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011

A CHARACTERIZATION OF BOUNDED-INPUT BOUNDED-OUTPUT STABILITY FOR LINEAR TIME-INVARIANT SYSTEMS WITH DISTRIBUTIONAL INPUTS*

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Abstract. We consider linear time-invariant operators defined on the space of distributions with left-bounded support. We argue that in this setting the convolution operators constitute the most natural choice of objects for constructing a linear system theory based on the concept of impulse response. We extend the classical notion of bounded-input bounded-output stability to distributional convolution operators and determine precise conditions under which systems characterized by such maps are stable. A variety of expressions for the "gain" of a stable system is derived. We show that every stable system has a natural threefold decomposition based on the classical decomposition of functions of bounded variation. Our analysis involves certain extensions of the Banach spaces L^p in the space of distributions.

Key words. linear systems, stability, distributions

AMS subject classification. 93

1. Introduction. The concept of impulse response has traditionally played a central role in linear system theory. In spite of this fact, certain fundamental system-theoretic ideas have apparently not been developed on a mathematically rigorous level for systems with arbitrary distributional inputs and outputs. In particular, an extract characterization of the impulse and step responses that correspond to bounded-input bounded-output (BIBO) stable systems has not previously appeared in the literature. As an illustration of the problem, recall that if a linear time-invariant system is described by convolution of its inputs with a measurable function h, then the system is BIBO stable if and only if $h \in L^1$. (See, e.g., [1, p. 388].) This characterization is inadequate, however, for studying classes of systems where h may be a distribution since even a simple all-pass system has impulse response $\delta \notin L^1$. Obtaining a complete description of stable distributional systems is the primary goal of this paper.

A somewhat more limited framework than ours that addresses this problem appears in [2, p. 108], where systems are viewed as convolution operators and the impulse response is restricted to be a measurable function plus a linear combination of time shifts of the unit impulse. (In [2] the time-varying case is also included.) Thus, systems that differentiate the input are not included in [2], nor are more exotic cases such as the examples we present in §5. Our framework includes that of [2] (restricted to the time-invariant setting) and gives a more general framework for linear systems and, in particular, BIBO stable systems.

In §2, we consider the problem of meaningfully characterizing linear time-invariant systems in terms of their impulse responses. To set the stage for stability analysis, in §3 we pose and solve the problem of extending the Banach spaces L^p in the space of distributions for $1 \leq p \leq \infty$. In §4 we define BIBO stability for convolution operators on distribution space and obtain exact conditions on the impulse and step responses of a BIBO stable system. Expressions for the induced norm or "gain" of a stable operator with bounded inputs are also established. Section 5 contains a discussion of a three-fold decomposition applicable to all BIBO stable impulse responses. Our results are summarized in §6.

2. Preliminaries. We need a brief introduction to the theory of distributions. (See [3]–[6].) If $\varphi : \mathbb{R} \to \mathbb{R}$, define the support of φ, i.e., supp φ, as the closure of the set $\{t \mid \varphi(t) \neq 0\}$,

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and let $\sigma_\tau\varphi$ be the translation of φ defined by $(\sigma_\tau\varphi)(t) = \varphi(t-\tau)$. Let K be the space of C^∞ functions $\varphi : \mathbb{R} \to \mathbb{R}$ with supp φ bounded, and let K' be the dual space of K. (See [6] for an exact description of the topology of K.) A distribution f is an element of K', i.e., a continuous linear functional $\varphi \to \langle f, \varphi\rangle$ on K. For $f \in K'$, supp f is defined to be the complement of the largest open set $U \subset \mathbb{R}$ such that supp $\varphi \subset U$ implies $\langle f, \varphi\rangle = 0$. We may also define the time shift $\sigma_\tau f$ of a distribution f by $\langle \sigma_\tau f, \varphi\rangle = \langle f, \sigma_{-\tau}\varphi\rangle$ and the derivative $\dot{f}$ of f by $\langle \dot{f}, \varphi\rangle = -\langle f, \dot{\varphi}\rangle$. Denote the ith distributional derivative by $f^{(i)}$. It is easy to show that the time shift and differentiation operators commute and that

$$\frac{d}{d\tau}\langle \sigma_\tau f, \varphi\rangle = \langle \sigma_\tau \dot{f}, \varphi\rangle.$$

The unit impulse δ is defined by $\langle \delta, \varphi\rangle = \varphi(0)$. Also, any function f that is locally L^1 determines a distribution according to $\langle f, \varphi\rangle = \int f\varphi$. (Functions that coincide a.e. are identified.) In this way, we may view functions in L^p as distributions for $1 \le p \le \infty$. In particular, the unit step function θ may be considered a distribution. Define $\delta_\tau = \sigma_\tau\delta$ and $\theta_\tau = \sigma_\tau\theta$. If f is locally L^1 and differentiable a.e. in the classical sense, denote this derivative by f'. It is an important fact that there exist f such that $f' \neq \dot{f}$. This may occur in trivial ways (e.g., $\dot{\theta} = \delta$, but $\theta' = 0$ a.e.), but such cases also exist where f is continuous. (See §5.) Define $K'_\tau = \{f \in K' \mid \text{supp } f \subset [\tau, \infty)\}$ and

$$K'_+ = \bigcup_{\tau\in\mathbb{R}} K'_\tau.$$

Convergence in K' is defined via its weak* topology, which has a subbasis consisting of all sets of the form

$$U_{\varphi g} = \{f + g \mid |\langle f, \varphi\rangle| < 1\},$$

where $\varphi \in K$ and $g \in K'$. In terms of convergence, this means that a sequence (or net) f_n converges to f iff $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ for all $\varphi \in K$. Thus a linear operator $T : K'_+ \to K'_+$ is weak* continuous iff $\langle f_n, \varphi\rangle \to 0$ implies $\langle T(f_n), \varphi\rangle \to 0$. A linear operator $T : K'_+ \to K'_+$ is causal iff $\inf(\text{supp } T(f)) \ge \inf(\text{supp } f)$ for all $f \in K'_+$.

We are especially interested in convolution operators; the convolution of any pair $f, g \in K'_+$ is defined as follows. It is shown in [3, p. 100] that the map $\overline{\psi}(t) = \langle g, \sigma_{-t}\varphi\rangle$ defines a C^∞ function. Since φ has bounded support, supp $\overline{\psi}$ is bounded above. Choosing $\psi \in K$ to be any function in K such that $\psi(t) = \overline{\psi}(t)$ for all $t \ge \inf(\text{supp } f)$, we define $\langle f^*g, \varphi\rangle = \langle f, \psi\rangle$. This definition is unambiguous, since $\langle f, \gamma\rangle = 0$ whenever supp $\gamma \cap$ supp $\varphi = \phi$. Convolution can be shown to be commutative and to satisfy $f^*\delta^{(i)} = f^{(i)}$. Also, if $h = f^*g$, $\dot{h} = \dot{f}^*g = f^*\dot{g}$. A *convolution operator* $T : K'_+ \to K'_+$ is an operator of the form $T(u) = h^*u$, where $h \in K'_+$. For any convolution operator, $T(\dot{\theta}) = T(\delta)^*\dot{\theta} = T(\delta)$; T is causal iff $h \in K'_0$. We will often refer to the following basic result from [3, p. 105] concerning continuity of convolution operators.

LEMMA 2.1. *Let T be a convolution operator with $T(v) = h^*v$ for every v, and let $u_n \to u$ be a convergent sequence (or net) in K'_+. If there exists a $\tau \in \mathbb{R}$ such that either* supp $h \subset (-\infty, \tau]$ *or* supp $u_n \subset [\tau, \infty)$ $\forall n$, *then* $T(u_n) \to T(u)$.

Let $C_0 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}, f(-\infty) = f(\infty) = 0\}$ with norm

$$\|f\|_\infty = \sup_t |f(t)|.$$

We denote by BV the space of functions $f : \mathbb{R} \to \mathbb{R}$ with bounded variation. Set $NBV = \{f \in BV \mid f \text{ is left-continuous}, f(-\infty) = 0\}$, and let $\mathrm{Var}(f)$ be the variation of f. From [7, Ch. 6], NBV is the dual of C_0 with induced norm

$$\|f\|_n = \mathrm{Var}(f).$$

In addition, let $DBV = \{\dot{g} \in K' \mid g \in BV\}$ and define

$$\|\dot{g}\|_d = \mathrm{Var}(g).$$

It is easy to verify that $\|\cdot\|_d$ defines a norm on DBV and that NBV and DBV are isometrically isomorphic under the map $g \to \dot{g}$; hence DBV may also be viewed as the dual of C_0. It is easy to show that K is dense in C_0 and that the norm $\|\cdot\|_d$ satisfies

$$\|f\|_d = \sup_{\substack{\varphi \in K \\ \|\varphi\|_\infty = 1}} |\langle f, \varphi\rangle| = \sup_{\substack{\varphi \in K \\ \|\varphi\|_\infty = 1}} \left| \int_{-\infty}^{\infty} \varphi(t) dg(t) \right| \tag{1}$$

for any $f = \dot{g} \in DBV$.

Recall that every $f \in BV$ has a decomposition (unique a.e.) of the form $f = f_1 + f_2 + f_3$, where f_1 is bounded and absolutely continuous, f_2 is a bounded saltus function (i.e., $f_2 = \Sigma \alpha_i \theta_{\tau_i}$, where $\Sigma |\alpha_i| < \infty$), and f_3 is a singular function (i.e., f_3 is continuous and nonconstant, $f_3 \in BV$, and $f_3' = 0$ a.e.).

Define L^p_+, BV_+, and DBV_+ to consist of the L^p, BV, and DBV distributions $f \in K'$, respectively, with $\inf(\mathrm{supp} f) > -\infty$. For $p < \infty$, L^p_+ is a dense subspace of L^p. On the other hand, the closure of L^∞_+ is a proper subspace of L^∞, namely,

$$L^\infty_0 = \left\{ f \in L^\infty \mid \operatorname*{ess\ sup}_{t \in (-\infty, -n]} |f(t)| \to 0 \text{ as } n \to \infty \right\}.$$

Let $L^\infty_{[0,\infty)}$ denote the L^∞ functions f with $\mathrm{supp}\, f \subset [0, \infty)$. Note that $L^\infty_{[0,\infty)}$, L^p_+, BV_+, and DBV_+ may be viewed as subspaces of K'_+.

The first question we address is that of determining which operators $T : K'_+ \to K'_+$ can be justifiably called "linear systems." Clearly, T should be linear. Also, since we wish to develop a theory based on the concept of impulse response, we need to establish conditions under which $T(\delta)$ uniquely characterizes the operator T. We will limit ourselves to time-invariant operators, although the results of this section can be generalized considerably. As a first step, we might also impose continuity on T, since continuous linear operators are easier to work with. These constraints and the following lemma lead to Theorem 2.2.

LEMMA 2.2. *Let* $\tau \in \mathbb{R}$ *and*

$$I_\tau = \left\{ f \in K' \mid \exists t_i \geq \tau, \beta_i \text{ such that } f = \sum_{i=1}^{k} \beta_i \delta_{t_i} \right\}.$$

Then I_τ *is weak* dense in* K'_τ.

Proof. Using an overbar to denote weak* closure, it is clear that $I_\tau \subset K'_\tau = \overline{K}'_\tau$, so $\bar{I}_\tau \subset K'_\tau$. We will demonstrate that $\bar{I}_\tau \supset \overline{K}_\tau \supset \overline{K}'_{\tau b} \supset K'_\tau$, where

$$K_\tau = \{\varphi \in K \mid \mathrm{supp}\, \varphi \subset [\tau, \infty)\}, \qquad K'_{\tau b} = \{f \in K'_\tau \mid \mathrm{supp}\, f \text{ is bounded}\}.$$

Let $\varphi \in K$ with $\mathrm{supp}\, \varphi \subset [\tau, \tau + \Delta]$ and define

$$\gamma_n = \sum_{k=0}^{n} \frac{\Delta}{n} \varphi\left(\tau + k\frac{\Delta}{n}\right) \delta\left(t - \tau - k\frac{\Delta}{n}\right).$$

For any $\psi \in K$, $\langle \gamma_n, \Psi \rangle$ is a sequence of Riemann sums converging to $\langle \varphi, \psi \rangle$; hence, $\gamma_n \to \varphi$ weak*. Since φ is arbitrary, $\bar{I}_\tau \supset K_\tau$ and $\bar{I}_\tau \supset \overline{K}_\tau$. Let $\varphi_n \in K_0$, $\varphi_n \to \delta$ weak*, $f \in K'_{\tau b}$, and $\psi_n = f^* \varphi_n$. From Lemma 2.1, $\psi_n \in K_\tau$ and $\psi_n \to f$. Hence, $\overline{K}_\tau \supset K'_{\tau b}$ and $\overline{K}_\tau \supset \overline{K}'_{\tau b}$. Finally, let $g \in K'_\tau$ and let $\eta_n \in K$ satisfy $\eta_n(t) = 1$ for $0 \leq t \leq n$. Then $\eta_n g \in K'_{\tau b}$ and $\eta_n g \to g$ weak*, so $\overline{K}'_{\tau b} \supset K'_\tau$. □

THEOREM 2.3. *Let $T : K'_+ \to K'_+$ be a weak* continuous, linear, time-invariant operator. Then $T(u) = T(\delta)^* u$ for all $u \in K'_+$.*

Proof. Suppose $u \in K'_\tau$. From Lemma 2.2, for any weak* neighborhood U of u, there exists a $v \in U$ of the form

$$v = \sum_{i=1}^{k} \beta_i \delta_{t_i}$$

with $t_i \geq \tau$ for all i. Let $\varphi \in K$. From linearity and time-invariance of T,

$$\langle T(v), \varphi \rangle = \langle T(\delta), \psi \rangle,$$

where

$$\psi = \sum_{i=1}^{k} \beta_i \sigma_{-t_i} \varphi.$$

Note that $\psi \in K$ and $\psi(t) = \langle v, \sigma_{-t} \varphi \rangle$ for all $t \geq \tau$. Thus

$$\langle T(\delta), \psi \rangle = \langle T(\delta)^* v, \varphi \rangle.$$

Since φ is arbitrary, $T(v) = T(\delta)^* v$. From Lemma 2.1, $T(u) = T(\delta)^* u$. □

Unfortunately, the converse to Theorem 2.3 is false; i.e., a convolution operator may fail to be weak* continuous. From Lemma 2.1, boundedness of $T(\delta)$ is sufficient to guarantee continuity of T. The next result establishes the converse.

THEOREM 2.4. *Let $T : K'_+ \to K'_+$ be a weak* continuous convolution operator. Then* supp $T(\delta)$ *is bounded.*

Proof. Suppose supp $T(\delta)$ is unbounded. Then there exist sequences $\varphi_n \in K$ and $\alpha_n \in \mathbb{R}$ such that supp $\varphi_n \subset [\alpha_n, \alpha_n + 1]$, $\alpha_n \to \infty$, and $\langle T(\delta), \varphi_n \rangle = \beta_n \neq 0$. Let

$$\gamma_n = \max_{\substack{i \leq n \\ t \in \mathbb{R}}} \left| \frac{d^i \varphi_i(t)}{dt^i} \right|$$

and $\psi_n = \frac{1}{n\gamma_n} \sigma_{-n} \varphi_n$. From [3, p. 2], $\psi_n \to 0$ in K; hence, $\{\psi_n\}$ is a bounded subset of K. Let

$$f_n = \frac{n\gamma_n}{\beta_n} \delta_{-\alpha_n}.$$

Then $f_n \to 0$ weak*, but

$$\begin{aligned}
\sup_m \langle T(\delta)^* f_n, \psi_m \rangle &\geq \langle T(\delta)^* f_n, \psi_n \rangle \\
&= \langle \frac{n\gamma_n}{\beta_n} \sigma_{-\alpha_n} T(\delta), \psi_n \rangle \\
&= \frac{1}{\beta_n} \langle T(\delta), n\gamma_n \sigma_{\alpha_n} \psi_n \rangle \\
&= \frac{1}{\beta_n} \langle T(\delta), \varphi_n \rangle \\
&= 1,
\end{aligned}$$

so $T(\delta)^* f_n$ does not converge uniformly to 0 on bounded subsets of K. From [4, pp. 55–56], $T(\delta)^* f_n$ does not converge to 0 weak*, so T is not continuous. □

It follows from Theorem 2.4 that there exist many familiar examples of linear systems that are characterized by weak* discontinuous convolution operators. For example, θ has unbounded support, so a simple integrator is discontinuous. In particular, the sequence $\delta_{-n} \to 0$ weak* as $n \to \infty$, but its integrals θ_{-n} converge to the constant distribution 1. In view of such examples, we choose not to restrict ourselves to weak* continuous operators.

Unfortunately, an arbitrary class of discontinuous operators T in general is not uniquely characterized by the values $T(\delta)$, since $T(\delta)$ only determines the action of a linear time-invariant operator on the proper subspace span $\{\delta_\tau \mid \tau \in \mathbb{R}\} \subset K'_+$. On the other hand, the distributions $T(\delta)$ do uniquely characterize the family of convolution operators. In fact, it is easy to show that $h^* u = 0$ for every $u \in K'_+$ implies $h = 0$, so $T \to T(\delta)$ maps the convolution operators one-to-one onto K'_+. Thus any linear time-invariant nonconvolution operator has the same impulse response as some convolution operator.

Based on these observations, we define a *linear time-invariant system* to be a convolution operator $T : K'_+ \to K'_+$. In the next section, we develop the machinery that will enable us to define and characterize BIBO stability for such systems.

3. Extension of normed linear spaces. In this section we examine the problem of extending L^p in K' for arbitrary p. We do this because L^1 is known to play a role in characterizing BIBO stability of an operator and because L^∞ is used in the definition of stability. Values $p \in (1, \infty)$ are not directly related to stability but can be easily handled along with $p = \infty$ and are therefore included. In fact, the problem can be couched in much more general terms without substantially increasing the level of difficulty.

Let X be a Hausdorff topological vector space over $\mathbb{R}$, and let $Y \subset X$ be a normed linear space. Then Y has two topologies: the norm topology and the one inherited from X. Denote the topology of X by $\mathcal{T}$ (i.e., $\mathcal{T}$ is the family of all open subsets U of X), let $\mathcal{T}_Y$ be the relative topology on Y generated by $\mathcal{T}$ (i.e., $\mathcal{T}_Y$ consists of all sets $U \cap Y$). Also, let $B(y, r) \subset Y$ be the norm ball about y with radius r. We make the following assumptions.

A1) $\forall U \in \mathcal{T}, U \cap Y \neq \phi$.
A2) $\forall U \in \mathcal{T}$ and $\forall y \in U \cap Y, \exists \varepsilon > 0$ such that $B(y, \varepsilon) \subset U$.
A3) $\exists U \in \mathcal{T}$ such that $U \cap Y = Y - B(0, 1)$.

Assumption A1) states that Y is dense in X relative to $\mathcal{T}$. The other two assumptions give upper and lower bounds on $\mathcal{T}_Y$. Assumption A2) states that the norm topology on Y is stronger than or equal to $\mathcal{T}_Y$, while A3) says that $B(0, 1)$ is closed in $\mathcal{T}_Y$.

Suppose Y has norm $\|\cdot\|$, let $x \in X$, and let $\{U_\beta\} \subset \mathcal{T}$ be the family of all neighborhoods of x. Define

$$\|x\|^e = \sup_\beta \inf_{y \in U_\beta \cap Y} \|y\|. \tag{2}$$

In view of A1), $\|x\|^e$ is well defined and determines a function $\|\cdot\|^e : X \to [0, \infty]$. The next result establishes that $\|\cdot\|^e$ is a natural extension of $\|\cdot\|$ to all of X.

PROPOSITION 3.1. 1) $\|\cdot\|$ *and* $\|\cdot\|^e$ *coincide on* Y.

2) $\|\cdot\|^e$ *is lower semicontinuous on* X *relative to* $\mathcal{T}$.

3) *If* $\|x\|^e < \infty$, *then for every* $\varepsilon > 0$ *and* $\mathcal{T}$*-neighborhood* U *of* x *there exists a* $y \in U \cap Y$ *such that*

$$\|y\| < \|x\|^e + \varepsilon.$$

4) *If* $\|\cdot\|^f : X \to [0, \infty]$ *satisfies* 1)–3) *(replacing e with f throughout), then* $\|\cdot\|^f = \|\cdot\|^e$.

Proof. 1) For $x \in Y$,

$$\inf_{y \in U_\beta \cap Y} \|y\| \leq \|x\|$$

for all β, so $\|x\|^e \leq \|x\|$ follows immediately from (2). Suppose $\|x\|^e < a < \|x\|$. Then (2) states that for every β and $\varepsilon > 0$ there exists a $y \in U_\beta \cap Y$ such that $\|y\| < \|x\|^e + \varepsilon$. Setting $\varepsilon = a - \|x\|^e$ yields $\|y\| < a$. Hence $U_\beta \cap B(0, a) \neq \phi$, and $B(0, a)$ is not closed relative to $\mathcal{T}$. Thus $B(0, 1)$ is also not closed, contradicting A3). Therefore $\|x\|^e = \|x\|$.

2) We need to show that

$$\sum_R = \{x \in X \mid \|x\|^e > R\}$$

is $\mathcal{T}$-open for every $R < \infty$. (See [8, p. 84].) From (2) we have

$$\sum_R = \{x \in X \mid \exists \text{ a } \mathcal{T}\text{-neighborhood } U \text{ of } x \text{ and } \varepsilon > 0 \text{ such that } \|y\| > R + \varepsilon \ \forall y \in U \cap Y\}. \tag{3}$$

If $x \in \sum_R$ for some R, then U and ε are determined by (3). In fact, $U \subset \sum_R$, so $\sum_R$ is open.

3) This follows immediately from (2).

4) If $\|x\|^e = \infty$, $\|x\|^f \leq \|x\|^e$. Suppose that $\|x\|^e < \infty$ and $\varepsilon > 0$ are given. From 2), there exists a β such that $\|y\| \geq \|x\|^f - \frac{\varepsilon}{2}$ for every $y \in U_\beta$. Setting $U = U_\beta$ in 3) guarantees the existence of a $z \in U_\beta \cap Y$ such that $\|z\| < \|x\|^e + \frac{\varepsilon}{2}$. Setting $y = z$ yields $\|x\|^f < \|x\|^e + \varepsilon$. Since ε is arbitrary, $\|x\|^f \leq \|x\|^e$. Interchanging the roles of "e" and "f" and applying the same arguments gives $\|x\|^f \geq \|x\|^e$. □

Let $Y_e = \{x \in X \mid \|x\|^e < \infty\}$. From Proposition 3.1, 1), it is obvious that $Y_e \supset Y$. We refer to Y_e as the *X-extension* of Y. The next result further justifies this terminology.

PROPOSITION 3.2. 1) *Y_e is a subspace of X.*

2) *$\|\cdot\|^e$ is a norm on Y_e.*

Proof. 1) If $x \in Y_e$ and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} \|\alpha x\|^e &= \sup_\beta \inf_{y \in \alpha U_\beta \cap Y} \|y\| \\ &= |\alpha| \sup_\beta \inf_{y \in U_\beta \cap Y} \|y\| \\ &= |\alpha|\, \|x\|^e \\ &< \infty. \end{aligned} \tag{4}$$

Furthermore, if $x_1, x_2 \in Y_e$ and U_β is a neighborhood of $x_1 + x_2$, then there exist neighborhoods V_β and W_β of x_1 and x_2, respectively, such that $V_\beta + W_\beta \subset U_\beta$. Hence,

$$\begin{aligned} \|x_1 + x_2\|^e &= \sup_\beta \inf_{y \in U_\beta \cap Y} \|y\| \\ &\leq \sup_\beta \inf_{\substack{y_1 \in V_\beta \cap Y \\ y_2 \in W_\beta \cap Y}} \|y_1 + y_2\| \\ &\leq \|x_1\|^e + \|x_2\|^e \\ &< \infty. \end{aligned} \tag{5}$$

Thus αx and $x_1 + x_2$ belong to Y_e, and 1) follows.

2) In view of (4) and (5), to demonstrate 2) it remains to show that $\|x\|^e = 0$ implies $x = 0$. Indeed, $\|x\|^e = 0$ implies

$$\inf_{y \in U_\beta \cap Y} \|y\| = 0 \tag{6}$$

for every β. Since X is Hausdorff, for $x \neq 0$ there must exist disjoint $\mathcal{T}$-neighborhoods U_β and V of x and 0, respectively. From assumption A2), there exists an $\varepsilon > 0$ such that $B(0, \varepsilon) \subset V$. Thus $B(0, \varepsilon) \cap U_\beta = \phi$, contradicting (6). □

Roughly speaking, Propositions 3.1 and 3.2 say that 1) $\|\cdot\|^e$ is the smallest possible extension of $\|\cdot\|$ such that $\|x\|^e$ is consistent with $\mathcal{T}$-approximations to x from within Y, and 2) Y_e is the largest subspace of X upon which $\|\cdot\|^e$ is a norm.

We may now specialize these ideas to $X = K'_+$ and $Y = L^p_+$. First note that assumptions A1) and A2) follow easily from [4, §II.4.4] and [3, §I.1.8]. The next result verifies assumption A3).

PROPOSITION 3.3. *$B(0,1) \subset L^p_+$ is weak* closed (relative to K'_+) for $1 \leq p \leq \infty$.*

Proof. Suppose $B(0, 1)$ is not weak* closed. Then there exist $\varepsilon > 0$ and $f \in L^p_+$ such that $\|f\|_p = 1 + \varepsilon$ and such that, for each $\varphi \in K$, there exists a $g \in L^p_+$ with $\|g\|_p \leq 1$ and $|\langle f - g, \varphi\rangle| < \frac{\varepsilon}{4}$. Let q be conjugate to p. Since K is dense in L^q using $\|\cdot\|_q$ for $q < \infty$, we may choose $\varphi \in K$ such that $\|\varphi\|_q = 1$ and $|\langle f, \varphi\rangle| > 1 + \frac{\varepsilon}{2}$. To handle the case $q = \infty$, we note that K is dense in C_0, so [7, Thm. 6.19] guarantees the existence of a $\varphi \in K$ with $\|\varphi\|_\infty = 1$ and $|\langle f, \varphi\rangle > 1 + \frac{\varepsilon}{2}$. Thus, for arbitrary p, $|\langle g, \varphi\rangle| \leq 1$ and

$$\frac{\varepsilon}{2} < |\langle f, \varphi\rangle| - |\langle g, \varphi\rangle| \leq |\langle f - g, \varphi\rangle| < \frac{\varepsilon}{4}.$$

This is a contradiction, so $B(0, 1)$ is closed. □

Since A1)-A3) are satisfied, the K'_+-extension L^p_{+e} of L^p_+ and its norm $\|\cdot\|^e_p$ are well defined. The following two results characterize L^p_{+e} more precisely.

PROPOSITION 3.4. *Let $1 < p \leq \infty$. Then $L^p_{+e} = L^p_+$.*

Proof. Suppose $f \in K' - L^p_+$, let $M < \infty$ be given, and let q be conjugate to p. Since K is dense in L^q relative to $\|\cdot\|_q$, the dual of K with $\|\cdot\|^q$ imposed on it is just L^p. Thus there exists a $\varphi \in K$ with $\|\varphi\|_q = 1$ such that $|\langle f, \varphi\rangle| > M + 1$. Furthermore, there exists a weak* neighborhood U of f such that $\|g\|_p \geq |\langle g, \varphi\rangle| > M$ for all $g \in U \cap L^p_+$; thus $\|f\|^e_p \geq M$. Since M is arbitrary, $\|f\|^e_p = \infty$ and $f \notin L^p_{+e}$. Hence $L^p_{+e} = L^p_+$. □

The case $p = 1$ is somewhat more challenging.

PROPOSITION 3.5. *$L^1_{+e} = DBV_+$ and $\|x\|^e_1 = \|x\|_d$ for all $x \in DBV_+$.*

Proof. Let

$$\|f\|_D = \begin{cases} \|f\|_d, & f \in DBV, \\ \infty, & f \notin DBV. \end{cases}$$

It suffices to verify 1)-3) in Proposition 3.1. If $f \in L^1$, then $f = \dot{g}$ for some absolutely continuous g. Hence

$$\|f\|_D = \mathrm{Var}(g) = \int_{-\infty}^{\infty} 1|dg(t)| = \int_{-\infty}^{\infty} |\dot{g}(t)|\, dt = \|f\|_1,$$

and 1) holds.

To prove 2), let $R < \infty$ and $\sum_R = \{f \in K' \mid \|f\|_D > R\}$. For $f \in \sum_R$, we have from (1) that

$$\sup_{\substack{\varphi \in K \\ \|\varphi\|_\infty = 1}} |\langle f, \varphi\rangle| = \|f\|_D > R.$$

Hence, there exists a $\varphi \in K$ with $\|\varphi\|_\infty = 1$ such that $|\langle f, \varphi\rangle| > R$. Let

$$U = \{g \in K' \mid |\langle f - g, \varphi\rangle| < |\langle f, \varphi\rangle| - R\}.$$

U is a weak* neighborhood of f. If $g \in U$,

$$|\langle f, \varphi\rangle| - |\langle g, \varphi\rangle| \le |\langle f-g, \varphi\rangle| < |\langle f, \varphi\rangle| - R,$$

so $|\langle g, \varphi\rangle| > R$. Hence,

$$\|g\|_D = \sup_{\substack{\varphi\in K \\ \|\varphi\|_\infty=1}} |\langle g, \varphi\rangle| > R.$$

Thus $g \in \sum_R$ and $U \subset \sum_R$. Hence $\sum_R$ is weak* open, and $\|\cdot\|_D$ is lower semicontinuous.

Condition 3) can be proven directly using elementary analytic arguments based on the definition of Var$(\cdot)$, but here we supply a functional analytic proof that is more amenable to generalization. Let U be a weak* neighborhood of f, and let $\varepsilon > 0$. Then there exist $\varphi_1, \ldots, \varphi_n \in K$ such that $h \in U$ whenever $|\langle f-h, \varphi_i\rangle| < 1$ for all i. If $\beta_1, \ldots, \beta_n \in \mathbb{R}$, then

$$\left|\sum \beta_i \langle f, \varphi_i\rangle\right| = \left|\left\langle f, \sum \beta_i\varphi_i\right\rangle\right| \le \|f\|_d \left\|\sum \beta_i\varphi_i\right\|_\infty.$$

Noting that $K \subset L^\infty$ and that L^∞ is the dual of the Banach space L^1, it follows from [9, Thm. 5, p. 109] that there exists an $h \in L^1$ such that $\langle h, \varphi\rangle = \langle f, \varphi_i\rangle$ for all i and $\|h\|_1 \le \|f\|_d + \frac{\varepsilon}{2}$. Note that $|\langle f-h, \varphi_i\rangle| = 0$, so $h \in U$. Since L^1_+ is dense in L^1 relative to $\|\cdot\|_1$ (and therefore also weak*), there exists a $g \in U \cap L^1_+$ such that $\|g\|_1 < \|f\|_d + \varepsilon$. □

We can make slight modifications to the arguments of this section and construct an extension L^p_e of L^p in K'. In this way, results similar to Propositions 3.1, 3.4, and 3.5 are obtained; i.e., $L^p_e = L^p$ for $1 < p \le \infty$ and $L^1_e = DBV$. This construction has the advantage that L^1_e is a Banach space, while L^1_{+e} is not; however, convolution is not defined on all of L^1_e, so we must restrict ourselves to L^1_{+e}.

Besides stability analysis, another important application of our extension theory occurs in treating minimum-norm optimization problems over K'. For example, the issue of extending a quadratic cost functional on L^2 to K' arose naturally in the earlier work of one of us [10]. It is easily seen that [10, Prop. 1] follows immediately from Proposition 3.1, part 2) and Proposition 3.4.

4. BIBO stability. Proposition 3.4 shows that "boundedness" of a distribution $f \in K'_+$ is most naturally interpreted to mean that $f \in L^\infty_+$. Hence, we define a linear operator $T : K'_+ \to K'_+$ to be *BIBO stable* if $T(L^\infty_+) \subset L^\infty_+$. Clearly, this definition extends the classical one, as long as $u, T(u) \in K'_+$.

Since convolution operators satisfying $T(\delta) \in L^1_+$ are known to be BIBO stable, a natural conjecture is that the convolution operators with kernels in the extension space L^1_{+e} described in Proposition 3.5 coincide exactly with the stable operators. This idea is supported by the fact that $\delta \in L^1_{+e}$, since $\delta = \dot\theta$ and $\theta \in BV_+$, and $\delta^{(i)} \notin L^1_{+e}$ for $i = 1, 2, 3, \ldots$, since $\delta^{(i-1)} \notin BV$. It is easy to show that $T(u) = \delta^{(i)} * u$ defines a stable operator iff $i = 0$.

Corresponding to each convolution operator T we may associate a $\widetilde{T} : K \to C^\infty$ defined by

$$\widetilde{T}(\varphi)(t) = \langle T(\delta), \sigma_{-t}\varphi\rangle.$$

Indeed, it is established in [3, p. 100] that $\widetilde{T}$ takes values in C^∞ with $\widetilde{T}(\varphi)(t) = 0$ for all t in some interval $[\tau_\varphi, \infty)$. The following result provides preliminary information about BIBO stable operators.

LEMMA 4.1. *Let $T : K'_+ \to K'_+$ be a convolution operator.*

1) *If T is BIBO stable, then*

$$\sup_{\substack{u\in L^\infty_+ \\ \|u\|_\infty=1}} \|T(u)\|_\infty < \infty.$$

2) *T is BIBO stable iff*

$$\sup_{\substack{\varphi\in K\\ \|\varphi\|_1=1}} \int_{-\infty}^{\infty} |\widetilde{T}(\varphi)(t)|\, dt < \infty.$$

Proof. 1) Let P be the restriction of T to $L^\infty_{[0,\infty)}$. We begin by showing that P is continuous relative to $\|\cdot\|_\infty$.

Let $u, u_i \in L^\infty_{[0,\infty)}$ and $\|u_i - u\|_\infty \to 0$. Suppose there exists a $v \in L^\infty$ such that $\|P(u_i) - v\|_\infty \to 0$. Since weak* topology is weaker than norm topology on L^∞, $u_i \to u$ weak* and $P(u_i) \to v$ weak*. From Lemma 2.1, P is weak* continuous, so $P(u_i) \to P(u)$ weak*. Since weak* topology is Hausdorff, $P(u) = v$. The continuity of P follows from the closed graph theorem.

Now let $u_i \in L^\infty_+$ with $\|u_i\|_\infty \to 0$. For each u_i, there exist τ_i such that $\sigma_{\tau_i}u_i \in L^\infty_{[0,\infty)}$. Also, $\|\sigma_{\tau_i}u_i\|_\infty = \|u_i\|_\infty \to 0$. From the continuity of P and time-invariance of T,

$$\|T(u_i)\|_\infty = \|\sigma_{-\tau_i}P(\sigma_{\tau_i}u_i)\|_\infty = \|P(\sigma_{\tau_i}u_i)\|_\infty \to 0.$$

This shows that T is continuous or, equivalently,

$$\sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \|T(u)\|_\infty < \infty.$$

2) (sufficient) Let $u \in L^\infty_+$. From the definition of convolution on K', we have

$$\langle T(u), \varphi\rangle = \int_{-\infty}^{\infty} u(t)\widetilde{T}(\varphi)(t)\, dt;$$

hence

$$|\langle T(u), \varphi\rangle| \le \|u\|_\infty \int_{-\infty}^{\infty} |\widetilde{T}(\varphi)(t)|\, dt$$

and

$$\sup_{\substack{\varphi\in K\\ \|\varphi\|_1=1}} |\langle T(u), \varphi\rangle| < \infty.$$

Since K is dense in L^1, $T(u)$ extends continuously to a unique linear functional on L^1. Hence, $T(u) \in L^\infty_+$.

(necessary) From 1),

$$\begin{aligned}\sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \sup_{\substack{\varphi\in K\\ \|\varphi\|_1=1}} \left|\int_{-\infty}^{\infty} u(t)\widetilde{T}(\varphi)(t)\, dt\right| &= \sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \sup_{\substack{\varphi\in K\\ \|\varphi\|_1=1}} |\langle T(u), \varphi\rangle| \\ &\le \sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \|T(u)\|_\infty \\ &< \infty.\end{aligned}$$

Let $\varphi \in K$ and $u_i(t) = \operatorname{sgn}(\widetilde{T}(\varphi)(t))\theta_{-i}(t)$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} |\widetilde{T}(\varphi)(t)|\, dt &= \lim_{i\to\infty} \int_{-\infty}^{\infty} u_i(t)\widetilde{T}(\varphi)(t)\, dt \\ &\le \sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \left|\int_{-\infty}^{\infty} u(t)\widetilde{T}(\varphi)(t)\, dt\right|,\end{aligned}$$

so

$$\sup_{\substack{\varphi \in K \\ \|\varphi\|_1 = 1}} \int_{-\infty}^{\infty} |\widetilde{T}(\varphi)(t)|\, dt < \infty. \qquad \square$$

Lemma 4.1, part 1) makes the striking statement that every stable convolution operator is also continuous relative to $\|\cdot\|_{\infty}$. In system-theoretic jargon, this means that, in the time-invariant case, BIBO stability implies that small changes in the system input give rise to only small changes in the output. It is an interesting fact that this statement is demonstrably false in the time-varying case.

We are now in a position to give our main result.

THEOREM 4.2. *Let $T : K'_+ \to K'_+$ be a convolution operator. The following statements are equivalent.*

1) *T is BIBO stable.*
2) *$T(\delta) \in L^1_{+e}$.*
3) *$T(\theta) \in BV_+$.*

Proof. The equivalence of 2) and 3) is obvious from Proposition 3.5. To prove that 3) implies 1), let $\varphi \in K$, $u \in L^{\infty}_{+}$, and $s = T(\theta)$ and note that

$$\begin{aligned}
\langle T(u), \varphi \rangle &= \int_{-\infty}^{\infty} u(t) \langle \dot{s}, \sigma_{-t}\varphi \rangle\, dt \\
&= -\int_{-\infty}^{\infty} u(t) \langle s, \sigma_{-t}\dot{\varphi} \rangle\, dt \\
&= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(t) s(\tau) \dot{\varphi}(t+\tau) d\tau\, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(t) \varphi(t+\tau) ds(\tau) dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} u(t-\tau) ds(\tau) \right) \varphi(t) dt,
\end{aligned}$$

so

$$T(u)(t) = \int_{-\infty}^{\infty} u(t-\tau) ds(\tau) \quad \text{a.e.}$$

and

$$|T(u)(t)| \le \|u\|_{\infty} \operatorname{Var}(s) \quad \text{a.e.}$$

Thus $T(u) \in L^{\infty}$.

Finally, we show that 1) implies 2). From [11, Thm. 2.3.9] and Lemma 4.1, part 2), we know that there exists a measurable function $g : \mathbb{R}^2 \to \mathbb{R}$ with $g(t, \cdot) \in L^{\infty}$ for all t, $\int_{-\infty}^{\infty} g(\cdot, \tau)\varphi(\tau)\, d\tau$ absolutely continuous,

$$\operatorname*{ess\,sup}_{\tau} \operatorname{Var}(g(\cdot, \tau)) < \infty$$

and

$$\widetilde{T}(\varphi)(t) = \frac{d}{dt} \int_{-\infty}^{\infty} g(t, \tau)\varphi(\tau)\, d\tau$$

for all $\varphi \in K$.

Since $\widetilde{T}$ is time-invariant,

$$\frac{d}{dt}\int_{-\infty}^{\infty} g(t,\tau)\varphi(\tau-t_0)\,d\tau = \frac{d}{dt}\int_{-\infty}^{\infty} g(t-t_0,\tau)\varphi(\tau)\,d\tau,$$

for all $t, t_0 \in \mathbb{R}$, $\varphi \in L^1$. Integration and a change of variables yield

$$\int_{-\infty}^{\infty} g(t,\tau+t_0)\varphi(\tau)\,d\tau = \int_{-\infty}^{\infty} g(t-t_0,\tau)\varphi(\tau)\,d\tau.$$

Thus $g(t,\tau+t_0) = g(t-t_0,\tau)$ for all t, t_0, τ. Set $s(t) = g(-t,0)$. Then $s \in BV$ and

$$g(t,\tau) = g(t-\tau,0) = s(\tau-t),$$

so

$$\widetilde{T}(\varphi)(t) = \frac{d}{dt}\int_{-\infty}^{\infty} s(\tau-t)\varphi(\tau)\,dt = \frac{d}{dt}\langle\sigma_t s,\varphi\rangle = \langle\sigma_t\dot{s},\varphi\rangle$$

for all $\varphi \in K$. Setting $t = 0$ yields $\langle\dot{s},\varphi\rangle = \langle T(\delta),\varphi\rangle$; hence $T(\delta) \in DBV \cap K'_+ = DBV_+$. $\square$

Our next objective is to obtain a more detailed picture of the additional structure imposed on a linear system by stability. We begin by considering certain extensions of the operators T and $\widetilde{T}$.

Since every stable T is continuous on L^∞_+ relative to $\|\cdot\|_\infty$, each such operator may be extended uniquely to a continuous linear operator $T_0 : L^\infty_0 \to L^\infty_0$. Similarly, Lemma 4.1, part 2) also states that T is BIBO stable iff $\widetilde{T}(K) \subseteq L^1$ and $\widetilde{T}$ is bounded using the L^1 norm throughout. In this case, since K is dense in L^1, $\widetilde{T}$ extends uniquely to a continuous linear operator $\widetilde{T}_e : L^1 \to L^1$. It is easy to show that T_0 and $\widetilde{T}_e$ are time-invariant.

THEOREM 4.3. *Suppose $T : K'_+ \to K'_+$ is a BIBO stable convolution operator and $s = T(\theta)$. Let $T_e : L^\infty \to L^\infty$ be defined by*

$$T_e(u)(t) = \int_{-\infty}^{\infty} u(t-\tau)\,ds(\tau).$$

Then

1) $T_e(u) = T_0(u)$ *for all* $u \in L^\infty_0$.
2) $\widetilde{T}_e(\varphi)(t) = \int_{-\infty}^{\infty} \varphi(t+\tau)\,ds(\tau)$ for all $\varphi \in L^1$.
3) T_e *is the adjoint of* $\widetilde{T}_e$.

Proof. 1) As in the proof of Theorem 4.2,

$$T(u)(t) = \int_{-\infty}^{\infty} u(t-\tau)\,ds(\tau)$$

for any $u \in L^\infty_+$. Since T_e is continuous relative to $\|\cdot\|_\infty$, 1) follows immediately.

2) For any $\varphi \in K$,

$$\begin{aligned}\widetilde{T}(\varphi)(t) &= \langle\dot{s},\sigma_{-t}\varphi\rangle \\ &= -\langle s,\sigma_{-t}\dot{\varphi}\rangle \\ &= -\int_{-\infty}^{\infty} s(\tau)\dot{\varphi}(t+\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} \varphi(t+\tau)\,ds(\tau).\end{aligned}$$

Thus $\widetilde{T}_e(\varphi) = \widetilde{T}(\varphi)$ for all $\varphi \in K$. Since $\widetilde{T}_e$ is continuous relative to $\|\cdot\|_1$, 2) follows.

3) Let $s \in BV$, $u \in L^\infty$, and $\varphi \in L^1$. Applying Fubini's theorem and a change of variable, we have

$$\int_{-\infty}^{\infty} u(t)\left(\int_{-\infty}^{\infty} \varphi(t+\tau)\,ds(\tau)\right)dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(t-\tau)\,ds(\tau)\right)\varphi(t)\,dt$$

or

$$\int_{-\infty}^{\infty} u(t)\widetilde{T}_e(\varphi)(t)\,dt = \int_{-\infty}^{\infty} T_e(u)(t)\varphi(t)\,dt. \qquad \square$$

Note that the proof of Theorem 4.3 applies even if $s \in BV - BV_+$. Hence, the idea of a stable system whose step or impulse response does not have left-bounded support is meaningful; the system may be viewed as an operator on L^∞. However, such an operator does not extend easily to K' or even K'_+.

To conclude this section, we give several equivalent expressions that quantify the "gain" of a stable operator.

THEOREM 4.4. *For any BIBO stable convolution operator $T : K'_+ \to K'_+$,*

$$\sup_{\substack{u\in L^\infty_+\\ \|u\|_\infty=1}} \|T(u)\|_\infty = \sup_{\substack{u\in L^\infty\\ \|u\|_\infty=1}} \|T_e(u)\|_\infty = \sup_{\substack{\varphi\in L^1\\ \|\varphi\|_1=1}} \|\widetilde{T}_e(\varphi)\|_1 = \mathrm{Var}(T(\theta)) = \|T(\delta)\|_1^e.$$

Proof. The first equality follows from continuity of T_e. Since the norms of adjoint operators must coincide, the second identity holds. The next equality follows from the representation of $\widetilde{T}_e$ established in Theorem 4.3. The last identity follows immediately from Proposition 3.5. $\square$

5. Additional properties of stable systems. In view of the Theorem 4.2, part 3) the step response of every BIBO system can be decomposed as $T(\theta) = s_1 + s_2 + s_3$, where s_1 is absolutely continuous, s_2 is saltus, and s_3 is singular. (See [12].) Thus, the corresponding impulse response is $T(\delta) = h_1 + h_2 + h_3$, where $h_i = \dot{s}_i$. Since s_1 is bounded and absolutely continuous, $h_1 \in L^1_+$. Also, since s_2 is a saltus function,

$$h_2 = \sum \alpha_i(\delta_{\tau_i}), \tag{7}$$

where $\sum |\alpha_i| < \infty$. Hence the impulse response of any stable system can be uniquely decomposed into the sum of an L^1 function, an impulsive distribution, and the (distributional) derivative of a singular function.

The distribution $h_3 = \dot{s}_3$ is particularly interesting and apparently has not been treated in the literature as a viable impulse response. Distributions of this type illustrate the fact that, for a function $s : \mathbb{R} \to \mathbb{R}$, the operations of differentiation and identification with a distribution do not in general commute, even if s is continuous. Indeed, the derivative s_3' of s_3 *as a function* vanishes a.e., so s_3' is identified with the distribution 0. On the other hand, s_3 is by definition not constant a.e., hence its *distributional derivative* $\dot{s}_3$ does not vanish.

A classical example of a singular function on [0, 1] is the Cantor function, which we denote by c_0. (See [12, p. 50].) Define

$$c(t) = \begin{cases} 0, & t < 0, \\ c_0(t), & 0 \le t \le 1, \\ 1, & t > 1. \end{cases}$$

Then c is nondecreasing and singular on $\mathbb{R}$. Since $\mathrm{Var}(c) = 1$, Proposition 3.5 gives $\|\dot{c}\|_1^e = 1$. The support of the distribution $\dot{c}$ is simply the Cantor ternary set, which is uncountable and has Lebesgue measure zero. (See [12, p. 49].) Note that $\dot{c}$ has a far more elusive structure than a conventional impulsive distribution (7). Nevertheless, Theorem 4.2 guarantees that the system governed by $T(u) = \dot{c} * u$ is BIBO stable.

On the other hand, suppose $T(\delta) = \ddot{c}$. Any attempt to decide the value $\int |T(\delta)|$ by intuitive means would be perilous at best. Using our theory, this case is easily handled by simply noting that $T(\theta) = \dot{c} \notin BV_+$.

Another characterization of stable linear time-invariant systems can be obtained by examining the set $\mathcal{H}$ of Fourier transforms of functions in L^1_e. It follows from [5, p. 189] that the Fourier transform of any $h \in L^1_e$ exists, is a function, and is given by

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\, ds(t), \tag{8}$$

where $s \in BV$ and $\dot{s} = h$. We refer to $\mathcal{H}$ as the set of *BIBO stable transfer functions*. Clearly, a rational function belongs to $\mathcal{H}$ iff it is BIBO stable in the usual sense.

According to Theorem 4.2, the stable transfer functions are generated by letting s vary over BV in (8). Since every function in BV can be written as the difference of two bounded nondecreasing functions, a substantial number of existing results in analysis come into play. For example, working from [13, Ch. VI], we find that all functions in $\mathcal{H}$ are bounded and uniformly continuous. Several complete, albeit abstruse, characterizations of $\mathcal{H}$ are available, perhaps the simplest following from Bochner's theorem: A function H belongs to $\mathcal{H}$ iff H is the difference of two continuous positive semidefinite functions. (See [13, p. 137].)

The Laplace transform

$$H(z) = \int_{-\infty}^{\infty} e^{-zt}\, ds(t)$$

of $h = \dot{s} \in L^1_{+e}$ also exists and is analytic on $\operatorname{Re} z > 0$. It is easy to show that the "boundary function"

$$\omega \to \lim_{\sigma \to 0} H(\sigma + i\omega)$$

is well defined and equals the Fourier transform (8). In fact, if $\operatorname{supp} h \subset [\tau, \infty)$,

$$|H(z)| \leq e^{-\operatorname{Re} z\tau} \operatorname{Var}(s)$$

for all right half-plane z. In particular, if $h \in L^1_{+e}$ with $\tau \geq 0$, then H belongs to the Hardy space $H^\infty(\mathbb{C}^+)$. The converse implication fails, however, since the function

$$H(z) = e^{-\frac{1}{z}}$$

belongs to H^∞, but $H(i\omega)$ is not continuous at $\omega = 0$ and therefore H is not stable. Sufficient conditions on $H(z)$ for stability (other than those on the boundary function) are difficult to obtain. Even analyticity on the whole plane is not sufficient (e.g., let $h = \dot{\delta}$).

Our final comment of this section addresses the issue of linear systems with multiple inputs and outputs. In this case, $T(\delta)$ and its Fourier transform are matrices. Extending the definition of BIBO stability in the obvious way, it is clear that stability simply corresponds to each entry of the matrix being stable in the sense described above.

6. Conclusions. We presented a coherent distributional theory for linear time-invrariant systems based on the concept of impulse response. The property of BIBO stability was shown to be equivalent to a simple condition on either the impulse or step response of the system. We also supplied a somewhat more difficult stability condition related to the system transfer function. The time-varying case is at present under investigation.

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