State Feedback Impulse Elimination for Singular Systems over a Hermite Domain

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Abstract

We reduce the problem of impulse elimination via state feedback in singular differential equations to algebra. Our results are developed for systems over an arbitrary Hermite domain. We show that the established theories for the time-invariant and the real analytic time-varying settings can be unified in this way. Besides the constant and real analytic functions, several other function rings are considered. Our algebraic theory is applied to these cases, providing solutions to the impulse elimination problem for classes of systems not previously studied. In particular, our work allows the restriction of the feedback matrix to certain function rings.

1 Introduction

We are interested in the problem of designing a state feedback law $u = K(t)\,x$ for a time-varying singular differential equation

$$E(t)\,\dot{x} = A(t)\,x + B(t)\,u \tag{1}$$

such that the closed-loop system

$$E(t)\,\dot{x} = (A(t) + B(t)\,K(t))\,x \tag{2}$$

exhibits no impulsive transients. The matrices E, A, and B are assumed to have entries in an appropriate set of functions on $\mathbb{R}$ (possibly constant) with $E(t), A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $K(t) \in \mathbb{R}^{m\times n}$. This problem has been treated in a variety of contexts over the past 25 years [10], [16], [12], [13], [4], [17], [18]. For example, we originally posed and solved the problem for the time-invariant (i.e. constant matrix) case in [10].

For time-invariant systems, the fact that solutions of (2) can exhibit impulsive behavior was originally established in [14] and [15], Ch.22. One method of analysis is based on the Weierstrass decomposition ([8], p.28, Theorem 3):.Given E, A with $\det(sE - A) \not\equiv 0$, there exist nonsingular $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix},$$

where N is nilpotent. If $N \neq 0$, the solution of (1) contains an impulsive term of the form

$$z = -\sum \delta^{(k-1)} N^k z_o. \tag{3}$$

(See [19] for details.) More generally, when $E(t)$ and $A(t)$ are analytic functions, it is shown in [3] that an expression similar to (3) holds under mild assumptions

Since impulses must be interpreted as unbounded, conventional notions of closed-loop stability dictate that K be chosen to make (2) impulse free. For the time-invariant case, we established a necessary and sufficient condition ([10], Theorem 6) under which such a matrix K exists. This condition can be written

$$\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = \mathbb{R}^n.$$

Since then, two alternative proofs of this result have appeared. (See [12], Theorem 2.5.1 and [13], Theorem 3-2.1.)

The work of Campbell and Petzold [3] extended the theory of singular systems (1) to the time-varying setting, where E, A, and B are matrices over the real analytic functions on $\mathbb{R}$. More recently, the corresponding impulse elimination problem has been solved by Wang in ([4], Theorem 4.1). In this case, necessary and sufficient conditions for impulse elimination are

$$\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n \quad \forall t,$$

$$\operatorname{rank} E(t) = \text{constant}.$$

Our contention is that the impulse elimination problem is primarily a problem in algebra. Indeed, after careful examination (and some modification), the arguments in [4] can be reduced to algebraic manipulations over a certain class of rings. Pursuing this idea not only leads to a unification of the time-invariant and analytic time-varying theories, but also yields a more general framework in which the impulse elimination problem for other classes of time-varying systems can be solved with little extra effort.

An important consequence of our approach is that it allows the entries of K to be restricted to certain function rings (although E, A, and B must share the same restriction). Hence, we are able to solve a wide variety of constrained feedback problems which have not been considered in the literature.

Our algebraic theory is the subject of Sections 2 and 3. In Section 4, we apply our results to various types of time-varying singular systems.

2 Algebraic Preliminaries

Let R be a commutative ring (with identity). If $x_1, \ldots, x_k \in R$, a *Bezout identity* is an equation of the form $\sum a_i x_i = 1$ ($a_i \in R$). For a matrix $M \in R^{p\times q}$, let

$$\operatorname{rank} M = \max\left\{k \,\middle|\, M \text{ has a nonzero } k\text{th-order minor}\right\} \tag{4}$$

and

$$\rho M = \max\left\{k \,\middle|\, \text{the } k\text{th-order minors of } M \text{ satisfy a Bezout identity}\right\}. \tag{5}$$

Obviously, $\operatorname{rank} M \geq \rho M$ for any M. It can be shown that $\operatorname{rank} M$ and ρM are invariant under left and right unimodular transformations. (See [1], p.25.) If $R = \mathbb{R}$, then $\operatorname{rank} M = \rho M$. We denote this common value by $\operatorname{rank}_{\mathbb{R}} M$.

Consider the set G of all triples (P, Q, D), where $P, Q, D \in R^{n\times n}$ and P, Q are unimodular. Define the binary operation

$$(P_1, Q_1, D_1) * (P_2, Q_2, D_2) = (P_2 P_1, Q_1 Q_2, D_1 Q_2 + Q_1 D_2).$$

It is routine to verify that G has the structure of a group. Now consider pairs (E, A), where $E, A \in R^{n\times n}$. We may define a right group action on the set of all (E, A) according to

$$(E, A) \cdot (P, Q, D) = (PEQ, P(AQ + ED)). \tag{6}$$

The *orbit* of particular (E, A) is the set of all pairs $\left(\widetilde{E}, \widetilde{A}\right)$ such that $\left(\widetilde{E}, \widetilde{A}\right) = (E, A) \cdot (P, Q, D)$ for some P, Q, D. It is easy to verify that the set of all orbits forms a partition of $R^{n\times n} \times R^{n\times n}$.

Following the terminology of Campbell and Petzold [3], we say (E, A) is in *standard canonical form*, if

$$E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \tag{7}$$

where N is strictly upper triangular with E, A identically partitioned. Similar to their notion of "analytic solvability" for systems (1), we say $(E, A) \in R^{n\times n} \times R^{n\times n}$ is *algebraically solvable*, if its orbit under (6) contains a member in standard canonical form. (The degenerate cases (I, X) and (N, I) are also allowed.) We say that (E, A) has *unit index* if the orbit of (E, A) contains a member in standard canonical form with $N = 0$. It is clear from the definitions that algebraically solvability are invariant under the group action (6).

The question arises whether a unit index orbit can contain a member in standard canonical form with $N \neq 0$.

Fortunately, the next result answers this question in the negative.

Theorem 1 *Suppose* (E,A) *has unit index and* $(E,A)\cdot(P,Q,D)$ *is in standard canonical form (7). Then* $N=0$.

Proof. Suppose (E,A) belongs to an orbit with two members in standard canonical form, one with $N=0$ and the other with $N\neq 0$. Then there exist D and unimodular P and Q such that

$$P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Q^{-1},$$

$$P\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} DQ^{-1}\right) = \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} Q^{-1}$$

for some X and Y and some strictly upper triangular $N\neq 0$. Let

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = DQ^{-1}, \quad P_1 = P\begin{bmatrix} I & D_{12} \\ 0 & I \end{bmatrix},$$

and $X_1 = X + D_{11}$. Then

$$P_1\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = P\begin{bmatrix} I & D_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = P\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Q^{-1}, \tag{8}$$

$$\begin{aligned} P_1\begin{bmatrix} X_1 & 0 \\ 0 & I \end{bmatrix} &= P\begin{bmatrix} I & D_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} X_1 & 0 \\ 0 & I \end{bmatrix} \\ &= P\begin{bmatrix} X+D_{11} & D_{12} \\ 0 & I \end{bmatrix} \\ &= P\left(\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} DQ^{-1}\right) \\ &= \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} Q^{-1}. \end{aligned} \tag{9}$$

Let

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = P_1, \quad \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = Q^{-1}.$$

From (8), $P_{21} = NQ_{21}$ and $NQ_{22} = 0$. From (9), $P_{22} = Q_{22}$. Since N is strictly triangular and nonzero, there exist an integer $q > 1$ and x such that $N^q = 0$ and $N^{q-1}x \neq 0$. Since P_1 is unimodular, there exist y and z such that

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = P_1 \begin{bmatrix} y \\ z \end{bmatrix},$$

$$x = P_{21}y + P_{22}z = NQ_{21}y + Q_{22}z.$$

Multiplying by N^{q-1} yields

$$N^{q-1}x = N^q Q_{21}y + N^{q-1}Q_{22}z = 0,$$

which is a contradiction. ■

In practice, algebraic solvability may be difficult to establish, so we introduce a more direct condition that will suit our purposes just as well. We say that (E, A) is *presolvable* if **any one** of the following conditions holds:

PS1) $\operatorname{Im} E + A \operatorname{Ker} E = R^n$,

PS2) $\operatorname{Im} E \cap A \operatorname{Ker} E \neq 0$,

PS3) $\operatorname{Ker} E \cap \operatorname{Ker} A \neq 0$.

Algebraic solvability and standard canonical form are related to existence and uniqueness of solutions of (1), as discussed in [3]. However, presolvability is a purely algebraic condition, having no simple connection to the dynamics of (1). Nevertheless, we can prove the following.

Theorem 2 *1) Algebraic solvability implies presolvability.*

2) Presolvability is invariant under the group action (6).

Proof. 1) There exist P, Q, and D that put (E, A) in standard canonical form. Suppose $N = 0$. Then

$$\begin{aligned} P(\operatorname{Im} E + A \operatorname{Ker} E) &= P\left(\operatorname{Im} E + A \operatorname{Ker} E + EDQ^{-1} \operatorname{Ker} E\right) \\ &= \operatorname{Im} PEQ + PAQ \operatorname{Ker} PEQ + PED \operatorname{Ker} PEQ \\ &\supset \operatorname{Im} PEQ + P(AQ + ED) \operatorname{Ker} PEQ \\ &= \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= R^n, \end{aligned}$$

so PS1) holds.

If $N \neq 0$, there exists an integer $q > 1$ such that $N^q = 0$ and $N^{q-1} \neq 0$. Choose any $x \in R^n$ such that $N^{q-1}x \neq 0$, set $y = N^{q-2}x$, and $z = Ny$. Then $z \neq 0$. Let

$$v = Q \begin{bmatrix} 0 \\ y \end{bmatrix} - D \begin{bmatrix} 0 \\ z \end{bmatrix}, \quad w = Q \begin{bmatrix} 0 \\ z \end{bmatrix}.$$

Then $w \neq 0$ and

$$P(Ev - Aw) = PEQ \begin{bmatrix} 0 \\ y \end{bmatrix} - P(AQ + ED) \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ Ny - z \end{bmatrix} = 0,$$

so $Ev = Aw$. Also,

$$P(Ew) = PEQ \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ Nz \end{bmatrix} = 0,$$

so $w \in \operatorname{Ker} E$ and $Aw \in \operatorname{Im} E \cap A \operatorname{Ker} E$. If $Aw \neq 0$, PS2) holds; if $Aw = 0$, PS3) holds.

2) To prove invariance of presolvability, first suppose PS1) holds for (E, A). Then

$$\begin{aligned} \operatorname{Im} PEQ + P(AQ + ED) \operatorname{Ker} PEQ &= P\left(\operatorname{Im} E + \left(A + EDQ^{-1}\right) \operatorname{Ker} E\right) \\ &= P\left(\operatorname{Im} E + \left(A + EDQ^{-1}\right) \operatorname{Ker} E + EDQ^{-1} \operatorname{Ker} E\right) \\ &\supset P\left(\operatorname{Im} E + \left(\left(A + EDQ^{-1}\right) - EDQ^{-1}\right) \operatorname{Ker} E\right) \\ &= R^n, \end{aligned}$$

so PS1) also holds for $(E, A) \cdot (P, Q, D)$ and $(E, A) \cdot (P, Q, D)$ is presolvable.

Now assume that PS2) holds for (E, A), but not for $(E, A) \cdot (P, Q, D)$. Then there exist x, y such that $Ex = 0$ and $Ey = Ax \neq 0$. Hence, $x \neq 0$,

$$P(AQ + ED) Q^{-1} x = PE\left(y + DQ^{-1} x\right) \in \operatorname{Im} PEQ \cap P(AQ + ED) \operatorname{Ker} PEQ = 0,$$

$$0 \neq Q^{-1} x \in \operatorname{Ker} PEQ \cap \operatorname{Ker} P(AQ + ED). \tag{10}$$

This establishes PS3), and therefore, presolvability, relative to $(E, A) \cdot (P, Q, D)$.

Finally, suppose that PS3) holds for (E, A), but $(E, A) \cdot (P, Q, D)$ fails to satisfy PS2). Then there exists $x \neq 0$ such that $Ex = Ax = 0$ and

$$P(AQ + ED) Q^{-1} x = PEDQ^{-1} x \in \operatorname{Im} PEQ \cap P(AQ + ED) \operatorname{Ker} PEQ = 0.$$

Hence, (10) again holds, verifying PS3) and presolvability of $(E, A) \cdot (P, Q, D)$. ■

If (E, A) has unit index, it turns out that the matrix D plays no essential role in establishing standard canonical form. This is made precise in the next theorem.

Theorem 3 *If (E, A) has unit index, then there exists a unimodular $Q \in R^{n \times n}$ such that, **for every** $D \in R^{n \times n}$, there exists a unimodular $P \in R^{n \times n}$ which yields standard canonical form (7) with $N = 0$.*

Proof. Suppose (P_1, Q_1, D_1) achieves standard canonical form for some X_1 and with $N = 0$. Let $Q = Q_1$ and let D be given. Setting

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = Q_1^{-1}(D - D_1), \quad P_2 = \begin{bmatrix} I & -D_{12} \\ 0 & I \end{bmatrix},$$

$X = X_1 + D_{11}$, and $P = P_2P_1$ yields

$$PEQ = P_2(P_1EQ_1) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned} P(AQ + ED) &= P_2\left(P_1(AQ_1 + ED_1) + (P_1EQ_1)Q_1^{-1}(D - D_1)\right) \\ &= \begin{bmatrix} I & -D_{12} \\ 0 & I \end{bmatrix}\left(\begin{bmatrix} X_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

■

For an arbitrary commutative ring R, we can establish necessary conditions under which (E, A) has unit index. First we need a lemma.

Lemma 4 *Let $M \in R^{n \times n}$. If there exist unimodular $P, Q \in R^{n \times n}$ such that*

$$PMQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

then $\operatorname{rank} M = \rho M$.

Proof. Suppose the identity matrix in (11) is $n_1 \times n_1$. Then it is clear by the definitions (4) and (5) that

$$\operatorname{rank} PMQ = n_1 = \rho PMQ.$$

The result follows from invariance of rank and ρ under unimodular transformations. ■

Theorem 5 *If (E, A) has unit index, then*

1) $\operatorname{rank} E = \rho E$,

2) $\operatorname{Im} E + A \operatorname{Ker} E = R^n$,

3) (E, A) *is presolvable.*

Proof. 1) This follows from standard canonical form and Lemma 4.

2) Invoking standard canonical form,

$$
\begin{aligned}
P\left(\operatorname{Im} E + A \operatorname{Ker} E\right) &= P\left(\operatorname{Im} E + A \operatorname{Ker} E + EDQ^{-1} \operatorname{Ker} E\right) \\
&= \operatorname{Im} PEQ + PAQ \operatorname{Ker} PEQ + PED \operatorname{Ker} PEQ \\
&\supset \operatorname{Im} PEQ + P\left(AQ + ED\right) \operatorname{Ker} PEQ \\
&= \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} + \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \\
&= R^n.
\end{aligned}
$$

3) This is obvious from part 2). ∎

Let $B \in R^{n \times m}$. The group action (6) may be extended to triples (E, A, B) according to

$$(E, A, B) \cdot (P, Q, D) = \left(PEQ, P\left(AQ + ED\right), PB\right). \tag{12}$$

In [16] we introduced the concept of "impulse controllability", which is fundamental to the study of state feedback in singular systems. We can adapt this idea to the algebraic setting by taking its feedback characterization as the definition. We say that $K \in R^{m \times n}$ is *impulse eliminating*, if $(E, A + BK)$ has unit index. The triple (E, A, B) is *impulse controllable*, if there exists an impulse eliminating K.

Theorem 6 *Impulse controllability is invariant under (12).*

Proof. Suppose (E, A, B) is impulse controllable, and let K be impulse eliminating. Choose any P, Q, D, and let $K_1 = KQ$. Then

$$\left(PEQ, P\left(AQ + ED\right) + \left(PB\right) K_1\right) = \left(PEQ, P\left(\left(A + BK\right) Q + ED\right)\right),$$

which lies in the same orbit as $(E, A + BK)$ and, hence, has unit index. Thus $\left(PEQ, P\left(AQ + ED\right), PB\right)$ is impulse controllable. ∎

Theorem 7 *If* (E, A, B) *is impulse controllable, then*

1) $\operatorname{rank} E = \rho E$,

2) $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$.

Proof. Suppose $(E, A + BK)$ has unit index. From Theorem 5, part 1), $\operatorname{rank} E = \rho E$. By Theorem 5, part 2),

$$\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B \supset \operatorname{Im} E + A \operatorname{Ker} E + BK \operatorname{Ker} E \supset \operatorname{Im} E + \left(A + BK\right) \operatorname{Ker} E = R^n.$$

■

We conclude this section by proving a pair of lemmas which will be useful in the sequel, and which hold for any commutative ring.

Lemma 8 *Let $M \in R^{p\times q}$. The following statements are equivalent:*

1) $\operatorname{Im} M = R^p$,

2) M has a right inverse,

3) $\rho M = p$.

Proof. 1)⇒2) Let $e_1, \ldots, e_p$ be the canonical unit vectors in R^p. Since $\operatorname{Im} M = R^p$, there exist $x_1, \ldots, x_p \in R^q$ such that $Mx_i = e_i$. Let $L = \begin{bmatrix} x_1 & \cdots & x_p \end{bmatrix}$. Then $MLe_i = Mx_i = e_i$, so $ML = I$.

2)⇒3) Suppose $ML = I$. From the Binet-Cauchy formula,

$$\sum_{1\le j_1<\ldots<j_p\le q} M\begin{pmatrix} 1 & \cdots & p \\ j_1 & \cdots & j_p \end{pmatrix} L\begin{pmatrix} j_1 & \cdots & j_p \\ 1 & \cdots & p \end{pmatrix} = \det I = 1,$$

so $\rho M = p$.

3)⇒1) There exist $x_{j_1\cdots j_p} \in R$ such that

$$\sum_{1\le j_1<\ldots<j_p\le q} x_{j_1\cdots j_p} M\begin{pmatrix} 1 & \cdots & p \\ j_1 & \cdots & j_p \end{pmatrix} = 1. \tag{13}$$

Traversing the ith row and expanding by minors yields

$$M\begin{pmatrix} 1 & \cdots & p \\ j_1 & \cdots & j_p \end{pmatrix} = \sum_{l=1}^{p} (-1)^{i+j_l} m_{ij_l} M\begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & p \\ j_1 & \cdots & j_{l-1} & j_{l+1} & \cdots & j_p \end{pmatrix}, \tag{14}$$

where $M = [m_{ij}]$. Combining (13) and (14), we obtain $y_{ij} \in R$ such that $\sum_j y_{ij} m_{ij} = 1$. Let $k \ne i$ and replace the ith row of M with the kth row. This yields the calculation

$$\sum_{l=1}^{p} (-1)^{i+j_l} m_{kj_l} M\begin{pmatrix} 1 & \cdots & i-1 & i+1 & \cdots & p \\ j_1 & \cdots & j_{l-1} & j_{l+1} & \cdots & j_p \end{pmatrix} = M\begin{pmatrix} 1 & \cdots & i-1 & k & i+1 & \cdots & p \\ & j_1 & & \cdots & & j_p & \end{pmatrix} = 0.$$

Hence, $\sum_j y_{ij} m_{kj} = 0$. Let

$$y_i = \begin{bmatrix} y_{i1} \\ \vdots \\ y_{iq} \end{bmatrix}.$$

Then My_i is equal to the ith unit vector e_i. Let $x \in R^p$ and

$$z = \begin{bmatrix} y_1 & \cdots & y_p \end{bmatrix} x.$$

Then

$$Mz = \begin{bmatrix} My_1 & \cdots & My_p \end{bmatrix} x = \begin{bmatrix} e_1 & \cdots & e_p \end{bmatrix} x = x.$$

Since x is arbitrary, $\operatorname{Im} M = R^p$. ■

Lemma 9 *Let*

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $E_{11} \in R^{n_1 \times n_1}$, $A_{ij} \in R^{n_i \times n_j}$, *and* $B_i \in R^{n_i \times m}$.

1) (E, A) *has unit index iff* E_{11} *and* A_{22} *are unimodular.*

2) $\rho E = n_1$ *and* $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$ *iff* E_{11} *is unimodular and* $\rho \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n_2$.

Proof. 1) (Necessary) From Theorem 3, there exist unimodular P and Q so that (PEQ, PAQ) is in standard canonical form with $N = 0$. Let

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = P, \quad \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = Q^{-1}.$$

Then

$$PE = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$$

implies $Q_{12} = 0$, so Q_{11} and Q_{22} are unimodular. Also, $P_{11}E_{11} = Q_{11}$ and $P_{21}E_{11} = 0$, so E_{11} is unimodular and $P_{21} = 0$. It follows from

$$PA = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} Q^{-1}$$

that $P_{22}A_{22} = Q_{22}$, so A_{22} is unimodular.

(Sufficient) Let

$$P = \begin{bmatrix} E_{11}^{-1} & -E_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}, \quad Q = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix},$$

and $D = 0$. Then

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad P(AQ + ED) = \begin{bmatrix} E_{11}^{-1}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right) & 0 \\ 0 & I \end{bmatrix}.$$

2) (Necessary) Unimodularity of E_{11} follows from the definition of ρ. Thus

$$\operatorname{Ker} E = \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \operatorname{Im} \begin{bmatrix} E_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \end{bmatrix} = \operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n.$$

For any $w \in R^{n_2}$, there exist x, y, z such that

$$\begin{bmatrix} 0 \\ w \end{bmatrix} = \begin{bmatrix} E_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

so

$$w = \begin{bmatrix} A_{22} & B_2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

and $\operatorname{Im} \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = R^{n_2}$. The result follows from Lemma 8.

(Sufficient) The definition of ρ gives $\rho E = n_1$. Let $v \in R^{n_1}$ and $w \in R^{n_2}$. Then there exist y and z such that $A_{22}y + B_2 z = w$. Set $x = E_{11}^{-1} (v - A_{12}y - B_1 z)$. Then

$$\begin{bmatrix} E_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} v \\ w \end{bmatrix},$$

so

$$\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = \operatorname{Im} \begin{bmatrix} E_{11} & A_{12} & B_1 \\ 0 & A_{22} & B_2 \end{bmatrix} = R^n.$$

■

3 Pencils over an Hermite Domain

We say R is an *Hermite domain* if it is an integral domain and, for every $a, b \in R$, there exist $u, v, x, y \in R$ such that $ux + vy = 1$ and $ax + by = 0$ ([2], p.469). It should be noted that the definition of an Hermite domain varies in the literature. For example, [6] gives a definition (p.345) which is different from, but is implied by, the one given in [2]. In particular, every Bezout domain is Hermite ([2], Theorem 3.2), and, therefore, every principal ideal domain, field, etc. is also an Hermite domain. *For the remainder of this section, our standing assumption is that R is an Hermite domain (as in [2]).*

One advantage of working in an Hermite domain is that matrices over R can be triangularized: For any $M \in R^{p \times q}$

$(p \neq q)$, there exists a lower triangular $L \in R^{\min\{p,q\}\times\min\{p,q\}}$ and a unimodular $Q \in R^{q\times q}$ such that

$$MQ = \begin{cases} \begin{bmatrix} L & 0 \end{bmatrix}, & p < q \\ \begin{bmatrix} L \\ 0 \end{bmatrix}, & p > q \end{cases}.$$

A similar result, in which $\operatorname{Ker} M$ plays a special role, was established for real analytic functions in [5]. The arguments used in [5] are essentially algebraic and can be adapted to any Hermite domain. Since these ideas are central to our results, we develop the underlying algebraic arguments in detail, culminating in Theorem 12 and its corollaries.

Lemma 10 *Let $M \in R^{2\times 2}$ with at least one first-row entry nonzero. There exists a unimodular $Q \in R^{2\times 2}$ such that MQ is lower triangular with its $1,1$ entry nonzero.*

Proof. Let $\begin{bmatrix} a & b \end{bmatrix}$ be the first row of M and choose $u, v, x, y \in R$ such that $ux + vy = 1$ and $ax + by = 0$. Let

$$Q = \begin{bmatrix} v & x \\ -u & y \end{bmatrix}.$$

Then MQ is lower triangular and $\det Q = 1$, so Q is unimodular. The first row of MQ is $\begin{bmatrix} a & b \end{bmatrix} Q \neq 0$, but the $1,2$ entry of MQ is zero, so its $1,1$ entry must be nonzero. ∎

Lemma 11 *Let $M \in R^{p\times q}$ with at least one first-row entry nonzero. There exists a unimodular $Q \in R^{q\times q}$ such that MQ has the form*

$$MQ = \begin{bmatrix} a & 0 \\ b & C \end{bmatrix}, \tag{15}$$

with $a \neq 0$.

Proof. Q will be constructed as a series of column permutations and transformations of the form

$$\begin{bmatrix} v & & x & \\ & I & & \\ -u & & y & \\ & & & I \end{bmatrix},$$

where u, v, x, and y are as in the proof of Lemma 10. The product of such transformations is unimodular.

Begin operating on M by permuting its columns so that either the $1,1$ or $1,2$ entry is nonzero. Applying Lemma

10 to the upper left 2×2 submatrix yields a matrix of the form

$$\begin{bmatrix} d & 0 & e \\ f & g & h \\ j & k & L \end{bmatrix},$$

where $d, f, g \in R$ and $d \neq 0$. The $1, 3$ entry may be brought to zero by applying Lemma 10 to the 2×2 submatrix formed from the first two rows and the first and third columns. Proceeding inductively across the first row, we achieve the form (15) with $a \neq 0$. ■

Theorem 12 *Let $M \in R^{p \times q}$. If $\operatorname{rank} M = k > 0$, then there exist $L \in R^{p \times k}$ with $\operatorname{rank} L = k$ and a unimodular $Q \in R^{q \times q}$ such that*

$$MQ = \begin{bmatrix} L & 0 \end{bmatrix}. \tag{16}$$

Proof. Although we will make use of row permutations in achieving our result, these may be reversed at the end without disturbing the form (16). Since $M \neq 0$, there exists a row permutation that places a nonzero entry in the first row. Applying Lemma 11, we achieve the form (15) with $a \neq 0$. Suppose $\operatorname{rank} C \geq k$. Then C has a kth-order minor $\mu \neq 0$, and $a\mu$ is a $(k+1)$th-order minor of MQ. Since R is an integral domain, $a\mu \neq 0$, which contradicts $\operatorname{rank} M = k$. Thus $\operatorname{rank} C \leq k - 1$.

If $k > 1$, the same arguments may then be applied to C, yielding a matrix of the form

$$\begin{bmatrix} a & 0 & 0 \\ d & e & 0 \\ f & g & H \end{bmatrix},$$

where $e \in R - \{0\}$ and $\operatorname{rank} H \leq k - 2$. Proceeding inductively, we eventually achieve (16) with k nonzero columns. Since Q is unimodular, $\operatorname{rank} L = \operatorname{rank} M = k$. ■

Corollary 13 *Let $M \in R^{p \times q}$.*

1) If $\rho M = p$, then there exists a unimodular Q such that

$$MQ = \begin{bmatrix} I & 0 \end{bmatrix}.$$

2) If $\operatorname{rank} M = k$, then there exist $L \in R^{k \times k}$ with $\operatorname{rank} L = k$ and unimodular P and Q such that

$$PMQ = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

3) If $\operatorname{rank} M = \rho M$*, then there exist unimodular* P *and* Q *such that*

$$PMQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

4) If $\operatorname{rank} M = \rho M = p$*, then there exists* $L \in R^{(q-p)\times q}$ *such that* $\begin{bmatrix} M \\ L \end{bmatrix}$ *is unimodular.*

Proof. 1) From Theorem 12, there exists Q_1 and L such that

$$MQ_1 = \begin{bmatrix} L & 0 \end{bmatrix}.$$

But $\rho L = \rho M = p$, so L is unimodular. Let

$$Q = Q_1 \begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

2) From Theorem 12, there exist $L_1 \in R^{k\times k}$and $L_2 \in R^{(p-k)\times k}$ with

$$\operatorname{rank} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = k$$

and a unimodular Q such that

$$MQ = \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \end{bmatrix}.$$

Also, there exist $L \in R^{k\times k}$ with $\operatorname{rank} L = k$ and a unimodular P such that

$$\begin{bmatrix} L_1^T & L_2^T \end{bmatrix} P^T = \begin{bmatrix} L^T & 0 \end{bmatrix}.$$

Hence,

$$PMQ = \left((MQ)^T P^T\right)^T = \left(\begin{bmatrix} L_1^T & L_2^T \\ 0 & 0 \end{bmatrix} P^T\right)^T = \begin{bmatrix} L^T & 0 \\ 0 & 0 \end{bmatrix}^T.$$

3) Suppose $\operatorname{rank} M = k$. From part 2), there exist L, P_1, Q such that

$$P_1 MQ = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},$$

where $L \in R^{k \times k}$. Since $\rho M = k$, L is unimodular. Let

$$P = \begin{bmatrix} L^{-1} & 0 \\ 0 & I \end{bmatrix} P_1.$$

4) From part 1), there exists a unimodular Q such that

$$MQ = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} J \\ L \end{bmatrix} = Q^{-1}$$

is unimodular, and

$$J = \begin{bmatrix} I & 0 \end{bmatrix} Q^{-1} = M.$$

■

Corollary 13, part 4) is contained in Lemma 59, p.345 in [6]. However, our proof is more directly applicable to our development.

Another advantage of working in an integral domain is that, if $M \in R^{p \times p}$, $x \in R^p$, and $Mx = 0$, then either $x = 0$ or $\det M = 0$, since

$$(\det M)\, x = (\operatorname{adj} M)\, Mx = 0.$$

We will make frequent use of this fact in developing our main results.

The next result is complimentary to Theorem 5, part 2).

Theorem 14 *If* $\operatorname{Im} E + A \operatorname{Ker} E = R^n$, *then* (E, A) *has unit index.*

Proof. If $E = 0$, then $\operatorname{Im} A = R^n$. From Lemma 8, A is unimodular. Then standard canonical form with $N = 0$ is achieved by letting $P = A^{-1}$, $Q = I$, and $D = 0$. If $E \neq 0$, we apply Corollary 13, part 2) to obtain

$$PEQ = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $\det E_{11} \neq 0$. Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x \in R^n.$$

Since R is an integral domain, $PEQx = 0$ implies $x_1 = 0$, so

$$\operatorname{Ker} PEQ = \operatorname{Im} \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \operatorname{Im} \begin{bmatrix} E_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \operatorname{Im} PEQ + PAQ \operatorname{Ker} PEQ = P\left(\operatorname{Im} E + A \operatorname{Ker} E\right) = R^n.$$

From Lemma 8, E_{11} and A_{22} are unimodular. From Lemma 9, part 1), (E, A) has unit index. ■

The next theorem, complementary to Theorem 7, is our main result.

Theorem 15 *If*

1) $\operatorname{rank} E = \rho E$,

2) $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$,

3) (E, A) *is presolvable,*

then (E, A, B) *is impulse controllable.*

Proof. Presolvability of (E, A) admits three cases. If PS1) holds, (E, A) has unit index from Theorem 14. Setting $K = 0$, $(E, A + BK)$ has unit index and (E, A, B) is impulse controllable. To analyze the remaining cases, we invoke Corollary 13, part 3). Let

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = PEQ, \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = PAQ, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB,$$

where partitioning conforms to $n = n_1 + n_2$. If PS2) holds,

$$\operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} \cap \operatorname{Im} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \operatorname{Im} PEQ \cap PAQ \operatorname{Ker} PEQ = P\left(\operatorname{Im} E \cap A \operatorname{Ker} E\right) \neq 0,$$

so there exist x and y such that

$$\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x \neq 0.$$

Hence, $x \neq 0$ and $A_{22}x = 0$. Since R is an integral domain, $\det A_{22} = 0$. Similarly, if PS3) holds,

$$\operatorname{Ker} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} = \operatorname{Ker} PEQ \cap \operatorname{Ker} PAQ = Q^{-1}\left(\operatorname{Ker} E \cap \operatorname{Ker} A\right) \neq 0,$$

so there exists $x \neq 0$ such that

$$\begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x = 0.$$

Hence, $A_{22}x = 0$ and $\det A_{22} = 0$. In either case, we need consider only singular A_{22}.

Note that

$$\operatorname{Im}\begin{bmatrix} I & A_{12} & B_1 \\ 0 & A_{22} & B_2 \end{bmatrix} = \operatorname{Im} PEQ + PAQ \operatorname{Ker} PEQ + \operatorname{Im} PB = P\left(\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B\right) = R^n,$$

so $\operatorname{Im}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = R^{n_2}$. Let $r = \operatorname{rank} A_{22}$. From Corollary 13, part 2), there exist P_1 and Q_1 such that

$$P_1 A_{22} Q_1 = \begin{bmatrix} \widehat{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\widehat{A} \in R^{r\times r}$. Let

$$\begin{bmatrix} \overline{B} \\ \overline{C} \end{bmatrix} = P_1 B_2.$$

Then

$$\operatorname{Im}\begin{bmatrix} \widehat{A} & \overline{B} \\ 0 & \overline{C} \end{bmatrix} = \operatorname{Im}\begin{bmatrix} P_1 A_{22} Q_1 & P_1 B_2 \end{bmatrix} = P_1 \operatorname{Im}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = R^{n_2}$$

and $\operatorname{Im}\overline{C} = R^{n_2 - r}$. From Corollary 13, part 1) and Lemma 8, there exists a unimodular Q_2 such that

$$\overline{C} Q_2 = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Let

$$\begin{bmatrix} \widetilde{B} & \widehat{B} \end{bmatrix} = \overline{B} Q_2$$

and

$$P_2 = \begin{bmatrix} I & -\widetilde{B} \\ 0 & I \end{bmatrix}.$$

Then

$$\operatorname{Im}\begin{bmatrix} \widehat{A} & 0 & \widehat{B} \\ 0 & I & 0 \end{bmatrix} = \operatorname{Im}\left(P_2 \begin{bmatrix} \widehat{A} & \overline{B} \\ 0 & \overline{C} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Q_2 \end{bmatrix}\right) = P_2 \operatorname{Im}\begin{bmatrix} \widehat{A} & \overline{B} \\ 0 & \overline{C} \end{bmatrix} = R^{n_2},$$

and $\operatorname{Im}\begin{bmatrix} \widehat{A} & \widehat{B} \end{bmatrix} = R^r$, so Lemma 8 guarantees the existence of a right inverse. From Corollary 13, part 4), there are W and Y such that

$$U = \begin{bmatrix} \widehat{A} & \widehat{B} \\ W & Y \end{bmatrix}$$

is unimodular. Let $K_1 \in R^{m \times n_1}$ be arbitrary,

$$K_2 = Q_2 \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} Q_1^{-1}, \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} Q^{-1}.$$

Then

$$P_2 P_1 \left(A_{22} + B_2 K_2\right) Q_1 = P_2 P_1 A_{22} Q_1 + \left(P_2 P_1 B_2 Q_2\right) \left(Q_2^{-1} K_2 Q_1\right) = \begin{bmatrix} \widehat{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widehat{B} \\ I & 0 \end{bmatrix} \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} = U,$$

so $A_{22} + B_2 K_2$ is unimodular. From Lemma 9, part 1),

$$\left(PEQ, P\left(A + BK\right)Q\right) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} + B_1 K_1 & A_{12} + B_1 K_2 \\ A_{21} + B_2 K_1 & A_{22} + B_2 K_2 \end{bmatrix} \right)$$

has unit index, so $(E, A + BK)$ has unit index and (E, A, B) is impulse controllable. ∎

Let $\mathcal{I}$ be the set of all impulse eliminating K. The arguments used in the proof of Theorem 15 can be generalized to construct a large subset of $\mathcal{I}$. We begin by fixing $P_1, P_2, Q, Q_1, Q_2, A_{22}, B_2, \widehat{A}, \widehat{B}$ as above. Then, for any K_1, W, Y, T, V with V and

$$U = \begin{bmatrix} \widehat{A} & \widehat{B} \\ W & Y \end{bmatrix}$$

unimodular, we set

$$K_2 = Q_2 \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T & V \end{bmatrix} Q_1^{-1}.$$

It follows that

$$P_2 P_1 \left(A_{22} + B_2 K_2\right) Q_1 = \begin{bmatrix} \widehat{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widehat{B} \\ I & 0 \end{bmatrix} \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} = U \begin{bmatrix} I & 0 \\ T & V \end{bmatrix}$$

is unimodular. Setting $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} Q^{-1}$ guarantees that $(E, A + BK)$ has unit index.

We note that the map $\pi\left(K_1, W, Y, T, V\right) = K$ is one-to-one. Indeed, if we choose K in the range of π, then K_1 is uniquely determined, and setting $L = Q_2^{-1} K_2 Q_1$ yields

$$\begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T & V \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

so

$$T = L_{21}, \quad V = L_{22}, \quad Y = L_{12}L_{22}^{-1}, \quad W = L_{11} - L_{12}L_{22}^{-1}L_{21}.$$

Hence, π may be considered a parametrization of the set of all impulse eliminating K with unimodular V (i.e. the 2,2 block of $Q_2^{-1}K_2Q_1$). Unfortunately, this may not be a complete parametrization of $\mathcal{I}$, as the example

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

illustrates. Here, direct calculation shows that $\mathcal{I}$ consists of all matrices of the form

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

with k_{12} a unit. However, π only yields those matrices of the form

$$K = \begin{bmatrix} W + YT & YV \\ T & V \end{bmatrix}$$

with V and Y units. Although π does predict that $k_{12} = YV$ must be a unit, it does not allow $k_{22} = V$ to be a non-unit, in spite the admissibility of such values. Hence, the range of π is a proper subset of $\mathcal{I}$.

4 Applications to Time-Varying Singular Systems

In this section, we consider time-varying differential equations

$$E(t)\,\dot{x} = A(t)\,x + B(t)\,u, \tag{17}$$

where the entries of E, A, and B belong to a ring of real-valued functions on $\mathbb{R}$. We assume $E(t), A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$. The interesting case occurs when $E(t)$ is singular on a subset of $\mathbb{R}$. Such systems have been studied at length under the assumption that E, A, and B are either constant [7] or real analytic [3], [4]. We will show that these cases fit into our algebraic framework, and examine certain additional classes of functions that can be treated in our setting. Our work does not apply to problems where E, A, B, and K are allowed to have arbitrary entries in C^n (as in [17] and [18]), since C^n is not Hermite.

In studying (17), it is useful to consider a change of variables of the form $x = Q(t)\,z$, where $Q(t)$ is everywhere nonsingular and where both Q and Q^{-1} belong to a given class of functions. Assuming differentiability of Q, direct substitution yields the equivalent system

$$P(t)\,E(t)\,Q(t)\,\dot{z} = P(t)\left(A(t)\,Q(t) - E(t)\,\dot{Q}(t)\right)z + P(t)\,B(t)\,u, \tag{18}$$

where $P(t)$ is also nonsingular for every t. (Note the relationship of (18) to the group action (12).)

Another important consideration in working with any kind of differential equation is that of solvability. Roughly, this means that (17) exhibits existence and uniqueness of solutions over a large class of forcing functions u. In the case of equations based on matrices over the real analytic functions $\mathcal{A}(\mathbb{R})$, Campbell and Petzold [3] define (E, A) to be *analytically solvable* if, for every C^n function u, the system

$$E(t)\,\dot{x} = A(t)\,x + u \tag{19}$$

has at least one C^1 solution x on $\mathbb{R}$ and no two distinct solutions coincide for any t. They then proceed to show that analytic solvability is equivalent to the existence of analytic nonsingular matrices P and Q that put (18) into standard canonical form. Hence, analytic solvability is equivalent to algebraic solvability.

In the time-invariant setting, analytic solvability of (17) reduces to the condition that the matrix pencil (E, A) be *regular* – i.e.

$$\det(sE - A) \not\equiv 0. \tag{20}$$

(See [8], pp.45-49.) From [8], p.28, Theorem 3, (20) is equivalent to the existence of nonsingular $P, Q \in \mathbb{R}^{n\times n}$ that put the pencil into *Weierstrass canonical form*:

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \tag{21}$$

where N is nilpotent. Since $\dot{Q} = 0$, (21) and (7) are the same, so (20) is equivalent to algebraic solvability.

In addition to solvability, we note that the unit index property is a natural concept in both the constant and real analytic settings, occurring iff $N \equiv 0$.

In order to study the impulsive behavior of singular systems, we must adopt a more sophisticated viewpoint based on distribution theory. In (19) we may investigate the consequences of applying an input u, which is arbitrary C^1 up to time $t = t_0$ and drops abruptly to 0 at t_0. As discussed in [15], Chapter 22, the resulting solution exists as a distribution and is, in fact, the unique distribution x satisfying $x(t) = 0$ for $t < t_0$ and

$$E(t)\,\dot{x} = A(t)\,x + \delta_{t_0} E(t_0)\,x_0, \tag{22}$$

where δ_{t_0} is the unit impulse and $x_0 = \lim_{t\to t_0^-} x(t)$. Equation (22) gives a precise meaning to the natural response of (17) with arbitrary initial conditions.

Our principal objective is to find a matrix $K(t)$, whose entries reside in the *same ring of functions as the entries of* E, A, *and* B, and such that the state feedback law $u = K(t)x$ yields a unit index closed-loop system

$$E(t)\dot{x} = (A(t) + B(t)K(t))x + \delta_{t_0}E(t_0)x_0. \tag{23}$$

Thus we are simultaneously treating a wide variety of constrained feedback problems, which have not been considered in the literature.

In order to apply our results to (17), we first need to identify a function ring R that satisfies the conditions that 1) R is an Hermite domain, 2) R is closed under differentiation, 3) solvability in the classical sense implies presolvability, and 4) the analytic and algebraic notions of the unit index property coincide. Note that it follows from 4) that the analytic and algebraic notions of impulse controllability must also coincide. Once these conditions are established, we are guaranteed that the results of Sections 2 and 3 apply to systems over R. In particular, Theorems 7 and 15 give necessary and sufficient algebraic conditions under which (17) is impulse controllable. It remains only to translate conditions 1) and 2) from Theorems 7 and 15 into analytic terms.

For the remainder of this paper, we restrict ourselves to subrings R (with identity) of $\mathcal{A}(\mathbb{R})$. Properties 1) and 2) will have to be established case-by-case. On the other hand, 3) and 4) hold automatically for $\mathcal{A}(\mathbb{R})$ as a consequence of previous results. Indeed, condition 3) may be established by examining the proof of Theorem 2 in [3]. In the light of our Theorem 12 and its corollaries, the arguments used by Campbell and Petzold carry over verbatim to R, demonstrating that analytic solvability of (E, A) guarantees algebraic solvability and, therefore, presolvability. To establish 4), suppose (E, A) is analytically (and algebraically) solvable. If $N \equiv 0$, then (E, A) has unit index in the algebraic sense with $D = -\dot{Q}$. Conversely, suppose (E, A) has algebraic unit index. Then, from Theorem 3, we may choose Q such that setting $D = -\dot{Q}$ yields P that achieves (21) with $N = 0$. Hence, the two notions of unit index coincide. This establishes that our algebraic theory applies to any Hermite subring of $\mathcal{A}(\mathbb{R})$ which is closed under differentiation.

Time-Invariant Systems: To treat time-invariant systems

$$E\dot{x} = Ax + Bu,$$

set $R = \mathbb{R}$. Since $\mathbb{R}$ is a field, it is Hermite. Viewing $\mathbb{R}$ as the set of constant functions, it is closed under differentiation. We therefore conclude that Theorems 7 and 15 specialize to the characterization of time-invariant impulse controllability first established in [16]. The proofs of Theorems 7 and 15 thus constitute an alternative to the known proofs of this result, as presented in [10] Theorem 6, [12], Theorem 2.5.1, and [13], Theorem 3-2.1.

General Analytic Systems: For $R = \mathcal{A}(\mathbb{R})$, [5], Lemma 1 shows that $\mathcal{A}(\mathbb{R})$ is Hermite. (In fact, it is shown in [11], Theorem 1.19 that $\mathcal{A}(\mathbb{R})$ is a Bezout domain.) R is closed under differentiation, so conditions 1) and 2) of

Theorems 7 and 15 are necessary and sufficient for impulse controllability. It remains to link the algebraic conditions to analytic conditions on $E(t)$, $A(t)$, and $B(t)$.

Theorem 16 *Conditions 1) and 2) of Theorems 7 and 15 hold for* $R = \mathcal{A}(\mathbb{R})$ *iff* $\operatorname{rank}_{\mathbb{R}} E(t)$ *is constant and* $\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n$ *for every* $t \in \mathbb{R}$.

Proof. (Sufficient) Suppose $\operatorname{rank}_{\mathbb{R}} E(t) = k$. Then $\operatorname{rank} E = k$ and, from Corollary 13, part 2), there exist unimodular P and Q such that

$$PEQ = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where $E_{11} \in R^{k \times k}$ and $\operatorname{rank} E_{11} = k$. But $\operatorname{rank}_{\mathbb{R}} E_{11}(t)$ must also be constant, so E_{11} is unimodular. Let

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = PAQ, \qquad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB.$$

Then

$$\operatorname{Im} \begin{bmatrix} E_{11}(t) & A_{12}(t) & B_1(t) \\ 0 & A_{22}(t) & B_2(t) \end{bmatrix} = \operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n$$

for every t, so $\operatorname{rank}_{\mathbb{R}} \begin{bmatrix} A_{22}(t) & B_2(t) \end{bmatrix} = n - k$. Let $\{\mu_i(t)\}$ be the $(n-k)$th-order minors of $\begin{bmatrix} A_{22}(t) & B_2(t) \end{bmatrix}$. Each μ_i is an analytic function and the μ_i have no common zero. Hence, $u = \sum \mu_1^2$ has no zero and is therefore a unit of R. Also,

$$\sum \left(\frac{\mu_i}{u} \right) \mu_i = 1,$$

so $\rho \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = n - k$. From Corollary 13, part 1), there exists a unimodular Q_1 such that

$$\begin{bmatrix} A_{22} & B_2 \end{bmatrix} Q_1 = \begin{bmatrix} I & 0 \end{bmatrix}.$$

If $x \in R^{n-k}$, then

$$\begin{bmatrix} A_{22} & B_2 \end{bmatrix} Q_1 \begin{bmatrix} x \\ 0 \end{bmatrix} = x,$$

so $x \in \operatorname{Im} \begin{bmatrix} A_{22} & B_2 \end{bmatrix}$. But x is arbitrary, so $\operatorname{Im} \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = R^{n-k}$. The theorem follows from Lemma 8 and Lemma 9, part 2).

(Necessary) From Corollary 13, part 3), there exist unimodular P and Q such that

$$P(t) E(t) Q(t) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

for every t. Hence, $\operatorname{rank}_{\mathbb{R}} E(t)$ is constant. Let $x \in \mathbb{R}^n$. Viewing x as a constant function, it follows from $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$ that there exist $u \in R^m$ and $y, z \in R^n$ such that $Ez = 0$ and $Ey + Az + Bu = x$. But this means $E(t) z(t) = 0$ and $E(t) y(t) + A(t) z(t) + B(t) u(t) = x$ for every t, so $\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n$. ■

Theorem 16 shows that Theorems 7 and 15 specialize to Theorem 4.1 of [4] for systems over the real analytic functions.

Now we apply our theory to classes of time-varying singular systems (17) which have not been previously studied.

Polynomial Systems: Let $R = \mathbb{R}[t]$ be the polynomials on $\mathbb{R}$ with real coefficients. $\mathbb{R}[t]$ is a subring of $\mathcal{A}(\mathbb{R})$ containing 1 and a principal ideal domain, so it is Hermite. $\mathbb{R}[t]$ is closed under differentiation. Theorem 16 applies to $\mathbb{R}[t]$ without modification.

Periodic Systems: Let $\mathcal{P}(\tau)$ be the analytic functions on $\mathbb{R}$ with period $\tau > 0$. (τ need not be the fundamental period.) $\mathcal{P}(\tau)$ is a subring of $\mathcal{A}(\mathbb{R})$ containing 1 and is closed under differentiation.

Theorem 17 *$\mathcal{P}(\tau)$ is a Bezout domain.*

Proof. We need to show that every finitely generated ideal in $\mathcal{P}(\tau)$ is principal. It suffices to show that, for every $a, b \in \mathcal{P}(\tau)$, there exists $c \in \mathcal{P}(\tau)$ such that $cR = aR + bR$. In view of [9], Theorem 3.7, p.78, a and b have finitely many zeros in any bounded interval. Let $\{z_1, \ldots, z_q\}$ be the common zeros of a and b in the interval $[0, \tau)$, counting multiplicities, and define

$$c(t) = \prod_k \left(e^{2\pi i \frac{t}{\tau}} - e^{2\pi i \frac{z_k}{\tau}} \right).$$

Then $c \in \mathcal{P}(\tau)$ with zeros $\{z_k\}$, and c is a common divisor of a and b. Let $\overline{a} = a/c$ and $\overline{b} = b/c$. If $x, y \in R$, then

$$ax + by = c\left(\overline{a}x + \overline{b}y\right) \in cR,$$

so $aR + bR \subset cR$. To prove the converse, note that $\overline{a}$ and $\overline{b}$ have no common zero, so $u = \overline{a}^2 + \overline{b}^2$ has no zero and is, therefore, a unit of R. For any $r \in R$, set $x = \overline{a}r/u$ and $y = \overline{b}r/u$. Then

$$cr = cr\frac{\overline{a}^2 + \overline{b}^2}{u} = ax + by \in aR + bR,$$

so $cR \subset aR + bR$. ■

It follows from Theorem 17 that $\mathcal{P}(\tau)$ is an Hermite domain. It can be further shown that $\mathcal{P}(\tau)$ is a principal ideal domain. Theorem 16 applies to $\mathcal{P}(\tau)$ without modification.

Systems Analytic at ∞: Let $\mathcal{A}_\infty(\mathbb{R})$ be the subring of $\mathcal{A}(\mathbb{R})$ consisting of all functions analytic at ∞. (x analytic at ∞ means that $x\left(\frac{1}{t}\right)$ is analytic at 0.) From the chain rule,

$$\dot{x}\left(\frac{1}{t}\right) = -t^2 \frac{d}{dt}\left(x\left(\frac{1}{t}\right)\right),$$

so $\mathcal{A}_\infty(\mathbb{R})$ is closed under differentiation.

Theorem 18 *$\mathcal{A}_\infty(\mathbb{R})$ and $\mathcal{P}(\tau)$ are isomorphic.*

Proof. Let

$$\phi(t) = \begin{cases} \tan\left(\pi \frac{t}{\tau}\right), & t \neq \left(k+\frac{1}{2}\right)\tau \\ \infty, & t = \left(k+\frac{1}{2}\right)\tau \end{cases}.$$

ϕ has period τ and is analytic, except for poles at $\left(k+\frac{1}{2}\right)\tau$. $1/\phi$ is analytic about $\left(k+\frac{1}{2}\right)\tau$, where it has a zero. For any $x \in \mathcal{A}_\infty(\mathbb{R})$, define $x_p(t) = x(\phi(t))$. Then x_p has period τ. Since $x\left(\frac{1}{t}\right)$ is analytic about 0, x_p is analytic about $\left(k+\frac{1}{2}\right)\tau$ and therefore on all of $\mathbb{R}$. Hence, the map $h : x \to x_p$ takes $\mathcal{A}_\infty(\mathbb{R})$ into $\mathcal{P}(\tau)$ and is obviously a ring homomorphism. Since the range of ϕ is $\mathbb{R}$, $x_p \equiv 0$ implies $x \equiv 0$, and h is $1-1$. Given any $x_p \in \mathcal{P}(\tau)$, $x(t) = x_p\left(\frac{\tau}{\pi}\arctan(t)\right)$ defines a function in $\mathcal{A}_\infty(\mathbb{R})$. But $\frac{\tau}{\pi}\arctan(\phi(t)) = t$, so $h(x) = x_p$ and h is onto. ■

It follows from Theorems 17 and 18 that $\mathcal{A}_\infty(\mathbb{R})$ is an Hermite domain.

The conditions of Theorem 16 must be augmented to handle analyticity at ∞.

Theorem 19 *Conditions 1) and 2) of Theorems 7 and 15 hold for $R = \mathcal{A}_\infty(\mathbb{R})$ iff*

$$\operatorname*{rank}_{\mathbb{R}} E(t) = \operatorname*{rank}_{\mathbb{R}} E(\infty),$$

$$\operatorname{Im} E(t) + A(t)\operatorname{Ker} E(t) + \operatorname{Im} B(t) = \operatorname{Im} E(\infty) + A(\infty)\operatorname{Ker} E(\infty) + \operatorname{Im} B(\infty) = \mathbb{R}^n$$

for every $t \in \mathbb{R}$.

Proof. (Sufficient) Suppose $\operatorname{rank}_{\mathbb{R}} E(t) = k$. As in the proof of Theorem 16, there exist unimodular P and Q such that

$$PEQ = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where $E_{11} \in R^{k\times k}$ and $\operatorname{rank} E_{11} = k$. But $\operatorname{rank}_{\mathbb{R}} E_{11}(t) = \operatorname{rank}_{\mathbb{R}} E_{11}(\infty) = k$, so E_{11} is unimodular. Then

$$\operatorname{Im}\begin{bmatrix} E_{11}(t) & A_{12}(t) & B_1(t) \\ 0 & A_{22}(t) & B_2(t) \end{bmatrix} = \operatorname{Im} E(t) + A(t)\operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n$$

for every t (including $t = \infty$), so $\operatorname{rank}_{\mathbb{R}}\begin{bmatrix} A_{22}(t) & B_2(t) \end{bmatrix} = n-k$ and the minors $\{\mu_i\}$ have no common finite or infinite zero. The remainder of the sufficiency proof proceeds without modification.

(Necessary) From Corollary 13, part 3), there exist unimodular P and Q such that

$$P(t)E(t)Q(t) = P(\infty)E(\infty)Q(\infty) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $\operatorname{rank}_{\mathbb{R}} E(t) = \operatorname{rank}_{\mathbb{R}} E(\infty)$. Let $x \in \mathbb{R}^n$. Viewing x as a constant function, there exist $u \in R^m$ and $y, z \in R^n$

such that $Ez = 0$ and $Ey + Az + Bu = x$. But this means

$$E(t)\,z(t) = E(\infty)\,z(\infty) = 0,$$

$$E(t)\,y(t) + A(t)\,z(t) + B(t)\,u(t) = E(\infty)\,y(\infty) + A(\infty)\,z(\infty) + B(\infty)\,u(\infty) = x.$$

Since x is arbitrary, the theorem follows. ■

Example: We close this section with a simple example illustrating how our results may be applied to periodic systems. Let

$$T(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

and note that T is unimodular over $\mathcal{P}(2\pi)$. Consider the singular system with

$$E = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

(E, A, B) is already in standard canonical form, so the it is analytically and algebraically solvable. A simple calculation shows that $u \equiv 0$ leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\delta_{t_0} x_{03} \\ -\delta_{t_0} x_{04} \\ 0 \\ 0 \end{bmatrix}.$$

We wish to find an *analytic periodic* state feedback matrix $K(t)$ to eliminate impulses in the closed-loop system.

Note that $\operatorname{rank}_{\mathbb{R}} E(t) = 2$,

$$\operatorname{Ker} E(t) = \operatorname{Im} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \operatorname{Im} \begin{bmatrix} T(t) & I & 0 \\ 0 & 0 & I \end{bmatrix} = \mathbb{R}^4$$

for every t. Theorems 15 and 16 guarantee that (E, A, B) is impulse controllable.

As in the proof of Theorem 15, we obtain the unimodular matrices

$$P = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & I \\ T^T & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = PAQ = Q,$$

so $A_{22} = 0$, $r = 0$, and $P_1 = Q_1 = P_2 = Q_2 = I$. Let $K_1, V, Y \in (\mathcal{P}(2\pi))^{2\times 2}$ with V, Y unimodular, and apply the parametrization π, as described at the end of Section 3. This yields the state feedback matrix

$$K = \begin{bmatrix} K_1 & YV \end{bmatrix} Q^{-1} = \begin{bmatrix} K_1 & YV \end{bmatrix} \begin{bmatrix} 0 & T \\ I & 0 \end{bmatrix} = \begin{bmatrix} YV & K_1 T \end{bmatrix} \in (\mathcal{P}(2\pi))^{2\times 4}$$

and the periodic closed-loop system

$$\begin{bmatrix} 0 & T(t) \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} I & 0 \\ Y(t)V(t) & I + K_1(t)T(t) \end{bmatrix} x. \tag{24}$$

Our theory guarantees that (24) has unit index. This can be verified directly by interchanging the block columns of (24) and applying Lemma 9, part 1).

5 Conclusion

Our work demonstrates that the solutions of the state feedback impulse elimination problem, as originally developed for the time-invariant and time-varying cases in [10] and [4], share a common algebraic basis. Once exposed, this structure lends itself naturally to numerous generalizations, requiring only a small amount of analytic effort to turn the problem into algebra. The rings discussed in this paper are only a few of the many possibilities. For example, it is easy to show that similar conclusions hold for the real analytic functions with an isolated singularity at ∞, those with a pole or removable singularity at ∞, those with a zero of order at least k at a fixed point in $\mathbb{R} \cup \{\infty\}$, rational functions with no pole in $\mathbb{R}$, etc. Perhaps the greatest challenge is to fully exploit our theory by proposing an Hermite domain which is not PID, Bezout, etc. We leave this question for further research.

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