, 2] \to \mathbb{R}$, $x_0 = 1$, and*

$$f(x, u) = x^2.$$

By separation of variables,

$$x(t) = \frac{1}{1 - t}.$$

Since $x(t) \to \infty$ as $t \to 1^-$, x can only be defined on $[0, 1]$.

4.2.3 PMP with Fixed End Points

In the most elementary version of PMP, we are given $x_0, x_1 \in \mathbb{R}^n$ and an arbitrary set $V \subset \mathbb{R}^m$. The constraint set Ω is the set of pairs $(x, u) \in PC^1 \times PC^0$ satisfying

1) $x : [0, t_1] \to \mathbb{R}^n$ and $u : [0, t_1] \to \mathbb{R}^m$,
2) $x(0) = x_0$,

3) $x(t_1) = x_1$,
4) $u(t) \in V$ for every $t \in [0, t_1]$,
5) $\dot{x}(t) = f(x(t), u(t))$ for every t where x is differentiable.

The problem is to find the constrained global extrema of the cost

$$J(x, u) = \int_0^{t_1} F(x, u)\, dt$$

subject to Ω. Define the *Hamiltonian*

$$H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},$$

$$H(x, u, \mu, p) = \mu F(x, u) + p^T f(x, u).$$

Theorem 4.14 *(PMP with End Points) If J achieves a global minimum at (x^*, u^*) subject to Ω, then there exist $\mu \in \mathbb{R}$ and $p \in PC^1$ with $p : [0, t_1^*] \to \mathbb{R}^n$ (not both 0) such that*
1) $\dot{p} = -\left(\frac{\partial H}{\partial x}\Big|_{(x^*, u^*)}\right)^T$,
2) $\mu \leq 0$,
3) $H(x^*(t), u^*(t), \mu, p(t)) = \max_{\omega \in V} H(x^*(t), \omega, \mu, p(t)) = 0$ *for every* $t \in [0, t_1^*]$.

Proof. Pontryagin, Chapter II ■

We note a couple of similarities between the Maximum Principle obtained from Lagrange multipliers and PMP: Necessary condition 1) is exactly the same. If $V = \mathbb{R}^m$, then condition 3) implies $\frac{\partial H}{\partial u} = 0$, taking us back to the classical setting. Also, condition 2) is reminiscent of the sign of λ in the Kuhn-Tucker theorem. This is reasonable, since the constraint $u(t) \in V$ is often specified as a non-strict inequality. The extra multiplier μ may be 0. If $\mu \neq 0$, we say that (x^*, u^*) is *regular*. The fact that the Hamiltonian vanishes along the optimal solution is unique to PMP.

4.2.4 Time Optimal Control

A *time optimal control* problem is one with t_1 variable and $F(x, u) = 1$ for every $x \in PC^1$ and $u \in PC^0$. Then

$$J(x, u) = \int_0^{t_1} 1 dt = t_1,$$

$$H(x, u, \mu, p) = \mu + p^T f(x, u).$$

Without any further constraint on $u(t)$, the differential equation (4.34) would typically admit solutions x with $x(t_1) = x_1$ for arbitrarily small t_1. This would result in u approximating an impulse, which in most applications is unnaceptable. Hence, it is customary to impose a bound on the input $u(t) \in V$ for some appropriate set V.

For fixed end points, Theorem 4.14 gives necessary conditions

$$\dot{p} = -\left(\frac{\partial f}{\partial x}\Big|_{(x^*, u^*)}\right)^T p,$$

$$p^T f(x^*, u^*) = \max_{\omega \in V} p^T f(x^*, \omega) = -\mu \geq 0.$$

$p \equiv 0$ is not allowed, since this would require $\mu = 0$ as well.

4.2.5 Time Optimal Control of an LTI Plant

When

$$f(x,u) = Ax + Bu$$

for constant A and B, we can be more specific about time optimal solutions. Here the necessary conditions become

$$\dot{p} = -A^T p,$$

$$\mu \leq 0,$$

$$p^T B u^* = \max_{\omega \in V} p^T B \omega = -\mu - p^T A x^*.$$

We may solve

$$p(t) = \exp\left(-tA^T\right) p(0).$$

Assume V is a *polyhedron*

$$V = \left\{ \omega \in \mathbb{R}^m \,\middle|\, \alpha \leq \omega \leq \beta \right\}$$

for some $\alpha, \beta \in \mathbb{R}^m$ with $\alpha < \beta$. Writing

$$\begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix} = B, \qquad \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = u,$$

we obtain

$$p^T B u^* = \max_{\omega \in V} \sum_{i=1}^m p^T b_i \omega_i = \sum_{i=1}^m \max_{\alpha_i \leq \omega_i \leq \beta_i} p^T b_i \omega_i. \tag{4.35}$$

Theorem 4.15 *If (A, b_i) is controllable for every i, then the set*

$$ST = \left\{ t \in [0, t_1^*] \,\middle|\, p^T(t)\, b_i = 0 \text{ for some } i \right\}$$

is finite.

Proof. Pontryagin, Chapter III, Theorem 9 ■

Applying Theorem 4.15 to (4.35), the optimal control is given by

$$u_i^*(t) = \begin{cases} \alpha_i, & p^T(t)\, b_i < 0 \\ \beta_i, & p^T(t)\, b_i > 0 \end{cases} = \frac{1}{2}\left(\alpha_i + \beta_i + (\beta_i - \alpha_i)\operatorname{sgn}\left(p^T(t)\, b_i\right)\right).$$

The members of ST are called *switching times.* Since x is obtained through a convolution (i.e. integration) involving u, the values $u(t)$ for $t \in ST$ are not relevant. In some problems, the number of switchings may be large. However, we can bound this number for certain plants.

Theorem 4.16 *If (A, b_i) is controllable for every i and the eigenvalues of A are all real, then ST contains at most $n - 1$ elements.*

Proof. Pontryagin, Chapter III, Theorem 10 ■

For LTI plants, the questions of existence and uniqueness of the optimal control can be answered in the affirmative.

Theorem 4.17 *If (A, b_i) is controllable for every i, then there exists a unique u^* that drives the system from x_0 at $t = t_0$ to x_1 in minimum time.*

Proof. Pontryagin, Chapter III, Theorems 11 and 13. ■

Note that Theorem 4.17 does *not* state that there is only one solution (x, u, μ, p) of 1)-3) in PMP, but merely that there is only one solution of 1)-3) *that minimizes* t_1.

Example 4.6 *(Discharging a Capacitor) Find a control u^* that drives the system*

$$\dot{x} = -x + u$$

from x_0 at $t = 0$ to $x_1 = 0$ in minimum time subject to $|u(t)| \leq 1$.
Theorem 4.16 implies that there are $n - 1 = 0$ switchings. Hence,

$$u^* = \pm 1.$$

Solving for x,

$$x^*(t) = e^{-t}x_0 + u^* \int_0^t e^{-(t-\tau)} d\tau = e^{-t}x_0 + u^*\left(1 - e^{-t}\right).$$

In order to drive the system to $x_1 = 0$, we need

$$\begin{aligned} u^* &= -\operatorname{sgn}(x_0), \\ x^*(t_1^*) &= e^{-t_1^*}x_0 - \operatorname{sgn}(x_0)\left(1 - e^{-t_1^*}\right) = 0, \\ e^{-t_1^*}&\left(1 + \frac{x_0}{\operatorname{sgn}(x_0)}\right) = 1, \\ t_1^* &= \ln(1 + |x_0|). \end{aligned}$$

Example 4.7 *Drive the system*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

from x_0 to 0 in minimum time subject to $|u(t)| \leq 1$.
From Theorem 4.16, there can be at most $n - 1 = 1$ switching. The differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u = \pm 1, \end{aligned}$$

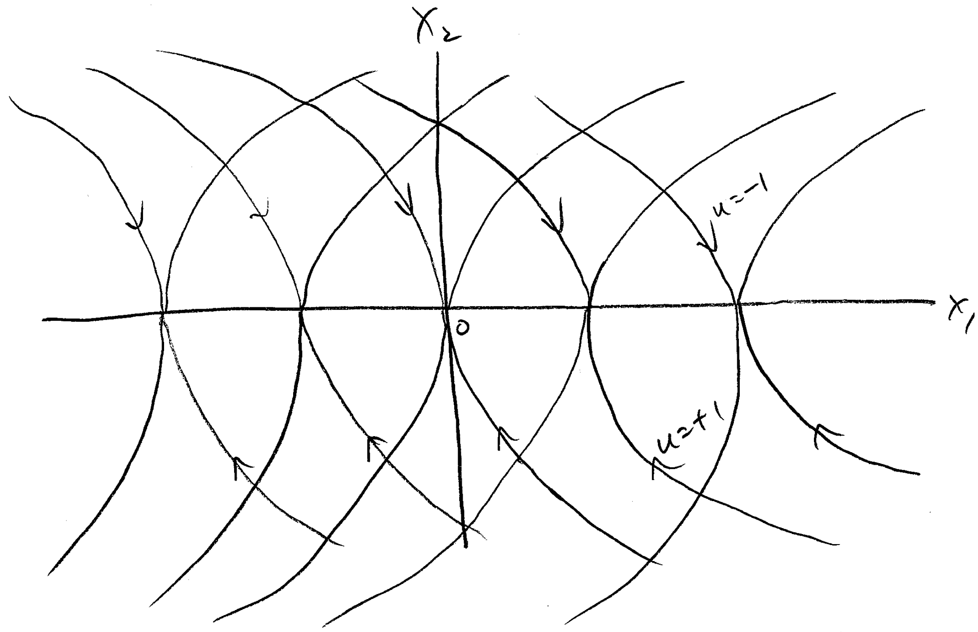
so

$$\frac{dx_1}{dx_2} = \pm x_2.$$

By separation of variables,

$$\int dx_1 = \pm \int x_2 dx_2,$$

$$x_1 = \pm \frac{x_2^2}{2} + a.$$



Optimal solutions can be worked out in specific cases. For example, let

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Switching occurs at

$$x_{s1} = -\frac{1}{2}x_{s2}^2 + \frac{1}{2},$$
$$x_{s1} = \frac{1}{2}x_{s2}^2,$$

or

$$x_s = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

For $x(t_0) = x_0$,

$$x(t) = \begin{bmatrix} 1 & t-t_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} \pm \int_{t_0}^{t} \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau = \begin{bmatrix} x_1(t_0) + x_2(t_0)(t-t_0) \pm \frac{1}{2}(t-t_0)^2 \\ x_2(t_0) \pm (t-t_0) \end{bmatrix}. \tag{4.36}$$

Setting $t_0 = 0,\ t = t_s,$ *and* $u = -1$,

$$x_s = \begin{bmatrix} t_s - \frac{1}{2}t_s^2 \\ 1 - t_s \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{\sqrt{2}} \end{bmatrix},$$
$$t_s = 1 + \frac{1}{\sqrt{2}}.$$

For $t_0 = t_s, x(t_0) = x_s,\ t = t_1^*,$ *and* $u = 1$,

$$x(t_1^*) = \begin{bmatrix} x_{s1} + x_{s2}(t_1^* - t_s) + \frac{1}{2}(t_1^* - t_s)^2 \\ x_{s2} + t_1^* - t_s \end{bmatrix} = 0,$$
$$t_1^* = 1 + \sqrt{2},$$
$$u^*(t) = \begin{cases} -1, & 0 < t < 1 + \frac{1}{\sqrt{2}} \\ 1, & 1 + \frac{1}{\sqrt{2}} < t < 1 + \sqrt{2} \end{cases}.$$

4.2.6 Terminal Manifolds

There is a version of PMP for terminal manifolds. Let $\phi_0, \phi_1 : \mathbb{R}^n \to \mathbb{R}^l$ be C^1 functions and

$$T_i = \left\{ w \in \mathbb{R}^n \,\middle|\, \phi_i(w) = 0 \right\}.$$

To ensure regularity, we assume

$$\operatorname{rank} \frac{\partial \phi_i(w)}{\partial x} = l$$

for every $w \in \mathbb{R}^n$. In this case, the constraint set Ω is the set of pairs $(x, u) \in PC^1 \times PC^0$ satisfying

1) $x : [0, t_1] \to \mathbb{R}^n$ and $u : [0, t_1] \to \mathbb{R}^m$,
2) $x(0) \in T_0$,
3) $x(t_1) \in T_1$,
4) $u(t) \in V$ for every $t \in [0, t_1]$,
5) $\dot{x}(t) = f(x(t), u(t))$ for every t where x is differentiable.

Theorem 4.18 *(PMP with Terminal Manifolds) If J achieves a global minimum at (x^*, u^*) subject to Ω, then there exist $\mu \in \mathbb{R}$ and $p \in PC^1$ with $p : [0, t_1^*] \to \mathbb{R}^n$ (not both 0) such that*
1) $\dot{p} = -\left(\left. \frac{\partial H}{\partial x} \right|_{(x^*, u^*)} \right)^T,$
2) $\mu \leq 0,$
3) $H(x^*(t), u^*(t), \mu, p(t)) = \max_{\omega \in V} H(x^*(t), \omega, \mu, p(t)) = 0$ *for every* $t \in [0, t_1^*],$
4) $\left. \frac{\partial \phi_0}{\partial x(0)} \right|_{x^*(0)} w = 0 \Longrightarrow p^T(0)\, w = 0,$
5) $\left. \frac{\partial \phi_1}{\partial x(t_1)} \right|_{x^*(t_1^*)} w = 0 \Longrightarrow p^T(t_1^*)\, w = 0.$

Proof. Pontryagin, Chapter II ■

The terminal manifold version of PMP reduces to the fixed end point version by setting

$$\phi_0(w) = w - x_0, \tag{4.37}$$

$$\phi_1(w) = w - x_1. \tag{4.38}$$

If one end point is fixed and the other specified to lie in a manifold, then Theorem 4.18 can be applied by using either (4.37) or (4.38). Conditions 4) and 5) are *transversality conditions.* The null space Ker $\frac{\partial \phi_0}{\partial x(0)}$ is the plane tangent to T_0 at $x(0)$. Hence, condition 4) states that $p(t)$ is orthogonal to the tangent plane at $x^*(0)$. The same holds for condition 5).

For LTI time-optimal problems, one can show that Theorems 4.15, 4.16, and 4.17 carry over to terminal manifolds.

Example 4.8 *Drive the system*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

from x_0 to

$$T_1 = \left\{ w \,\middle|\, w_1 = 0 \right\}$$

in minimum time subject to $|u(t)| \leq 1$.
As in Example 4.7, $u^* = \pm 1$ *with at most one switching. The transversality condition states*

$$\begin{bmatrix} 1 & 0 \end{bmatrix} w = 0 \Longrightarrow \begin{bmatrix} p_1(t_1^*) & p_2(t_1^*) \end{bmatrix} w = 0$$

or

$$p_2(t_1^*) = 0.$$

From PMP,

$$\dot{p} = -A^T p,$$

$$p(t) = e^{(t_1^* - t)A^T} p(t_1^*) = \begin{bmatrix} 1 & 0 \\ t_1^* - t & 1 \end{bmatrix} \begin{bmatrix} p_1(t_1^*) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ t_1^* - t \end{bmatrix} p_1(t_1^*).$$

Hence,

$$u^*(t) = \operatorname{sgn}\left(p^T(t) B\right) = \operatorname{sgn}\left((t_1^* - t) p_1(t_1^*)\right),$$

so there is no switching.
As in Example 4.7, the solution x *is restricted to curves*

$$x_1 = \pm \frac{x_2^2}{2} + a.$$

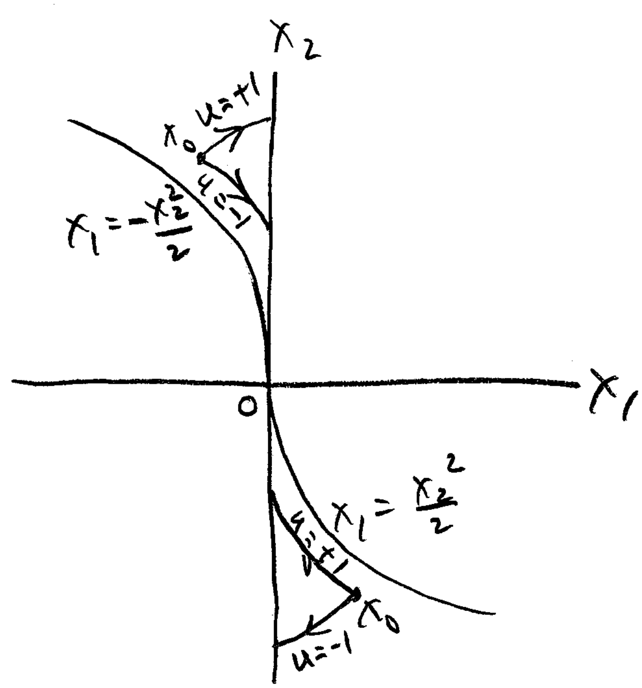
Each x_0 *corresponds to exactly one path to* T_1 *without switching, except when either*

$$0 < x_1 \leq \frac{x_2^2}{2}$$

or

$$-\frac{x_2^2}{2} \leq x_1 < 0.$$

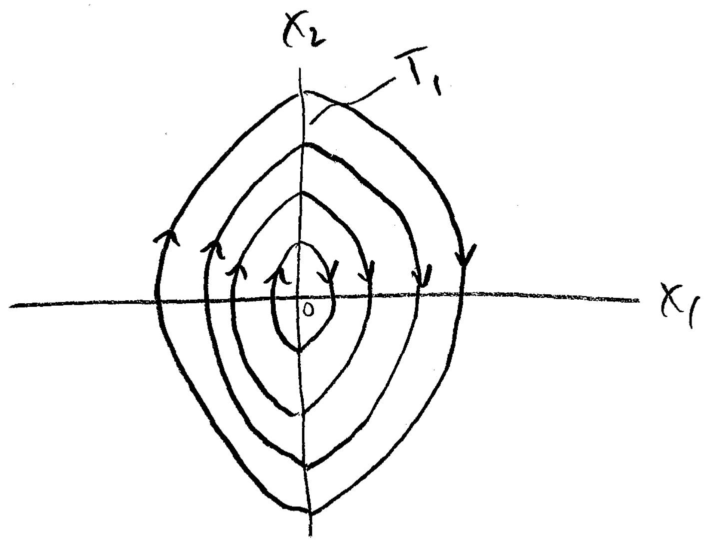
In these cases, two paths to T_1 *are possible without switching:*



Since $\dot{x}_2 = \pm 1$, the curve with the smallest vertical change is time-optimal. In the first case, one can show that this corresponds to $u = -1$. In the second case, $u = +1$. Hence, for every x_0 the optimal control is

$$u^*(t) = \begin{cases} -1, & x_{01} > 0 \\ +1, & x_{01} < 0 \end{cases}.$$

The optimal trajectories are shown below:



For example, let

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Then $u^* = -1$, and (4.36) yields

$$x^*(t) = \begin{bmatrix} 1 + t - \frac{1}{2}t^2 \\ 1 - t \end{bmatrix}.$$

Setting $x_1^*(t_1^*) = 0$ implies

$$1 + t - \frac{1}{2}t^2 = 0,$$

$$t_1^* = 1 + \sqrt{3},$$

$$x^*(t_1^*) = \begin{bmatrix} 0 \\ -\sqrt{3} \end{bmatrix}.$$

4.3 State Feedback Implementation

4.3.1 Background and Examples

Thus far, the solution of an optimal control problem has been described as a pair (x^*, u^*) which minimizes the cost functional and satisfies all the given constraints, including a differential equation. This is called an "open loop" solution. The problem with open loop control is that it is sensitive to errors in the system model.

The alternative is to develop a "closed-loop" optimal solution. This requires a pair of observations. First, suppose we solve the original problem with all its constraints, but with arbitrary initial time and state $x(t_0) = x_0$. In principle, this yields a solution $(x^*(x_0, t_0; t), u^*(x_0, t_0; t))$ for every x_0 and t_0. (Note that this is different from an initial manifold problem, since each choice of (x_0, t_0) yields a different (x^*, u^*).) Second, along any such optimal solution (x^*, u^*), we may choose any $\tau > t_0$ and think of the control process as beginning at initial state $x^*(x_0, t_0; \tau)$ and initial time τ. In other words, the current state and time may be viewed as the initial state and initial time for the remainder of the process. Obviously, the process must minimize J from time τ onward. In symbols,

$$u^*(x^*(x_0, t_0; \tau), \tau; t) = u^*(x_0, t_0; t)$$

for any $t > \tau$. This is called the *principle of optimality.* Let

$$v(x, \tau) = \lim_{t \to \tau^+} u^*(x, \tau; t).$$

Along any optimal solution,

$$v(x^*(x_0, t_0; \tau), \tau) = \lim_{t \to \tau^+} u^*(x^*(x_0, t_0; \tau), \tau; t) = \lim_{t \to \tau^+} u^*(x_0, t_0; t) = u^*(x_0, t_0; \tau).$$

Hence, the optimal control is implemented by the "feedback law"

$$u^*(t) = v(x, t).$$

In some cases, v is independent of t, so we write $v(x)$.

Example 4.9 *Returning to Example 4.7, regardless of x_0 and t_0, the optimal control is given by*

$$u^*(t) = \begin{cases} +1, & x_1 > -\frac{1}{2}x_2|x_2| \\ -1, & x_1 < -\frac{1}{2}x_2|x_2| \end{cases}.$$

Hence,

$$v(x) = \begin{cases} +1, & x_1 > -\frac{1}{2}x_2|x_2| \\ -1, & x_1 < -\frac{1}{2}x_2|x_2| \end{cases}.$$

Example 4.10 *Consider the linear, time optimal problem with*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$V[-1, 1]$, *and the end point constraint $x(t_1) = 0$. The system is controllable through its single input, so Theorem 4.17 guarantees that a unique optimal control u^* exists. The function u^* must be piecewise constant, taking on only the extreme values ± 1. However, the characteristic polynomial is*

$$\Delta(s) = \det \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2 + 1,$$

yielding imaginary eigenvalues, so Theorem 4.15 does not apply. Theorem 4.16 merely guarantees finitely many switchings from any x_0 and t_0.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 \pm 1$$

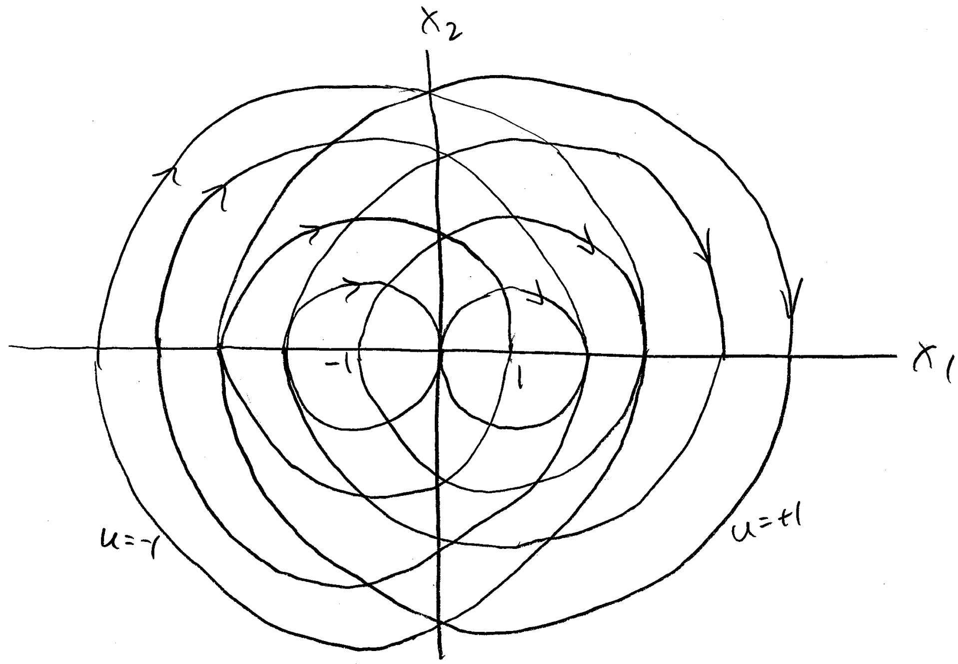
$$\frac{dx_2}{dx_1} = -\frac{x_1 \mp 1}{x_2}$$

$$\int (\pm 1 - x_1)\, dx_1 = \int x_2 dx_2$$

$$\pm x_1 - \frac{1}{2} x_1^2 = \frac{1}{2} x_2^2 + \frac{1-c}{2}$$

$$(x_1 \mp 1)^2 + x_2^2 = c \geq 0.$$

The possible optimal solutions are circles centered at $u = \pm 1$:



The state equations may be solved for $u = \pm 1$:

$$\begin{aligned} x(t) &= e^{(t-t_0)A} x_0 \pm \int_{t_0}^{t} e^{(t-\tau)A} d\tau B \\ &= \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} x_0 \pm \begin{bmatrix} 1-\cos(t-t_0) \\ \sin(t-t_0) \end{bmatrix} \\ &= \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} \begin{bmatrix} x_{01} \mp 1 \\ x_{02} \end{bmatrix} + \begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} \end{aligned}$$

This indicates that motion around each circle is uniform with angular velocity

$$\dot{\theta} = -1.$$

From (4.36),

$$u^*(t) = \operatorname{sgn}\left(p^T(t)\, B\right) = \operatorname{sgn}(p_2(t)).$$

From

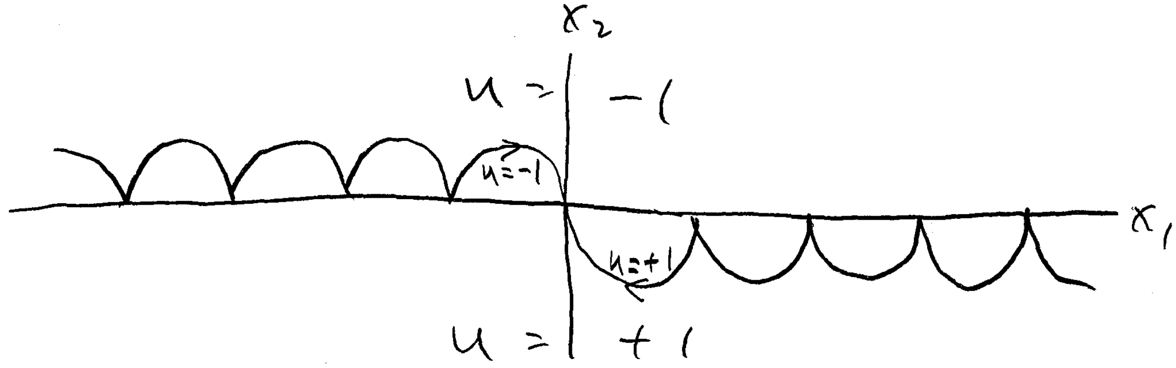
$$\dot{p} = -A^T p,$$

we obtain

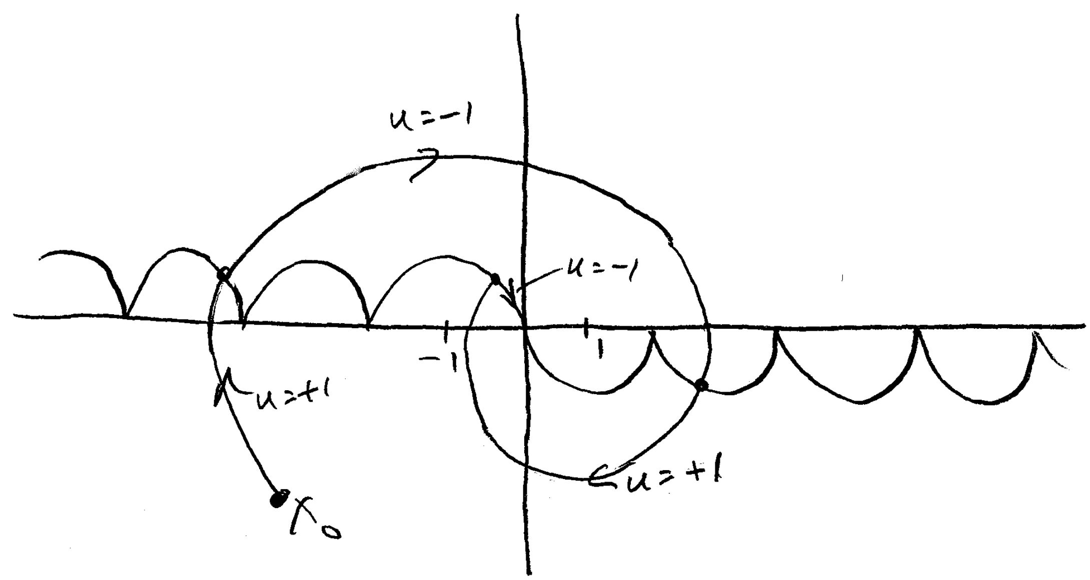
$$p(t) = e^{-(t-t_0)A^T} p(0) = \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} p(0),$$

$$p_2(t) = \begin{bmatrix} -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} p(0).$$

Note that p_2 is a sinusoid with period 2π. Hence, p_2 changes sign every π units of t, so u^* is a square wave with period 2π. Under these rules, the only possible path from x_0 to 0 is governed by the switching curve shown:



A typical optimal solution is also depicted:



4.3.2 State Regulation with Feedback

As in Section 4.1.6, the state regulation problem

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0$$

$$J(x,u) = \int_{t_0}^{1} \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q(t) & N(t) \\ N^T(t) & R(t) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt$$

has open-loop solution

$$u = -R^{-1}\left(N^T x + \frac{1}{2}B^T p\right),$$

obtained by solving the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}N^T & -\frac{1}{2}BR^{-1}B^T \\ -2\left(Q - NR^{-1}N^T\right) & -\left(A - BR^{-1}N\right)^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix},$$

$$x(t_0) = x_0, \quad p(1) = 0.$$

Consider the *Riccati matrix differential equation*

$$\dot{K} + K\left(A - BR^{-1}N^T\right) + \left(A - BR^{-1}N^T\right)^T K - KBR^{-1}B^T K + Q - NR^{-1}N^T = 0$$

with boundary condition

$$K(1) = 0, \tag{4.39}$$

where $K : [t_0, 1] \to \mathbb{R}^{n \times n}$.

Theorem 4.19 *If A, B, Q, N, and R are continuous functions of t, then the Riccati equation has a unique solution K satisfying (4.39). Furthermore, $K \in C^1$. If $Q(t)$ and $R(t)$ are symmetric and*

$$R(t) > 0,$$

$$Q(t) - N(t)R^{-1}(t)N^T(t) \geq 0$$

for every $t \in [t_0, 1]$, then $K(t)$ is symmetric, positive semidefinite for every t.

Suppose we solve the Riccati equation and the state equation

$$\dot{x} = \left(A - BR^{-1}(KB + N)^T\right)x, \quad x(t_0) = x_0 \tag{4.40}$$

and set

$$p = 2Kx.$$

Then

$$p(1) = 2K(1)x(1) = 0,$$

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} \dot{x} \\ 2\dot{K}x + 2K\dot{x} \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}(KB + N)^T \\ 2\dot{K} + 2K\left(A - BR^{-1}(KB + N)^T\right) \end{bmatrix} x \\ &= \begin{bmatrix} A - BR^{-1}N^T & -\frac{1}{2}BR^{-1}B^T \\ -2\left(Q - NR^{-1}N^T\right) & -\left(A - BR^{-1}N^T\right)^T \end{bmatrix} \begin{bmatrix} I \\ 2K \end{bmatrix} x \\ &= \begin{bmatrix} A - BR^{-1}N^T & -\frac{1}{2}BR^{-1}B^T \\ -2\left(Q - NR^{-1}N^T\right) & -\left(A - BR^{-1}N^T\right)^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \end{aligned}$$

Hence, solving (4.40) yields x^* and

$$u^* = -R^{-1}\left(N^T x^* + \frac{1}{2}B^T p\right) = -R^{-1}(KB + N)^T x^*.$$

The feedback law is

$$\begin{aligned} v(x, \tau) &= \lim_{t \to \tau^+} \left(-R^{-1}(t)\left(K(t)B(t) + N(t)\right)^T x\right) \\ &= -R^{-1}(\tau)\left(K(\tau)B(\tau) + N(\tau)\right)^T x. \end{aligned} \tag{4.41}$$

Theorem 4.20 *The feedback law (4.41) minimizes J with cost*

$$J(x^*, u^*) = x_0^T K(t_0) x_0.$$

Example 4.11 *Minimize*

$$J(x,u) = \int_{t_0}^{1} \left(x^2 + u^2\right) dt$$

subject to

$$\dot{x} = u.$$

The Riccati equation is

$$\dot{K} - K^2 + 1 = 0,$$

By separation of variables,

$$\int \frac{1}{K^2 - 1} dK = t + a.$$

Since K is continuous, the integral exists with $K(1) = 0$ iff $|K| < 1$. In this case,

$$\frac{1}{2} \ln \frac{1-K}{1+K} = t + a,$$

$$K(t) = -\frac{e^{2(t+a)} - 1}{e^{2(t+a)} + 1} = -\tanh(t + a).$$

From the boundary condition,

$$K(1) = -\tanh(1 + a) = 0,$$

so $a = -1$,

$$K(t) = \tanh(1 - t),$$

$$v(x,t) = -x \tanh(1 - t),$$

$$J(x^*, u^*) = x_0^2 \tanh(1 - t_0).$$