High-Gain State Feedback Analysis Based on Singular System Theory

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Abstract

We consider linear, time-invariant state-space systems under high-gain state feedback. The analysis is couched in terms of singular system theory and Grassman manifolds. Our work is distinguished from that of other authors by the fact that we do not allow a gain-dependent state coordinate change. Simple necessary and sufficient conditions are proven under which a singular system is a high-gain limit of a given state-space system. It is shown that the feedback matrix achieves a limit on an appropriate Grassmanian, so infinite gains constitute well-defined mathematical objects. The special cases of minimum-order stable and zeroth-order limits are studied in depth, including an analysis of solution behavior. Finally, the classical "cheap control" problem is interpreted within the context of our results.

1 Introduction

Consider the linear, time-invariant state-space system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For any $K \in \mathbb{R}^{m\times n}$, we may apply state feedback

$$u = -Kx + v, \tag{2}$$

yielding the closed loop system

$$\dot{x} = (A - BK)x + Bv. \tag{3}$$

In this paper, we are interested in the “high-gain limits” of (3) as $\|K\| \to \infty$. We seek a characterization of all such limits for a given system (1). In addition, we will specialize our results to certain important classes of limits, and develop conditions under which a limit of (2) constitutes a well-defined system in its own right. We will then apply our results to the classical “cheap control” problem.

Numerous references deal with the issue of high gain limits under state feedback. For example, early papers such as [1] treat high gain in a classical singular perturbation context. Much of this work can be viewed largely as a special case of our results. The details will be provided in Sections 4-6.

More recent efforts, such as [2], [3], and [4], study high gain limits in great depth. However, this body of work is fundamentally different from ours in that a K-dependent coordinate change is allowed, while our approach admits no coordinate change. The consequences of the two approaches are strikingly different. Indeed, consider the 1st-order system

$$\dot{x} = u$$

with feedback

$$u = -kx + v.$$

Our analysis (and that of [1]) dictates that the closed-loop system be written

$$-\frac{1}{k}\dot{x} = x - \frac{1}{k}v,$$

yielding $x = 0$ in the limit. Note that controllability is progressively weakened as k increases, and lost entirely for $k = \infty$. This is precisely the effect one would observe in practice, with the variable x representing the fixed (i.e. K-*independent*) state of the plant.

On the other hand, the analyses in [2], [3], and [4] allow a K-*dependent* coordinate change. In this case, the kth closed-loop system becomes

$$p_k q_k \dot{z} = -p_k k q_k z + p_k v,$$

where $x = q_k z$, and p_k, q_k are arbitrary nonzero sequences. For any $g \neq 0$, setting

$$p_k = 1, \quad q_k = \frac{1}{kg}$$

yields the controllable limit $z = gv$. The problem here is that the loss of controllability is masked by the coordinate change $z = kgx$, which scales the physical state x progressively higher as $k \to \infty$.

Another phenomenon that can occur with a k-dependent coordinate change is illustrated by the example

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ u &= -\begin{bmatrix} k^2 & 1 \end{bmatrix} x. \end{aligned} \tag{4}$$

Let $x = Q_k z$ and premultiply (4) by P_k, where P_k, Q_k are nonsingular. Then

$$P_k Q_k \dot{z} = P_k \begin{bmatrix} 0 & 1 \\ -k^2 & -1 \end{bmatrix} Q_k z, \tag{5}$$

which is equivalent to a system of the form

$$X_k \dot{z} = z \tag{6}$$

If $Q_k = I$,

$$X_k = \begin{bmatrix} -\frac{1}{k^2} & -\frac{1}{k^2} \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{7}$$

irrespective of P_k. On the other hand, setting $P_k = I$ and

$$Q_k = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$X_k = \begin{bmatrix} -\frac{1}{k^2} & -\frac{1}{k} \\ \frac{1}{k} & 0 \end{bmatrix} \to 0. \tag{8}$$

Substituting (7) and (8) into (6) produces vastly different results. In particular, (7) produces impulses, while (8) does not. (See [9], Ch. 22.) Losing track of the impulsive behavior in (6),(8) is again due to the progressive redefinition of the state.

Our approach disallows coordinate changes of the state x. A moment's reflection indicates that, in our setting, the high-gain limits of (3) form a subset of those in [2], [3], and [4]. Nevertheless, characterization of these "fixed coordinate" limits requires an independent analysis. Although the limits we obtain must satisfy the necessary conditions proven in ([2]) and ([3]), we will establish alternative conditions, which are arguably simpler and both necessary and sufficient. We will also conduct a careful analysis of stable and "zeroth order" limits, which have heretofore not been explicitly studied in the literature, at least at this level of generality.

One of our objectives is to establish results which are dual to those we developed for observers in [6]. To this end, much of our work relies on the theory of differentiable manifolds. (See e.g. [10].)

Throughout the paper, we assume for convenience that $\operatorname{rank} B = m$. For a system where this is not the case, an input coordinate change $\widehat{u} = Tu$ can be used to reduce the problem to our framework.

2 Preliminaries

Before we can talk about the limits of (1), we need some elementary results from singular system theory. Consider the matrix differential equation

$$E\dot{x} = Fx + Gu, \tag{9}$$

where $E, F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times m}$. We assume the matrix pencil (E, F) is *regular* – i.e.

$$\Delta(s) = |sE - F| \not\equiv 0.$$

The roots of Δ are the *eigenvalues* of the system. Consider the *Stiefel manifold* $\mathcal{V}_n\left(\mathbb{R}^{n\times(2n+m)}\right)$ of all $\left[\begin{array}{ccc} E & F & G \end{array}\right] \in \mathbb{R}^{n\times(2n+m)}$ with full rank. Also, let

$$\Sigma(n,m) = \left\{ \left[\begin{array}{ccc} E & F & G \end{array}\right] \Big| \Delta \not\equiv 0 \right\}.$$

Since $\Delta \not\equiv 0$ implies $\left[\begin{array}{cc} E & F \end{array}\right]$ has full rank, $\Sigma(n,m) \subset \mathcal{V}_n\left(\mathbb{R}^{n\times(2n+m)}\right)$. Both $\Sigma(n,m)$ and $\mathcal{V}_n\left(\mathbb{R}^{n\times(2n+m)}\right)$ are complementary to algebraic varieties in $\mathbb{R}^{n\times(2n+m)}$ and are, therefore, open and dense in $\mathbb{R}^{n\times(2n+m)}$.

Since premultiplication of (9) by a nonsingular matrix M does not affect the dynamics of (9), it is natural to identify systems (9) related by such a transformation. On the other hand, right multiplication of E and A amounts to a coordinate change, so we avoid such transformations, retaining the coordinate-dependent nature of conventional state-space theory. We claim that this approach leads to a simpler theory overall.

With these ideas in mind, we couch our problem in terms the *Grassman manifold* $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$. A Grassmanian is obtained by applying the equivalence relation

$$\left[\begin{array}{ccc} E_1 & F_1 & G_1 \end{array}\right] \approx \left[\begin{array}{ccc} E_2 & F_2 & G_2 \end{array}\right] \iff \exists \text{ nonsingular } M \ni \left[\begin{array}{ccc} E_1 & F_1 & G_1 \end{array}\right] = M\left[\begin{array}{ccc} E_2 & F_2 & G_2 \end{array}\right] \tag{10}$$

to $\mathcal{V}_n\left(\mathbb{R}^{n\times(2n+m)}\right)$ and forming the quotient manifold $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ with dimension $n(n+m)$. Charts on $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ may be constructed by setting n columns of $\left[\begin{array}{ccc} E & F & G \end{array}\right]$ to the $n\times n$ identity matrix and varying the remaining entries. Doing this in all $\left(\begin{array}{c} 2n+m \\ n \end{array}\right)$ ways generates an atlas for $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$. We denote points in $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ by $[E, F, G]$. Setting

$$\mathcal{L}(n,m) = \left\{ [E,F,G] \in \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right) \Big| \Delta \not\equiv 0 \right\}$$

is consistent with the quotient structure of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, since premultiplication of $\left[\begin{array}{ccc} E & F & G \end{array}\right]$ by a nonsingular M scales Δ by a nonzero constant. Let $\mu : \mathcal{V}_n\left(\mathbb{R}^{2n+m}\right) \to \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ be the submersion defined by $\left[\begin{array}{ccc} E & F & G \end{array}\right] \to [E, F, G]$. Then μ is continuous and open ([10], Proposition 6.1.5). Hence, $\mathcal{L}(n,m) = \mu(\Sigma(n,m))$ is an open, dense submanifold of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$. This makes $\mathcal{L}(n,m)$ an analytic manifold of dimension $n(n+m)$. We studied $\mathcal{L}(n,m)$ in [5].

We will make frequent use of the Weierstrass Decomposition ([8], pp. 24-28): For any regular pencil (E, F), there exists nonsingular M, N such that

$$MEN = \left[\begin{array}{cc} I & 0 \\ 0 & E_f \end{array}\right] \quad MFN = \left[\begin{array}{cc} F_s & 0 \\ 0 & I \end{array}\right], \tag{11}$$

where E_f is nilpotent. E_f and F_s are unique up to similarity. Define the *order* of (E, F) to be $\operatorname{ord}(E,F) = \deg\Delta$ (i.e. the dimension of F_s) and the *index* $\operatorname{ind}(E,F)$ to be the smallest integer $q \geq 1$ such that $E_f^q = 0$. The functions ord and ind are uniquely defined on $\Sigma(n,m)$. In fact, both are invariant under the equivalence (10), so we may apply them to points in $\mathcal{L}(n,m)$:

$$\operatorname{ord}[E,F,G] = \operatorname{ord}(E,F),$$
$$\operatorname{ind}[E,F,G] = \operatorname{ind}(E,F).$$

Eigenvalues are also invariant over orbits in $\mathcal{V}_n\left(\mathbb{R}^{n\times(2n+m)}\right)$, so we may refer to a point $[E, F, G]$ as being *stable*, if all its eigenvalues λ satisfy $\mathrm{Re}\lambda < 0$ and $\operatorname{ind}[E,F,G] = 1$.

We will need to consider solutions of (9). To this end, we review some basic facts from the theory of distributions. (See e.g. [11]). Let $\mathcal{D}$ be the space of C^∞ functions $\phi : \mathbb{R} \to \mathbb{R}$ with bounded support, and let $\mathcal{D}'$ denote the dual space of $\mathcal{D}$. A *distribution* f is any member of $\mathcal{D}'$. Each locally L^1 function f (i.e. L^1 on bounded intervals) may be considered a distribution, since it determines a functional $\phi \to \int f\phi$. The unit impulse δ is defined to be the evaluation functional $< \delta, \phi >= \phi(0)$. Every distribution has a derivative defined by $< \dot{f}, \phi >= - < f, \dot{\phi} >$; thus $< \delta^{(i)}, \phi >= (-1)^i \phi^{(i)}(0)$. A sequence of distributions f_k is said to converge *weak** to f if $\langle f_k, \phi \rangle \to \langle f, \phi \rangle$ for every $\phi \in \mathcal{D}$. One advantage of working with distributions is that differentiation is a weak*-continuous operation. Besides weak* convergence, we will sometimes refer to uniform convergence $f_k \to f$ on an interval in $\mathcal{I} \subset \mathbb{R}$. This simply means that there exist locally L^1 functions g_k, g defined on $\mathcal{I}$ such that $< f_k, \phi >=< g_k, \phi >, < f, \phi >=< g, \phi >$ for all ϕ with support in $\mathcal{I}$ and $g_k \to g$ uniformly. Let $U \subset \mathbb{R}$ be the largest open set such that $\operatorname{supp} \phi \subset U$ implies $< f, \phi >= 0$. The *support* of f is $\operatorname{supp} f = U^c$. Let $\mathcal{D}'_+$ be the distributions with support in $[0, \infty)$.

In order to apply arbitrary initial conditions x_0 to (9), it is convenient to consider the augmented system

$$E\dot{x} = Fx + Gu + \delta E x_0, \tag{12}$$

which yields a unique solution $x \in \mathcal{D}'_+$. (See [9], Ch.22 for details.). Let

$$\begin{bmatrix} G_s \\ G_f \end{bmatrix} = MG, \quad \begin{bmatrix} x_s \\ x_f \end{bmatrix} = N^{-1}x, \quad \begin{bmatrix} x_{0s} \\ x_{0f} \end{bmatrix} = N^{-1}x_0 \tag{13}$$

and $\exp(F_s) : \mathbb{R} \to \mathbb{R}^{\deg \Delta \times \deg \Delta}$ be given by

$$\exp(F_s)t = \begin{cases} e^{tF_s}, & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

Define the *state-transition matrix*

$$\Phi = N \begin{bmatrix} \exp(F_s) & 0 \\ 0 & -\sum_{i=0}^{q-1} \delta^{(i)} E_f^i \end{bmatrix} M. \tag{14}$$

Direct calculation shows that Φ is the inverse Laplace transform of $(sE - A)^{-1}$, so Φ may be viewed as a map on $\Sigma(n, 0)$, obviously $1-1$. Since Φ is $1-1$, it varies over each orbit in $\Sigma(n, 0)$, so Φ cannot be defined consistently on $\mathcal{L}(n, 0)$. Φ may be extended trivially to $\Sigma(n, m)$, with similar consequences. The state transition matrix relates to the system (12) as follows:

Theorem 1 *1)* $E\dot{\Phi} = A\Phi + \delta I$
2) The solution of (12) is $x = \Phi E x_0 + \Phi * Gu$.
3) The system (12) is asymptotically stable iff ΦE *is bounded and decays asymptotically to* 0.

Proof. *1) and 2) follow by direct calculation.*

3) By asymptotic stability, we mean that, for $u \equiv 0$, *we have the conditions a)* $x(t) \to 0$ *as* $t \to \infty$,*for every* x_0, *and b)* $\sup_t |x(t)| \to 0$ *as* $x_0 \to 0$. *Boundedness and decay of* ΦE *are equivalent to the eigenvalues* λ *of* F_s *satisfying* $\operatorname{Re} \lambda < 0$ *and* $E_f = 0$.

(Sufficient) From (11) and (14),

$$\Phi E = N \begin{bmatrix} \exp(F_s) & 0 \\ 0 & 0 \end{bmatrix} N^{-1},$$

so conditions a) and b) are met relative to $\Phi E x_0$.

(Necessary) We have $\Phi(t) E x_0 \to 0$ *for every* x_0, *so*

$$\Phi E = N \begin{bmatrix} \exp(F_s) & 0 \\ 0 & -\sum_{i=0}^{q-1} \delta^{(i)} E_f^{i+1} \end{bmatrix} N^{-1} \to 0,$$

which implies F_s *is stable. Furthermore,* $\Phi E x_0$ *is bounded for every* x_0, *so* ΦE *is bounded, which implies it contains no impulses – i.e.* $E_f = 0$. ■

3 The Manifold of Closed-Loop Systems

The present paper closely follows the development of [6], where the dual problem of the limiting behavior of state observers under high gain feedback was studied. One might speculate that the state feedback case should be obtained from [6] merely be taking the "transpose" of all theorems. While some theorems do transfer over in this way, much of the state feedback theory is different. One way to see that this must be true is to observe that, in both cases, systems are identified when they are related by left multiplication by a nonsingular M. In contrast, pure transposition of the observer problem would require *right* multiplication by M, leading to a K-dependent coordinate change, which we explicitly avoid.

The closed-loop systems (3) for a given plant (1) imbed naturally into $\mathcal{L}(n,m)$ via the map $K \rightarrow [I, A-BK, B]$. We denote the image of $\mathbb{R}^{m\times n}$ under this map by $\mathcal{C}_r$. We further denote the closure of $\mathcal{C}_r$ in $\mathcal{L}(n,m)$ by $\mathcal{C}$ and consider the set $\mathcal{C}_s = \mathcal{C}$ - $\mathcal{C}_r$. $\mathcal{C}$ may be regarded as the set all limits of (3), $\mathcal{C}_r$ the full-order limits (i.e. ordinary state space systems) and $\mathcal{C}_s$ the singular limits (i.e. generalized state space systems). Another way to define $\mathcal{C}$, $\mathcal{C}_r$, and $\mathcal{C}_s$ is via the submersion μ. Let

$$\Omega_r = \left\{ \begin{bmatrix} M & M(A-BK) & MB \end{bmatrix} \Big| M \text{ nonsingular} \right\}.$$

Obviously, $\Omega_r \subset \Sigma(n,m)$. Let Ω be the closure of Ω_r in $\Sigma(n,m)$, and $\Omega_s = \Omega - \Omega_r$. It is easy to see that $\mathcal{C} = \mu(\Omega)$, $\mathcal{C}_r = \mu(\Omega_r)$, and $\mathcal{C}_s = \mu(\Omega_s)$.

We need the following lemma to prove Theorem 3, which establishes the basic structure of $\mathcal{C}$.

Lemma 2 *Let $X, Y \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank}\begin{bmatrix} X & Y \end{bmatrix} = n$. *There exist $K_k \in \mathbb{R}^{m\times n}$ and nonsingular $X_k \in \mathbb{R}^{n\times n}$ such that $X_k \rightarrow X$ and $X_k B K_k \rightarrow Y$ iff* $\operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m$.

Proof. *(Necessary) For large k,*

$$\operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} \leq \operatorname{rank}\begin{bmatrix} X_k B & X_k B K_k \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B & BK_k \end{bmatrix} = m.$$

Suppose

$$\operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} < m,$$

and let $R \subset \mathbb{R}^n$ be a subspace such that

$$\operatorname{Im} B \oplus R = \mathbb{R}^n.$$

Then $\dim R = n - m$, *and*

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} X & Y \end{bmatrix} &= \dim(\operatorname{Im} X + \operatorname{Im} Y) \\ &= \dim(XR + \operatorname{Im} XB + \operatorname{Im} Y) \\ &\leq \dim XR + \dim(\operatorname{Im} XB + \operatorname{Im} Y) \\ &\leq \dim R + \operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} \\ &< n. \end{aligned}$$

From this contradiction, we conclude

$$\operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m.$$

(Sufficient) Let

$$\begin{aligned} R &= \operatorname{Im} XB \cap \operatorname{Im} Y, \\ S &= \operatorname{Ker} X \cap \operatorname{Im} B, \end{aligned}$$

$p = \dim R$, *and* $q = \dim S$. *Then*

$$m = q + \operatorname{rank} XB,$$

and there exists a nonsingular $T \in \mathbb{R}^{n\times n}$ such that

$$YT = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

with $\operatorname{Im} Y_1 = R$ *and*

$$\operatorname{Im} XB \cap \operatorname{Im} Y_2 = 0.$$

Hence, we may select $H \in \mathbb{R}^{m\times p}$ such that $XBH = Y_1$. Also,

$$\operatorname{rank} XB + \operatorname{rank} Y_2 = \operatorname{rank}\begin{bmatrix} XB & Y_2 \end{bmatrix} \leq \operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m,$$

so

$$\operatorname{rank} Y_2 \leq q.$$

We may choose $J \in \mathbb{R}^{m\times q}$ such that $\operatorname{Im} BJ = S$. Then $XBJ = 0$, and

$$\operatorname{rank} BJ = q \geq \operatorname{rank} Y_2.$$

Thus there exists $Z \in \mathbb{R}^{n\times n}$ such that $ZBJ = Y_2$.

Let $Z_k = X + \frac{1}{k}Z$ and, for each k, select nonsingular $Z_{kj} \to Z_k$ as $j \to \infty$. We may select a sequence $j_k \uparrow \infty$ such that

$$\|Z_{kj_k} - Z_k\| < \frac{1}{k^2}$$

for every k. Setting $X_k = Z_{kj_k}$, we have

$$\|X_k - X\| \leq \|X_k - Z_k\| + \|Z_k - X\| = \frac{1}{k^2} + \frac{1}{k}\|Z\|,$$

so $X_k \to X$. Let $K_k = \left[\begin{array}{cc} H & kJ \end{array}\right] T^{-1}$. Then

$$\begin{aligned} X_k B K_k &= \left[\begin{array}{cc} X_k BH & k(X_k - Z_k)BJ + kZ_k BJ \end{array}\right] T^{-1} \\ &= \left[\begin{array}{cc} X_k BH & k(Z_{kj_k} - Z_k)BJ + kXBJ + ZBJ \end{array}\right] T^{-1} \\ &\to \left[\begin{array}{cc} Y_1 & Y_2 \end{array}\right] T^{-1} \\ &= Y. \end{aligned}$$

■

Theorem 3 *1) $\mathcal{C} = \left\{[X, XA - Y, XB] \in \mathcal{L}(n,m) \,\middle|\, \operatorname{rank}\left[\begin{array}{cc} XB & Y \end{array}\right] = m\right\}$*
2) $\mathcal{C}$ is a regular submanifold of $\mathcal{L}(n,m)$ with dimension nm.
3) $\mathcal{C}_r$ is a (relatively) open, dense submanifold of $\mathcal{C}$
4) $[X, XA - Y, XB] \in \mathcal{C}_s$ iff $\operatorname{rank}\left[\begin{array}{cc} XB & Y \end{array}\right] = m$ with X singular.

Proof. 1) Let

$$\Omega_e = \left\{\left[\begin{array}{ccc} X & XA - Y & XB \end{array}\right] \in \mathcal{V}_n\left(\mathbb{R}^{2n+m}\right) \,\middle|\, \operatorname{rank}\left[\begin{array}{cc} XB & Y \end{array}\right] = m\right\}.$$

Setting $X = M$ and $Y = MBK$ yields

$$\left[\begin{array}{ccc} X & XA - Y & XB \end{array}\right] = \left[\begin{array}{ccc} M & M(A - BK) & MB \end{array}\right],$$

$$\operatorname{rank}\left[\begin{array}{cc} XB & Y \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} MB & MBK \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} B & BK \end{array}\right] = m,$$

so $\Omega_r \subset \Omega_e \cap \Sigma(n,m)$. It suffices to show that the closure of Ω_r in $\mathcal{V}_n\left(\mathbb{R}^{2n+m}\right)$ is Ω_e, because then the closure of Ω_r in $\Sigma(n,m)$ is $\Omega = \Omega_e \cap \Sigma(n,m)$, and part 1) follows from $\mu(\Omega_r) = \mathcal{C}_r$, $\mu(\Omega) = \mathcal{C}$.

For any nonsingular $T \in \mathbb{R}^{n\times n}$, let

$$L_T = \left[\begin{array}{ccc} T^{-1} & T^{-1}A & T^{-1}B \\ 0 & -I & 0 \end{array}\right].$$

Choose X, Y such that

$$\left[\begin{array}{cc} X & Y \end{array}\right] L_I = \left[\begin{array}{ccc} X & XA - Y & XB \end{array}\right] \in \Omega_e.$$

L_T has independent rows, so $\operatorname{rank}\left[\begin{array}{cc} X & Y \end{array}\right] = n$. From Lemma 2, there exist sequences X_k and K_k, with X_k nonsingular, such that $X_k \to X$ and $X_k BK_k \to Y$. Hence,

$$\left[\begin{array}{ccc} X_k & X_k(A - BK_k) & X_k B \end{array}\right] \to \left[\begin{array}{ccc} X & XA - Y & XB \end{array}\right],$$

and the closure of Ω_r contains Ω_e. Conversely, if

$$\left[\begin{array}{ccc} X_k & X_k(A - BK_k) & X_k B \end{array}\right] \to \left[\begin{array}{ccc} X & F & G \end{array}\right] \in \mathcal{V}_n\left(\mathbb{R}^{2n+m}\right),$$

then $X_k \to X$ and $G = XB$. Let $Y = XA - F$. Then

$$\operatorname{rank}\left[\begin{array}{cc} X & Y \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} X & Y \end{array}\right]\left[\begin{array}{cc} I & A \\ 0 & -I \end{array}\right] = \operatorname{rank}\left[\begin{array}{cc} X & F \end{array}\right] = \operatorname{rank}\left[\begin{array}{ccc} X & F & G \end{array}\right] = n,$$

$$X_k B K_k = X_k A - X_k (A - BK_k) \to Y,$$

so, from Lemma 2,

$$\operatorname{rank} \begin{bmatrix} XB & Y \end{bmatrix} = m.$$

Hence, $\begin{bmatrix} X & F & G \end{bmatrix} \in \Omega_e$, and Ω_e contains the closure of Ω_r.

2) This part of the proof will be based on the following construction. Choose a nonsingular T such that

$$T^{-1}B = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

and consider the diagram

$$\begin{array}{ccccc} & \widehat{h} & & \widehat{g} & \\ \mathcal{V}_m \left(\mathbb{R}^{m+n}\right) & \to & \mathcal{V}_n \left(\mathbb{R}^{2n}\right) & \to & \mathcal{V}_n \left(\mathbb{R}^{2n+m}\right) \\ \downarrow \pi & & \downarrow \nu & & \downarrow \mu \\ \mathcal{G}_m \left(\mathbb{R}^{m+n}\right) & \to & \mathcal{G}_n \left(\mathbb{R}^{2n}\right) & \to & \mathcal{G}_n \left(\mathbb{R}^{2n+m}\right) \\ & h & & g & \end{array}$$

where

$$\widehat{g}\left(\begin{bmatrix} \widetilde{X} & Y \end{bmatrix}\right) = \begin{bmatrix} \widetilde{X} & Y \end{bmatrix} L_T,$$

$$\widehat{h}(Z) = \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix},$$

and μ, ν, and π are the standard submersions. We note that

$$\widehat{g}\left(M \begin{bmatrix} \widetilde{X} & Y \end{bmatrix}\right) = M \widehat{g}\left(\begin{bmatrix} \widetilde{X} & Y \end{bmatrix}\right),$$

and

$$\widehat{h}(NZ) = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \widehat{h}(Z)$$

for any nonsingular M, N, so g and h may be defined to make the diagram commute. We are mainly interested in the compositions $f = g \circ h$ and $\widehat{f} = \widehat{g} \circ \widehat{h}$. Note that $\widehat{g}$, $\widehat{h}$, and hence $\widehat{f}$ are $1-1$ Furthermore, if

$$\widehat{g}\left(\begin{bmatrix} \widetilde{X}_a & Y_a \end{bmatrix}\right) = M \widehat{g}\left(\begin{bmatrix} \widetilde{X} & Y \end{bmatrix}\right),$$

we obtain

$$\widehat{g}\left(\begin{bmatrix} \widetilde{X}_a & Y_a \end{bmatrix}\right) = \widehat{g}\left(M \begin{bmatrix} \widetilde{X} & Y \end{bmatrix}\right),$$

so

$$\begin{bmatrix} \widetilde{X}_a & Y_a \end{bmatrix} = M \begin{bmatrix} \widetilde{X} & Y \end{bmatrix}$$

and g is $1-1$. Now suppose

$$\widehat{h}(Z_a) = M \widehat{h}(Z).$$

Then

$$\begin{bmatrix} I & 0 \\ 0 & Z_a \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix}.$$

Inspection of the block matrix equations yields $Z_a = M_{22} Z$ with M_{22} nonsingular, so h and f are $1-1$.

Let $\mathcal{C}_e = \mu(\Omega_e)$. Since $\mathcal{L}(n,m)$ is open in $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, it suffices to demonstrate that $\mathcal{C}_e$ satisfies 2), because then $\mathcal{C} = \mathcal{C}_e \cap \mathcal{L}(n,m)$ inherits the same properties. We begin by showing that $f\left(\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)\right) = \mathcal{C}_e$. Consider any point $[X, XA - Y, XB] \in \mathcal{C}_e$. Setting $\widetilde{X} = XT$ and partitioning

$$\begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 \end{bmatrix} = \widetilde{X},$$

with $\widetilde{X}_1 \in \mathbb{R}^{n \times (n-m)}$, $\widetilde{X}_2 \in \mathbb{R}^{n \times m}$, we obtain

$$\operatorname{rank} \begin{bmatrix} \widetilde{X}_2 & Y \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \widetilde{X} T^{-1} B & Y \end{bmatrix} = \operatorname{rank} \begin{bmatrix} XB & Y \end{bmatrix} = m,$$

$$\operatorname{rank}\begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 & Y \end{bmatrix} = \operatorname{rank}\begin{bmatrix} XT & Y \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X & Y \end{bmatrix} = n,$$

so $\operatorname{rank}\widetilde{X}_1 = n - m$. Hence, there exists $Z_1 \in \mathbb{R}^{m\times m}$, $Z_2 \in \mathbb{R}^{m\times n}$, and a nonsingular M such that

$$M\begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 & Y \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & Z_1 & Z_2 \end{bmatrix},$$

and

$$\operatorname{rank}\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} = m.$$

It follows that

$$f\left([Z_1, Z_2]\right) = g\left(\left[\widetilde{X}, Y\right]\right) = [X, XA - Y, XB],$$

which yields the desired result. In fact, letting functions ϕ range over an atlas of $\mathcal{G}_m\left(\mathbb{R}^{n+m}\right)$, $\left\{\phi \circ f^{-1}\right\}$ becomes an atlas for $\mathcal{C}_e$, making f an analytic diffeomorphism between $\mathcal{G}_m\left(\mathbb{R}^{n+m}\right)$ and $\mathcal{C}_e$.

As a map into $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, we can prove that f is analytic by showing that g and h are analytic. Let $\xi \in \mathcal{G}_n\left(\mathbb{R}^{2n}\right)$, and choose charts ϕ on $\mathcal{G}_n\left(\mathbb{R}^{2n}\right)$ and $\psi \in \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ such that ξ and $g(\xi)$ lie in the domains of ϕ and ψ, respectively. Then $\psi \circ g \circ \phi^{-1}$ is a rational function, where the denominator has no zero, and is thus analytic. Since ϕ, ψ are arbitrary, g is analytic. Analyticity of h is proved similarly.

To show that $\mathcal{C}_e$ is a submanifold of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, we must also prove that f has full rank. We need to show that the derived linear function f_* at each point of $\mathcal{G}_m\left(\mathbb{R}^{n+m}\right)$ is $1-1$. From [10], Proposition 4.3.1, $f_* = g_* \circ h_*$, so it suffices to prove that g_* and h_* are $1-1$. Since $g \circ \nu = \mu \circ \widehat{g}$, the same theorem guarantees

$$g_* \circ \nu_* = \mu_* \circ \widehat{g}_*.$$

Since $\widehat{g}$ is $1-1$ and μ_*, ν_* are onto,

$$\operatorname{rank} g_* = \operatorname{rank}\left(\mu_* \circ \widehat{g}_*\right) \geq \operatorname{rank}\widehat{g}_* - \left(\dim \mathcal{V}_n\left(\mathbb{R}^{2n+m}\right) - \operatorname{rank}\mu_*\right) = 2n^2 - \left(n\left(2n+m\right) - n\left(n+m\right)\right) = n^2,$$

so g_* is $1-1$. Unfortunately, this calculation does not work for h_*. To prove h_* is $1-1$, consider any point $\xi \in \mathcal{G}_m\left(\mathbb{R}^{n+m}\right)$ and a chart ϕ whose coordinate domain contains ξ. Applying ϕ amounts to choosing m columns $\{c_i\}$ of $\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}$, setting them equal to the $m\times m$ identity matrix, and allowing the remaining entries to vary, forming an $m\times n$ matrix $\widetilde{Z}$. In a neighborhood of $h(\xi)$, each point of $\mathcal{G}_n\left(\mathbb{R}^{2n}\right)$ may be represented as $\begin{bmatrix} I & S_1 \\ 0 & S_2 \end{bmatrix}$, where the columns $\{c_i\}$ of $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ are $\begin{bmatrix} 0 \\ I \end{bmatrix}$. This generates a chart ψ of $\mathcal{G}_n\left(\mathbb{R}^{2n}\right)$, whose coordinate domain contains $h(\xi)$. It is easy to see that

$$\psi\left(h\left(\phi^{-1}\left(\widetilde{Z}\right)\right)\right) = \begin{bmatrix} 0 \\ \widetilde{Z} \end{bmatrix}.$$

From [10], p.58, h_* has matrix representation $\frac{\partial\left(\psi\circ h\circ\phi^{-1}\right)}{\partial\widetilde{Z}}$. But $\psi \circ h \circ \phi^{-1}$ is linear, so

$$h_*\left(\widetilde{Z}\right) = \begin{bmatrix} 0 \\ \widetilde{Z} \end{bmatrix},$$

which is $1-1$. Hence, we conclude that $\mathcal{C}_e$ is an nm-dimensional submanifold of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$.

Finally we prove regularity of $\mathcal{C}_e$. We need to show that the topologies that $\mathcal{C}_e$ inherits from $\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$ (through f) and from $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ (as a subset) coincide. Since f is analytic, it is continuous, and $f^{-1}\left(W \cap f\left(\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)\right)\right) = f^{-1}(W)$ is open in $\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$ for every open $W \subset \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$. To prove the converse, let $U \subset \mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$ be open. Then $\pi^{-1}(U)$ is open. For any $Z \in \pi^{-1}(U)$ there exists $\varepsilon > 0$ such that the ball $B(Z,\varepsilon) \subset \pi^{-1}(U)$. Then $NB(Z,\varepsilon) \subset \pi^{-1}(U)$ for every nonsingular N. Let

$$\widetilde{L} = \begin{bmatrix} T & A \\ 0 & -I \\ 0 & 0 \end{bmatrix},$$

and define

$$W_Z = \left\{ M^{-1} B\left(\widehat{f}(Z), \frac{\varepsilon}{2\left\|\widetilde{L}\right\|}\right) \,\middle|\, M \text{ nonsingular} \right\} = \mu^{-1}\left(\mu\left(B\left(\widehat{f}(Z), \frac{\varepsilon}{2\left\|\widetilde{L}\right\|}\right)\right)\right),$$

$$W=\bigcup_{Z\in\pi^{-1}(U)} W_Z.$$

Since μ is open, each W_Z and, therefore, W are open. It suffices to show that

$$\widehat{f}\left(\pi^{-1}\left(U\right)\right)=W\cap\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right). \tag{15}$$

Indeed, since W is a union of orbits in $\mathcal{V}_n\left(\mathbb{R}^{2n+m}\right)$, $\mu\left(W\cap A\right)=\mu\left(W\right)\cap\mu\left(A\right)$ for any A,.from which it follows that

$$\begin{aligned}
f\left(U\right) &= \mu\left(\widehat{f}\left(\pi^{-1}\left(U\right)\right)\right)\\
&= \mu\left(W\cap\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right)\right)\\
&= \mu\left(W\right)\cap\mu\left(\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right)\right)\\
&= \mu\left(W\right)\cap\mathcal{C}_e,
\end{aligned}$$

so $f\left(U\right)$ is (relatively) open in $\mathcal{C}_e$.

To prove (15), first note that $Z\in\pi^{-1}\left(U\right)$ implies $\widehat{f}\left(Z\right)\in W_Z$, so

$$\widehat{f}\left(\pi^{-1}\left(U\right)\right)\subset W\cap\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right).$$

Conversely, suppose $Z_a\in\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)$, $\Delta\in B\left(0,\frac{\varepsilon}{2\|\widetilde{L}\|}\right)$, and nonsingular M satisfy

$$M^{-1}\left(\widehat{f}\left(Z\right)+\Delta\right)=\widehat{f}\left(Z_a\right).$$

Then

$$\left\|M\widehat{f}\left(Z_a\right)-\widehat{f}\left(Z\right)\right\|<\frac{\varepsilon}{2\left\|\widetilde{L}\right\|}.$$

But

$$M\widehat{f}\left(Z_a\right)-\widehat{f}\left(Z\right)=\left(M\begin{bmatrix} I & 0\\ 0 & Z_a\end{bmatrix}-\begin{bmatrix} I & 0\\ 0 & Z\end{bmatrix}\right)L_T$$

and $L_T\widetilde{L}=I$, so

$$\left\|M\begin{bmatrix} I & 0\\ 0 & Z_a\end{bmatrix}-\begin{bmatrix} I & 0\\ 0 & Z\end{bmatrix}\right\|\leq\left\|M\widehat{f}\left(Z_a\right)-\widehat{f}\left(Z\right)\right\|\left\|\widetilde{L}\right\|<\frac{\varepsilon}{2}.$$

Partitioning M, we obtain

$$\left\|M_{22}Z_a-Z\right\|<\frac{\varepsilon}{2}$$

(assuming an appropriate norm). Let N be a nonsingular matrix such that

$$\left\|N^{-1}-M_{22}\right\|<\frac{\varepsilon}{2\left\|Z_a\right\|}.$$

Then

$$\left\|N^{-1}Z_a-Z\right\|\leq\left\|N^{-1}-M_{22}\right\|\left\|Z_a\right\|+\left\|M_{22}Z_a-Z\right\|<\varepsilon,$$

so $Z_a\in NB\left(Z,\varepsilon\right)$. Hence,

$$\widehat{f}\left(\pi^{-1}\left(U\right)\right)\supset M^{-1}B\left(\widehat{f}\left(Z\right),\frac{\varepsilon}{2\left\|\widetilde{L}\right\|}\right)\cap\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right)$$

for every nonsingular M,.and

$$\widehat{f}\left(\pi^{-1}\left(U\right)\right)\supset W\cap\widehat{f}\left(\mathcal{V}_m\left(\mathbb{R}^{m+n}\right)\right).$$

3) Density of $\mathcal{C}_r$ follows from the definition of $\mathcal{C}$. To show $\mathcal{C}_r$ is open in $\mathcal{C}$, it suffices to show that Ω_r is open in Ω. Let $\sigma\in\Omega_r$ and $\sigma_k\in\Omega$ with $\sigma_k\to\sigma$. Then there exist $M,X_k,Y_k\in\mathbb{R}^{n\times n}$ and $K\in\mathbb{R}^{m\times n}$, with M nonsingular and

$$\operatorname{rank}\begin{bmatrix} X_kB & Y_k\end{bmatrix}=m,$$

such that

$$\sigma = \left[\begin{array}{ccc} M & M(A-BK) & MB \end{array} \right],$$
$$\sigma_k = \left[\begin{array}{ccc} X_k & X_k A - Y_k & X_k B \end{array} \right].$$

Since $\sigma_k \to \sigma$, $X_k \to M$, so X_k is nonsingular for large k and

$$\operatorname{rank} \left[\begin{array}{cc} B & X_k^{-1} Y_k \end{array} \right] = m.$$

Then $\operatorname{Im} X_k^{-1} Y_k \subset \operatorname{Im} B$, so there exists $K_k \in \mathbb{R}^{m \times n}$ such that $X_k^{-1} Y_k = B K_k$. Therefore,

$$\sigma_k = \left[\begin{array}{ccc} X_k & X_k (A - B K_k) & X_k B \end{array} \right] \in \Omega_r,$$

and Ω_r is open in Ω.

4) (Sufficient) This follows from the definition of Ω_s and $\mathcal{C}_s = \mu(\Omega_s)$.

(Necessary) Assume X is nonsingular. From part 1),

$$\operatorname{rank} \left[\begin{array}{cc} B & X^{-1} Y \end{array} \right] = \operatorname{rank} \left[\begin{array}{cc} XB & Y \end{array} \right] = m,$$

so, $\operatorname{Im} X^{-1} Y \subset \operatorname{Im} B$, and there exists $K \in \mathbb{R}^{m \times n}$ such that $X^{-1} Y = BK$. It follows that

$$[X, XA - Y, XB] = [X, X(A - BK), XB] \in \mathcal{C}_r,$$

which is a contradiction. $\blacksquare$

Theorem 3, part 4), characterizes all degenerate closed-loop systems $\mathcal{C}_s$. This corresponds to applying a sequence of feedback matrices K_k such that $\|K_k\| \to \infty$, driving some or all eigenvalues to ∞ in magnitude. Since $\mathcal{C}_s$ is obtained with no state coordinate change, $\mathcal{C}_s$ must be a subset of the high-gain limits considered in [2] and [3]. In particular, each point in $\mathcal{C}_s$ must satisfy the necessary conditions established in [2], Theorem 1 and [3], Corollary 4.3. Compared with these results, our characterization of $\mathcal{C}_s$ has a very different form, is necessary *and* sufficient, and is arguably simpler.

4 Stable and Zeroth Order Limits

In this section, we study certain subsets of $\mathcal{C}$ which have special significance. In particular, we examine those systems in $\mathcal{C}$ which are stable (i.e. all eigenvalues satisfy $\operatorname{Re} \lambda < 0$) and those with order 0. We begin with a discussion of an important submanifold of $\mathcal{C}$, which will help simplify the development. Let

$$\mathcal{C}_I = \{[X, I, XB] \in \mathcal{C}\}.$$

$\mathcal{C}_I$ is simply the set of points in $\mathcal{C}$ with no eigenvalue at 0. Each point in $\mathcal{C}_I$ corresponds to a system

$$X\dot{x} = x + XBv + \delta X x_0 \tag{16}$$

with state transition matrix determined by

$$X\dot{\Phi} = \Phi + \delta I.$$

From Theorem 3, part 1), we obtain

$$\mathcal{C}_I = \left\{ [X, I, XB] \in \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right) \;\middle|\; \operatorname{rank} \left[\begin{array}{cc} XB & XA - I \end{array} \right] = m \right\}.$$

The next result gives several alternative characterizations of $\mathcal{C}_I$.

Theorem 4 *For any $X \in \mathbb{R}^{n \times n}$, the following are equivalent:*
1) $\operatorname{rank} \left[\begin{array}{cc} XB & XA - I \end{array} \right] = m$
2) $\operatorname{Ker} \left[\begin{array}{cc} X & I \end{array} \right] \subset \operatorname{Im} \left[\begin{array}{cc} B & A \\ 0 & -I \end{array} \right]$
3) $\operatorname{Im}(AX - I) \subset \operatorname{Im} B$
4) There exists $U \in \mathbb{R}^{m \times n}$ such that $AX + BU = I$.

Proof. ($1 \iff 2$) From elementary linear algebra,

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} XB & XA-I \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} X & I \end{bmatrix}\begin{bmatrix} B & A \\ 0 & -I \end{bmatrix} \\ &\geq \operatorname{rank}\begin{bmatrix} B & A \\ 0 & -I \end{bmatrix} - \left(2n - \operatorname{rank}\begin{bmatrix} X & I \end{bmatrix}\right) \\ &= (n+m) - (2n-n) \\ &= m\end{aligned} \tag{17}$$

with equality iff

$$\operatorname{Ker}\begin{bmatrix} X & I \end{bmatrix} \subset \operatorname{Im}\begin{bmatrix} B & A \\ 0 & -I \end{bmatrix}.$$

($2 \iff 3$) Condition 2) is equivalent to saying that, for each x, there exist y, z such that

$$\begin{bmatrix} B & A \\ 0 & -I \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -x \\ Xx \end{bmatrix}. \tag{18}$$

Writing out the equations, (18) is the same as $By = (AX - I)\,x$, which is a restatement of 3).

($3 \iff 4$) Condition 3) says that there exists U such that $AX - I = -BU$, which is the same as 4). ∎

Theorem 4, part 4) indicates that $\mathcal{C}_I$ is nonempty iff $\begin{bmatrix} A & B \end{bmatrix}$ has full rank – i.e. iff 0 is a controllable mode of (1). In this case, the affine set

$$\mathcal{W} = \left\{ \begin{bmatrix} X \\ U \end{bmatrix} \in \mathbb{R}^{2n\times n} \,\middle|\, AX + BU = I \right\}$$

will prove central to our theory. The next result gives a precise relationship between $\mathcal{C}_I$ and $\mathcal{W}$.

Theorem 5 *1) $[X, I, XB] \in \mathcal{C}_I$ iff there exists $U \in \mathbb{R}^{m\times n}$ such that $\begin{bmatrix} X \\ U \end{bmatrix} \in \mathcal{W}$. In this case, U is unique.*
2) Let $K_k \in \mathbb{R}^{m\times n}$. Then $[I, A - BK_k, B] \to [X, I, XB] \in \mathcal{C}_I$ as $k \to \infty$ iff $A - BK_k$ is nonsingular for large k and $(A - BK_k)^{-1} \to X$. In this case, $-K_k\,(A - BK_k)^{-1} \to U$.
3) $\mathcal{C}_I$ is a (relatively) open, dense submanifold of $\mathcal{C}$, diffeomorphic to $\mathcal{W}$.

Proof. 1) All but uniqueness is a restatement of Theorem 4, part 4). Uniqueness follows from $BU = I - AX$ and $\operatorname{rank} B = m$.

2) If $(A - BK_k)^{-1} \to X$,

$$[I, A - BK_k, B] = \left[(A - BK_k)^{-1}, I, (A - BK_k)^{-1} B\right] \to [X, I, XB]. \tag{19}$$

To prove the converse, we note that μ is a submersion, so there exist nonsingular M_k such that

$$M_k \begin{bmatrix} I & A - BK_k & B \end{bmatrix} \to \begin{bmatrix} X & I & XB \end{bmatrix}.$$

Hence, $M_k \to X$ and $M_k\,(A - BK_k) \to I$, so $A - BK_k$ is nonsingular for large k, and

$$(A - BK_k)^{-1} = \left(M_k\,(A - BK_k)\right)^{-1} M_k \to X.$$

If $[X, I, XB] \in \mathcal{C}_I$, part 1) indicates that there exists a unique U such that $AX + BU = I$. Then

$$BK_k\,(A - BK_k)^{-1} = A\,(A - BK_k)^{-1} - I \to AX - I = -BU,$$

$$K_k\,(A - BK_k)^{-1} \to -U.$$

3) Consider the open, dense subset

$$\Omega_I = \left\{ \begin{bmatrix} X & Y \end{bmatrix} L_I \in \mathcal{V}_n\left(\mathbb{R}^{2n+m}\right) \,\middle|\, \operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m, \quad \det\left(AX - Y\right) \neq 0 \right\}$$

of Ω. Since μ is a submersion, $\mathcal{C}_I = \mu\left(\Omega_I\right)$ is open and dense in $\mathcal{C}$. The map

$$f : \begin{bmatrix} X \\ U \end{bmatrix} \to [X, I, XB]$$

takes $\mathcal{W}$ onto $\mathcal{C}_I$, by 1). Since U is uniquely determined by X, f is $1-1$. Both $\mathcal{W}$ and $\mathcal{C}_I$ are covered by single coordinate domains. One may construct an affine chart ϕ for $\mathcal{W}$ and apply the chart

$$\psi : [X, I, XB] \to X$$

to $\mathcal{C}_I$. Then $\psi \circ f \circ \phi^{-1}$ is an affine diffeomorphism, so f is a diffeomorphism. ∎

Since closed-loop systems in $\mathcal{C}_I$ (or, alternatively, $\mathcal{W}$) have no eigenvalue at 0, $\mathcal{C}_I$ contains all stable limits and all zeroth order limits. The structure of $\mathcal{W}$ is dual to the structure of the manifold $\mathcal{V}$ we studied in [5].

Restricting to $\mathcal{C}_I$ yields a surprising result related to controllability of the closed-loop system (16).

Theorem 6 *Let $[X, I, XB] \in \mathcal{C}_I$. Then* $\operatorname{rank} X \geq n-m$ *with equality iff* $XB = 0$.

Proof. $\begin{bmatrix} A & B \end{bmatrix}$ has full rank, so we may choose nonsingular M, N such that

$$MB = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad MAN = \begin{bmatrix} I & 0 \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}. \tag{20}$$

Let

$$\begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{bmatrix} = N^{-1} X M^{-1}, \quad \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{bmatrix} = U M^{-1}.$$

Then

$$\begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ A_{21}\widetilde{X}_{11} + A_{22}\widetilde{X}_{21} + \widetilde{U}_1 & A_{21}\widetilde{X}_{12} + A_{22}\widetilde{X}_{22} + \widetilde{U}_2 \end{bmatrix} = M\left(AX + BU\right)M^{-1} = I,$$

so X and XB have the form

$$X = N \begin{bmatrix} I & 0 \\ X_{21} & X_{22} \end{bmatrix} M, \quad XB = N \begin{bmatrix} 0 \\ X_{22} \end{bmatrix}.$$

Hence, $\operatorname{rank} X \geq n - m$ with equality iff $X_{22} = 0$. ∎

Theorem 6 states that high gain limits of (3) where the rank of X degenerates maximally have the unfortunate property that the input v exerts no control whatsoever on the system. This is undoubtedly a limitation for control problems where closed-loop tracking to a reference input is required.

Now we consider the special cases of minimum-order stable and zeroth order limits. By applying essentially the same arguments as in [5], several results are obtained immediately. These are summarized in Theorems 7 and 8. The first is based on the following construction. Choose any nonsingular matrix T such that

$$T^{-1}B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{21}$$

and let

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = T^{-1}AT, \tag{22}$$

where $\widetilde{A}_{22} \in \mathbb{R}^{m \times m}$. If (A, B) is stabilizable,

$$\operatorname{rank} \begin{bmatrix} \lambda I - \widetilde{A}_{11} & -\widetilde{A}_{12} & 0 \\ -\widetilde{A}_{21} & \lambda I - \widetilde{A}_{22} & I \end{bmatrix} = n$$

for every λ with $\operatorname{Re}\lambda \geq 0$. Hence, $\operatorname{rank} \begin{bmatrix} \lambda I - \widetilde{A}_{11} & \widetilde{A}_{12} \end{bmatrix} = n - m$ (i.e. $\left(\widetilde{A}_{11}, \widetilde{A}_{12}\right)$ is stabilizable). We may thus choose Λ such that $\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda$ is stable, and set

$$X = T \begin{bmatrix} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \\ -\Lambda\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \end{bmatrix} T^{-1}, \tag{23}$$

$$U = \begin{bmatrix} -\left(\widetilde{A}_{21} - \widetilde{A}_{22}\Lambda\right)\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & I \end{bmatrix} T^{-1}. \tag{24}$$

By direct calculation, $AX+BU=I$, so $\begin{bmatrix} X \\ U \end{bmatrix} \in \mathcal{W}$ and $\xi = [X, I, 0] \in \mathcal{C}_I$. Note that $\operatorname{ind}\xi = 1$ and $\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1}$ is stable, so ξ is stable. From Theorem 1, part 1), the state transition matrix is

$$\Phi = T \begin{bmatrix} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \\ -\Lambda\left(\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) + \delta I\right) & -\delta I \end{bmatrix} T^{-1}, \tag{25}$$

so

$$\Phi X = T \begin{bmatrix} \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \\ -\Lambda \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \end{bmatrix} T^{-1}. \tag{26}$$

Letting

$$\begin{bmatrix} \widetilde{x}_{01} \\ \widetilde{x}_{02} \end{bmatrix} = T^{-1} x_0,$$

we obtain the solution of (16):

$$x = T \begin{bmatrix} I \\ -\Lambda \end{bmatrix} \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) \widetilde{x}_{01}.$$

Theorem 7 *1) $\mathcal{C}_s$ contains a stable point iff (A, B) is stabilizable.*
2) If $\xi \in \mathcal{C}_s$ is stable, then $\operatorname{ord}\xi \geq n - m$ with equality iff $\xi = [X, I, 0]$, where X has the structure (23).

Proof. See [6], Theorems 4.2 and 4.3. ■

We are also interested in the zeroth order closed-loop limits

$$\mathcal{C}_0 = \left\{ \xi \in \mathcal{C} \,\middle|\, \operatorname{ord}\xi = 0 \right\}.$$

$\mathcal{C}_0$ corresponds precisely to those $\xi = [X, I, XB] \in \mathcal{C}_I$ with X nilpotent. From Theorem 1, part 1), the state transition matrix is

$$\Phi = -\sum_{i=0}^{q-1} \delta^{(i)} X^i, \tag{27}$$

so the solution of (16) is

$$x = \Phi X x_0 + \Phi * v = -\sum_{i=0}^{n-1} X^{i+1} B v^{(i)} - \sum_{i=1}^{n-1} \delta^{(i-1)} X^i x_0.$$

The system corresponds to successive differentiation of the input v plus a "noise" term.

Theorem 8 *1) $\mathcal{C}_0$ is nonempty iff (A, B) is controllable.*
2) If (A, B) is controllable and $m = 1$, $\mathcal{C}_0$ is a singleton.
3) If (A, B) is controllable, $m = 1$, $\xi_k \in \mathcal{C}_r$, and all eigenvalues λ_{ik} of ξ_k satisfy $|\lambda_{ik}| \to \infty$, then ξ_k converges to the unique point in $\mathcal{C}_0$.
4) If (A, B) is controllable and $m > 1$, $\mathcal{C}_0$ is uncountable and unbounded (as a subset of $\mathcal{W}$).
5) Every $\xi \in \mathcal{C}_0$ satisfies $\operatorname{ind}\xi \geq \frac{n}{m}$.

Proof. See [6], Theorems 5.1-5.3. ■

Next, we consider $\mathcal{C}_r$ approximations $[I, A - BK_k, B]$ to certain points in $\mathcal{C}_s$. This is important in applications, since points with singular X can only be achieved as limits as $\|K_k\| \to \infty$ in (3). In view of (12), the closed-loop system (3) can be written equivalently as

$$(A - BK_k)^{-1} \dot{x} = x + (A - BK_k)^{-1} Bv + \delta (A - BK_k)^{-1} x_0, \tag{28}$$

yielding state transition matrix

$$\Phi_k = (A - BK_k) \exp(A - BK_k) \tag{29}$$

and solution

$$x_k = \Phi_k (A - BK_k)^{-1} x_0 + \Phi_k * Bv. \tag{30}$$

We are interested in finding a sequence $\{K_k\}$ that yields not only convergence of $[I, A - BK_k, B]$ in $\mathcal{C}$, but also the strongest possible convergence of the forced and natural response in (30).

We begin by consider stable systems.

Theorem 9 *Let $\xi \in \mathcal{C}_s$ be stable with* $\operatorname{ord} \xi = n - m$, *and let*

$$K_k = \left[\begin{array}{cc} \widetilde{A}_{21} + k\Lambda & \widetilde{A}_{22} + kI \end{array} \right] T^{-1}, \tag{31}$$

$$\xi_k = \left[I, A - BK_k, B \right].$$

Then
1) $\xi_k \to \xi$,
2) $\Phi_k \left(A - BK_k \right)^{-1}$ *is uniformly bounded,*
3) $\Phi_k \to \Phi$ *uniformly on* $[\varepsilon, \infty)$ *for every* $\varepsilon > 0$,
4) $\Phi_k \to \Phi$ *weak*,*
where Φ *is given by (25).*

Proof. 1)-3) See [6], Theorem 6.2.
4) From 2),3), $\left(A - BK_k \right)^{-1} \Phi_k \to X\Phi$ weak*. Since differentiation is weak* continuous,

$$\Phi_k = \left(A - BK_k \right)^{-1} \dot{\Phi}_k - \delta I \to X\dot{\Phi} - \delta I = X.$$

■

The results of [1] can be interpreted in terms of Theorems 7 and 9. In [1], the special case

$$K_\mu = -\frac{1}{\mu} K \tag{32}$$

is considered, where K is a fixed matrix and $\mu > 0$ is small. Adopting (21) and (22) and setting

$$\left[\begin{array}{cc} \widetilde{K}_1 & \widetilde{K}_2 \end{array} \right] = KT,$$

it is assumed in [1] (equations (32) and (33)) that $\widetilde{K}_2$ and $\widetilde{A}_{11} - \widetilde{A}_{12} \widetilde{K}_2^{-1} \widetilde{K}_1$ are stable. Under these conditions, (32) constitutes an alternative to (31). Indeed, define

$$\Gamma_\mu = \mu \widetilde{A}_{22} + \widetilde{K}_2, \quad \Delta_\mu = \widetilde{A}_{11} - \widetilde{A}_{12} \Gamma_\mu^{-1} \left(\mu \widetilde{A}_{21} + \widetilde{K}_1 \right),$$

and note that Γ_μ and Δ_μ are stable for small $\mu > 0$. Block matrix inversion reveals

$$\begin{aligned} X &= T \left[\begin{array}{cc} \Delta_\mu^{-1} & -\mu \Delta_\mu^{-1} \widetilde{A}_{12} \Gamma_\mu^{-1} \\ -\Gamma_\mu^{-1} \left(\mu \widetilde{A}_{21} + \widetilde{K}_1 \right) \Delta_\mu^{-1} & \mu \left(\Gamma_\mu^{-1} + \Gamma_\mu^{-1} \left(\mu \widetilde{A}_{21} + \widetilde{K}_1 \right) \Delta_\mu^{-1} \widetilde{A}_{12} \Gamma_\mu^{-1} \right) \end{array} \right] T^{-1} \\ &\to T \left[\begin{array}{cc} \left(\widetilde{A}_{11} - \widetilde{A}_{12} \widetilde{K}_2^{-1} \widetilde{K}_1 \right)^{-1} & 0 \\ -\widetilde{K}_2^{-1} \widetilde{K}_1 \left(\widetilde{A}_{11} - \widetilde{A}_{12} \widetilde{K}_2^{-1} \widetilde{K}_1 \right)^{-1} & 0 \end{array} \right] T^{-1}, \end{aligned}$$

which is the same as (23) with $\Lambda = \widetilde{K}_2^{-1} \widetilde{K}_1$. Although the structures (31) and (32) are slightly different, the methods of [6], Theorem 6.2 can easily be modified to prove Theorem 9 relative to (32). Note that, in [1], only asymptotic stability for each $\mu > 0$ is actually proven.

Now consider zeroth order systems $\xi \in \mathcal{C}_0$. Theorem 8, part 1), guarantees that (A, B) is controllable. From [16], pp. 342-343, there exist $\widetilde{K} \in \mathbb{R}^{m \times n}, w \in \mathbb{R}^m$ such that $\left(A - B\widetilde{K}, Bw \right)$ is controllable with $A - B\widetilde{K}$ nilpotent. Thus there exists a nonsingular N such that

$$N^{-1} \left(A - B\widetilde{K} \right) N = \left[\begin{array}{cccc} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{array} \right], \quad N^{-1} B w = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \right].$$

Theorem 10 *Let*

$$\beta_{ik} = \left(\begin{array}{c} n \\ i \end{array} \right) k^{n-i}, \quad \widehat{K}_k = \left[\begin{array}{ccc} \beta_{0k} & \cdots & \beta_{n-1,k} \end{array} \right],$$

$$K_k = \widetilde{K} + w\widehat{K}_k N^{-1},$$
$$\xi_k = [I, A - BK_k, B].$$

Then
1) ξ_k converges to a point in $\mathcal{C}_0$,
2) $\Phi_k \to \Phi$ uniformly on $[\varepsilon, \infty)$ for every $\varepsilon > 0$,
3) $\Phi_k \to \Phi$ weak,*
where Φ is given by (27).

Proof. 1) From Theorem 5, part 2), it suffices to prove that $(A - BK_k)^{-1} \to X$ for some nilpotent X. This follows by the same arguments as in [6], Theorem 6.3.

2),3) See [6], Theorem 6.3. ■

Note that, in Theorem 10, boundedness of the natural response matrix $\Phi_k (A - BK_k)^{-1}$ was dropped. This is a consequence of the appearance of impulses in Φ when $\xi \in \mathcal{C}_0$ and $X \neq 0$. We can, in fact, prove a stronger result, which demonstrates the disastrous effect of driving the system to a limit with $\operatorname{ord} \xi < n - m$.

Theorem 11 *Let $m < n$, $1 < p \leq \infty$, and $\xi_k \in \mathcal{C}$ be stable for all k. If the eigenvalues λ_{ik} of ξ_k satisfy $\max_i \{|\lambda_{ik}|\} \to \infty$ as $k \to \infty$, then $\|\Phi_k X_k\|_p \to \infty$.*

Proof. See [6], Theorem 6.4. ■

5 The Limiting Compensator

The state feedback law (2) may be written

$$\begin{bmatrix} I & K \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} = v. \tag{33}$$

This suggests that compensators of the form (2) are naturally identified with points $[I, K]$ in the Grassmanian $\mathcal{G}_m (\mathbb{R}^{m+n})$. In the proof of Theorem 3, we considered the maps $g : \mathcal{G}_n (\mathbb{R}^{2n}) \to \mathcal{G}_n (\mathbb{R}^{2n+m})$ and $h : \mathcal{G}_m (\mathbb{R}^{m+n}) \to \mathcal{G}_n (\mathbb{R}^{2n})$ defined by

$$g\left(\left[\widetilde{X}, Y\right]\right) = \left[\widetilde{X}T^{-1}, \widetilde{X}T^{-1}A - Y, \widetilde{X}T^{-1}B\right], \tag{34}$$

$$h([Z_1, Z_2]) = \left(\begin{bmatrix} I & 0 \\ 0 & Z_1 \end{bmatrix}, \begin{bmatrix} 0 \\ Z_2 \end{bmatrix}\right), \tag{35}$$

where T is given by (21). The composition $f = g \circ h$ was shown to be an analytic diffeomorphism between the manifolds $\mathcal{G}_m (\mathbb{R}^{m+n})$ and $\mathcal{C}_e = f (\mathcal{G}_m (\mathbb{R}^{m+n}))$, with $\mathcal{C}_e$ regular in $\mathcal{G}_n (\mathbb{R}^{2n+m})$. Consider the open, dense submanifolds $\mathcal{F} = f^{-1}(\mathcal{C})$ and $\mathcal{F}_r = f^{-1}(\mathcal{C}_r)$ of $\mathcal{G}_m (\mathbb{R}^{m+n})$, and let $\mathcal{F}_s = f^{-1}(\mathcal{C}_s)$. The next result establishes basic properties of state feedback (33).

Theorem 12 *1) $\mathcal{F}_r = \left\{ [I, K] \in \mathcal{G}_m (\mathbb{R}^{m+n}) \,\middle|\, K \in \mathbb{R}^{m \times n} \right\}$*
2) $\mathcal{F}_s = \left\{ [Z_1, Z_2] \in \mathcal{F} \,\middle|\, \det Z_1 = 0 \right\}$

Proof. 1) The result follows by the calculation

$$f([I, K]) = g(h([I, K])) = g\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 \\ K \end{bmatrix}\right) = \left[T^{-1}, T^{-1}A - T^{-1}BK, T^{-1}B\right] = [I, A - BK, B].$$

2) This follows from $\mathcal{F}_r = \left\{ [Z_1, Z_2] \in \mathcal{G}_m (\mathbb{R}^{m+n}) \,\middle|\, \det Z_1 \neq 0 \right\}$ and $\mathcal{F}_s = \mathcal{F} - \mathcal{F}_r$. ■

The properties of f guarantee that, if K_k is any sequence of feedback matrices such that the closed-loop systems (3) converge in $\mathcal{C}$, then the sequence $[I, K_k]$ also converges in $\mathcal{G}_m (\mathbb{R}^{m+n})$. By Theorem 12, degeneration of (3) to a point in $\mathcal{C}_s$ occurs iff $[I, K_k]$ converges to a point in $\mathcal{F}_s$. In other words, the limiting compensator always exists, and it is singular iff the limiting closed-loop system is singular. Compensators in $\mathcal{F}_s$ are not physically realizable, since they correspond to feedback laws of the form

$$Z_1 u = -Z_2 x + v$$

with Z_1 singular. Yet, as a mathematical object, each compensator in $\mathcal{F}$ determines a well-defined closed-loop system.

For the special case of minimum-order stable limits, as in Theorem (7), we can obtain the form of Z_1 and Z_2 explicitly.

Theorem 13 *If $\xi = [X, I, XB]$ is given by (23), then $f^{-1}(\xi) = \left[0, \left[\begin{array}{cc} \Lambda & I \end{array}\right]\right]$.*

Proof. Choose a representative $\left[\begin{array}{cc} Z_1 & Z_2 \end{array}\right]$ for $f^{-1}(\xi)$. From (23), (34), and (35),

$$\left[\begin{array}{cc} I & 0 \\ 0 & Z_1 \end{array}\right] T^{-1} = MT \left[\begin{array}{cc} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \\ -\Lambda\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \end{array}\right] T^{-1},$$

for some nonsingular M. Hence, $Z_1 = 0$ and

$$\left[\begin{array}{c} \widetilde{A}_{11} - \widetilde{A}_{12}\Lambda \\ 0 \end{array}\right] = MT \left[\begin{array}{c} I \\ -\Lambda \end{array}\right].$$

Letting

$$\left[\begin{array}{cc} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{array}\right] = MT,$$

we obtain $\widetilde{M}_{21} = \widetilde{M}_{22}\Lambda$. Also, from (34) and (35),

$$\left[\begin{array}{c} 0 \\ Z_2 \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right] T^{-1}A - M,$$

so

$$Z_2 = -\widetilde{M}_{22}\left[\begin{array}{cc} \Lambda & I \end{array}\right].$$

Since rank $\left[\begin{array}{cc} Z_1 & Z_2 \end{array}\right] = m$, $\widetilde{M}_{22}$ is nonsingular. Premultiplication of $\left[\begin{array}{cc} Z_1 & Z_2 \end{array}\right]$ by $-\widetilde{M}_{22}^{-1}$ yields the desired result.
■

We conclude this section by examining behavior of the input function u under high-gain feedback. For simplicity, we will only consider the case where $v = 0$. If we apply the feedback gains K_k to (3), then both u and x depend on k, and are related by the feedback law

$$u_k = K_k x_k.$$

In Theorems 9 and 10, we established cases under which the state-transition matrix Φ_k converges in two different topologies. More generally, consider the linear subspace

$$\mathcal{D}_0' = C[0,\infty) + \operatorname{span}\left\{\delta, \dot{\delta}, \ddot{\delta}, \ldots\right\} \subset \mathcal{D}_+',$$

where $C[0,\infty)$ is the set of continuous functions on $\mathbb{R}$ with support in $[0,\infty)$. Both weak* convergence and uniform convergence on every $[\varepsilon,\infty)$ correspond to specific topologies on $\mathcal{D}_0$. It is easy to show that both make $\mathcal{D}_0$ a topological vector space.

Theorem 14 *Suppose $\mathcal{D}_0$ is given a topology that makes it a topological vector space. If $[I, A - BK_k, B] \to [X, I, XB] \in \mathcal{C}_I$ and $\Phi_k \to \Phi$ in $\mathcal{D}_0$, then $u_k \to U\Phi x_0$ in $\mathcal{D}_0$.*

Proof. From (29) and Theorem 5, part 2),

$$U\Phi + K_k(A - BK_k)^{-1}\Phi_k = \left(U + K_k(A - BK_k)^{-1}\right)\Phi + K_k(A - BK_k)^{-1}(\Phi_k - \Phi) \to 0,$$

so

$$u_k = -K_k(A - BK_k)^{-1}\Phi_k x_0 \to U\Phi x_0.$$

■

Theorem 14 can be extended to $v \neq 0$ through choice of an appropriate space of inputs v and exploiting the properties of the convolution operator. We leave the details to the reader.

6 Application to Cheap Control

A classical problem in the theory of linear-quadratic optimal control is the "cheap control" problem, where an input function $u^*(t)$ is sought to minimize the cost

$$J(\varepsilon) = \int_0^\infty x^T x + \varepsilon u^T u dt$$

subject to (1), with fixed initial condition x_0 and small $\varepsilon \geq 0$. For $\varepsilon > 0$, this problem has been extensively studied (e.g. see [14], [7], [12], [15]). The solution is obtained by constructing the unique positive definite symmetric solution $P(\varepsilon)$ of the algebraic Riccati Equation

$$P(\varepsilon)A + A^T P(\varepsilon) - \frac{1}{\varepsilon} P(\varepsilon) BB^T P(\varepsilon) + I = 0.$$

Then, for each x_0, the optimal u and x are related by the feedback law

$$u^* = -\frac{1}{\varepsilon} B^T P(\varepsilon) x^*,$$

yielding the closed-loop system

$$\left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} \dot{x}^* = x^* + \delta \left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} x_0$$

(cf. (28)).

For $\varepsilon = 0$, we adopt (21) and (22), let

$$\begin{bmatrix} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{12}^T & \widetilde{Q}_{22} \end{bmatrix} = T^T T,$$

and let Γ be the unique positive definite symmetric solution of the reduced Riccati equation

$$\Gamma \left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T\right) + \left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T\right)^T \Gamma - \Gamma \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{A}_{12}^T\Gamma + \widetilde{Q}_{11} - \widetilde{Q}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T = 0.$$

Setting

$$\Lambda = \widetilde{Q}_{22}^{-1}\left(A_{12}^T \Gamma + \widetilde{Q}_{12}^T\right) \tag{36}$$

leads to values of X, U, and Φ according to (23), (24), and (25). It is shown in [15], Corollary 2.6.1, that $J(0)$ is minimized, subject to (1), by $x^* = \Phi x_0$ and $u^* = U\Phi x_0$. Furthermore, [15], Theorem 2.7.1 indicates that

$$\left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} \to X$$

as $\varepsilon \to 0^+$. These facts are now interpreted in the context of the present paper.

Theorem 15 *For each $\varepsilon \geq 0$, let $\xi_\varepsilon^* \in \mathcal{C}_r$ be the optimal closed-loop system in the cheap control problem. Then $\xi_\varepsilon^* \to \xi_0^*$ in $\mathcal{C}$ as $\varepsilon \to 0^+$, where ξ_0^* is stable and $\operatorname{ord} \xi_0^* = n - m$. The limiting system ξ_0^* is determined uniquely by the singular compensator $\left[0, \begin{bmatrix} \Lambda & I \end{bmatrix}\right] \in \mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$ as in Theorem 13, where Λ is given by (36).*

7 Conclusions

In this paper, we have developed a general theory of high-gain state feedback, retaining a fixed state coordinate system. For many control problems, this approach lends itself to a more natural interpretation of results than if the coordinates were allowed to vary with the feedback gain. Relationships to other seminal work in the area have been drawn. As in our earlier similar work on high-gain observers, the present paper has focused primarily on system parameter convergence and behavior of solutions, particularly in the cases of stable and zeroth order limits. A unique aspect of our results is that even infinite state feedback gains are identified with specific mathematical objects. As future work, we hope to be able to extend our results to observer-based output feedback and, ultimately, to general output feedback.

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