A CHARACTERIZATION OF BOUNDED-INPUT 
BOUNDED-OUTPUT STABILITY FOR LINEAR TIME-VARYING 
SYSTEMS WITH DISTRIBUTIONAL INPUTS

DANIELI COBB† AND CHI-JO WANG‡

Abstract. We consider the problem of extending the concept of bounded-input bounded-output stability to linear time-varying systems with distributional inputs. In particular, the notion of impulse response is examined in a functional analytic setting. This requires that we first extend the classical notion of an integral operator to distribution space. Duality theory for several key normed spaces is then examined. Next, the adjoint operator corresponding to the given system is studied. Finally, necessary and sufficient conditions for stability are established, along with several expressions for the "gain" of the system.

Key words. bounded-input bounded-output stability, distributions, time-varying systems

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1. Introduction. The concept of impulse response has traditionally played a 
central role in linear system theory. In spite of this fact, certain fundamental system-
theoretic ideas have apparently not been developed on a mathematically rigorous level 
for systems with arbitrary distributional inputs and outputs. In a previous paper 
we addressed the problem of characterizing bounded-input bounded-output (BIBO) 
stability in the time-invariant case. In this paper we extend the theory to include 
time-varying linear systems.

To frame the problem, recall that in classical system theory a “system” is typically 
viewed as an integral operator

\[ y(t) = \int_{-\infty}^{\infty} h(t, \tau)u(\tau)d\tau. \]

It can be shown (e.g., see [2, p. 109]) that (1.1) determines a bounded linear operator 
on \( L^\infty \) if and only if

\[ \sup_t \int_{-\infty}^{\infty} |h(t, \tau)| d\tau < \infty. \]

Such a characterization is inadequate, however, for studying systems with distribu-
tional inputs \( u \), since the integral (1.1) is not defined. In spite of this fact, the kernel 
\( h(t, \tau) \) is often referred to as the system “impulse response.” Furthermore, there 
are many common systems where \( h \) itself is a distribution. For example, consider the 
“time-varying gain”

\[ y(t) = \beta(t)u(t). \]

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†Department of Electrical and Computer Engineering, University of Wisconsin, 1415 Engineering Drive, Madison, WI 53706-1691 (cobbe@engr.wisc.edu).
‡Department of Electrical Engineering, Southern Taiwan University of Technology, Yung-Kang, 
Taiwan 710, Taiwan (chiJo@ms2.hinet.net).
Based on formal manipulations,

\[ h(t, \tau) = \beta(t) \delta(t - \tau), \]

where \( \delta \) is the unit impulse; condition (1.2) cannot be applied directly to distributions. Our goal is to develop a more general theory that characterizes stability for systems with distributional inputs and distributional impulse responses.

In section 2, we present basic analytic results that will be required in subsequent sections. In section 3, we study families of distributions in one variable satisfying certain smoothness properties in the index. These are then interpreted as distributions in two variables and used to generalize the notion of an integral operator. Section 4 applies the theory of normed-space extensions, developed in [1], to distributions in two variables. It is shown in Theorem 4.3 that the space of BIBO stable kernels (i.e., functions satisfying (1.2)) extends naturally to the space of distributions which are derivatives of functions of uniformly bounded variation \( DUBV \). Section 5 contains the main results of the paper. Theorem 5.3 states that BIBO stable linear systems are precisely those with \( DUBV \) kernels, Theorem 5.4 gives an expression for the adjoint system, and Theorem 5.5 establishes several equivalent representations of the system gain.

2. Preliminaries. First we present some pertinent facts concerning the theory of distributions. See [3], [4], and [5] for more detail. If \( \phi : \mathbb{R}^n \to \mathbb{R} \), define the support of \( \phi \) (denoted supp \( \phi \)) to be the closure of the set \( \{(t_1, \ldots, t_n) \in \mathbb{R}^n | \phi(t_1, \ldots, t_n) \neq 0 \} \). Let \( K_n \) be the space of \( C^\infty \) functions \( \phi : \mathbb{R}^n \to \mathbb{R} \) with supp \( \phi \) bounded. Convergence in \( K_n \) can be defined in several ways. When we assign a norm \( \| \cdot \| \) to \( K_n \), we will refer to the pair \( (K_n, \| \cdot \|) \). For example, \( K_1 \subset L^p, 1 \leq p \leq \infty \), so we may consider \((K_1, \| \cdot \|_p)\). Also,\n
\[ \| \psi \|_{L^p} = \left( \int_{-\infty}^{\infty} |\psi(t, \tau)|^p dt \right)^{1/p} < \infty \]

for \( \psi \in K_2 \), so \((K_2, \| \cdot \|_{L^p})\) is well defined for \( 1 \leq p < \infty \). Strong convergence \( \phi_k \to 0 \) in \( K_n \) means that there exists \( a < \infty \) such that supp \( \phi_k \subset [-a, a] \) for every \( k \) and \( \| \phi_k \|_{C^p} \to 0 \) for every integer \( p \geq 0 \), where

\[ \| \phi \|_{C^p} = \max \left\{ \left| \frac{\partial^{i_1 + \cdots + i_n} \phi(t_1, \ldots, t_n)}{\partial t_1^{i_1} \cdots \partial t_n^{i_n}} \right| : 0 \leq i_1 + \cdots + i_n \leq p; t_1, \ldots, t_n \in \mathbb{R} \right\}. \]

A distribution \( f \) is an element of \( K_n' \), the dual space of \( K_n \) under strong convergence. For \( f \in K_n, \) supp \( f \) is defined to be the complement of the largest open set \( U \subset \mathbb{R}^n \) such that supp \( \phi \subset U \) implies \( \langle f, \phi \rangle = 0 \). Let \( K_{n+1} \) be the set of all \( f \in K_n \) such that there exists \( a \in \mathbb{R} \) with supp \( f \subset [a, \infty) \). Also let \( K_{2n+1} \) be the set of all \( f \in K_{2n+1} \) such that there exists a function \( a : \mathbb{R} \to \mathbb{R} \) with \( a(\tau) \to \infty \) as \( \tau \to \infty \) and supp \( f \subset \{ t \geq a(\tau) \} \). Note that \( K_{2n+1} \) is a subspace of \( K_n \). Weak* convergence \( f_k \to 0 \) in \( K_n \) means that \( \langle f_k, \phi \rangle \to 0 \) for every \( \phi \in K_n \). One basis of \( \text{weak}^* \) neighborhoods of 0 in \( K_n \) consists of all sets of the form \( \{ f \ | \ \| f, \phi \| < \epsilon; i = 1, \ldots, m \} \), where \( \epsilon > 0 \) and \( \phi_1, \ldots, \phi_m \in K_n \) are arbitrary.

The partial derivative of \( f \in K_n \) with respect to \( t_i \) is defined by \( \frac{\partial f}{\partial t_i} \phi = -\langle f, \frac{\partial \phi}{\partial t_i} \rangle \) for \( \phi \in K_n \). It follows that the differentiation operator \( f \mapsto \frac{\partial f}{\partial t_i} \) is (weak*) continuous. In case \( n = 1 \), we denote a derivative by \( \frac{df}{dt} \) or by an overdot \( \dot{f} \); the kth
derivative will be denoted \( \frac{d^k f}{dt^k} \) or \( f^{(k)} \). For any \( f \in K_1' \), define the to-translation \( \Delta t_0 f \) by \( \langle \Delta t_0 f, \phi \rangle = \langle f, \phi_{-t_0} \rangle \), where \( \phi_{-t_0}(t) = \phi(t + t_0) \). By a routine calculation, \( \frac{d}{dt} \Delta t_0 f = \Delta t_0 f \). Multiplication of \( f \in K_1 \) by a \( C^\infty \) function \( \gamma \) is defined by \( \langle f \gamma, \phi \rangle = \langle f, \gamma \phi \rangle \).

A Lebesgue measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) is locally integrable if \( \int_A |f| < \infty \) for every bounded interval \( A \subset \mathbb{R}^n \). Every locally integrable \( f \) determines a distribution according to

\[
\langle f, \phi \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n) \phi(t_1, \ldots, t_n) \, dt_1 \cdots dt_n.
\]

Note that the unit step function

\[
\theta(t) = \begin{cases} 
1, & t \geq 0, \\
0, & t < 0.
\end{cases}
\]

may be considered a distribution in \( K_1' \); the unit impulse \( \delta \in K_1' \) is defined by \( \langle \delta, \phi \rangle = \phi(0) \). Translations of \( \theta \) and \( \delta \) will be denoted by \( \theta_{t_0} \) and \( \delta_{t_0} \), respectively. It is easily verified that

\[
\theta_{t_0}(t) = \theta(t - t_0), \quad \langle \delta_{t_0}, \phi \rangle = \phi(t_0).
\]

If \( f \) is an absolutely continuous function, \( f \) and its classical derivative \( \dot{f} \) are locally integrable. In this case, the classical and distributional derivatives of \( f \) coincide, since

\[
\langle \dot{f}, \phi \rangle = \int_{-\infty}^{\infty} f(t) \phi(t) \, dt = -\int_{-\infty}^{\infty} f(t) \dot{\phi}(t) \, dt = \langle f, \dot{\phi} \rangle.
\]

Consider the spaces

\[
BV = \{ g : \mathbb{R} \to \mathbb{R} \mid \var_{t^\infty} g(t) < \infty \},
\]

\[
NBV = \{ g \in BV \mid g \text{ is left-continuous and } g(\infty) = 0 \}
\]

with norm \( \|g\|_{NBV} = \var_{t^\infty} g(t) \). (Note that we are deviating slightly from the conventional definition of \( NBV \), as in [9, p. 171].)

In [1] we also considered the space \( DBV = \{ \dot{g} \mid g \in NBV \} \) with norm \( \|\dot{g}\|_{DBV} = \|g\|_{NBV} \). We showed in [1, p. 980] that \( DBV \) is isometrically isomorphic to the dual space of \( (K_1, \| \cdot \|_\infty) \). Furthermore, for any \( g \in NBV \) and \( \phi \in K_1 \), \( \langle \dot{g}, \phi \rangle = \int_{-\infty}^{\infty} \dot{g}(t) \phi(t) \, dt \). We need to generalize these ideas to distributions on \( \mathbb{R}^2 \). The appropriate construction requires a preliminary result.

**Lemma 2.1.** Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be Lebesgue measurable and \( g(t, \cdot) \in NBV \) for a.e. \( t \). Then the function

\[
v(t, \tau) = \var_{\eta^2 \tau} g(t, \eta)
\]

is Lebesgue measurable on \( \mathbb{R}^2 \).

**Proof.** Enumerate the rationals \( \{r_n\} \), and consider the partition \( \pi_n = (r_{k_1}, \ldots, r_{k_n}) \) of \( \mathbb{R} \), where \( \{k_1, \ldots, k_n\} = \{1, \ldots, n\} \) and

\[
r_{k_1} < r_{k_2} < \cdots < r_{k_n}.
\]
Let \( A = \{ t \mid g(t, \cdot) \in NBV \} \). For each \( n, j = 1, \ldots, n-1 \), and \((t, \tau) \in A \times (r_{kj}, r_{kj+1})\), define
\[
v_n(t, \tau) = |g(t, r_{kj})| + \sum_{i=j+1}^{n-1} |g(t, r_{ki}) - g(t, r_{ki+1})| + |g(t, \tau) - g(t, r_{kj})|.
\]

For other \((t, \tau)\), set \( v_n(t, \tau) = 0 \). Each \( v_n \) is Lebesgue measurable. Let \( \varepsilon > 0 \), \((t, \tau) \in A \times \mathbb{R} \), and \( \pi = (\tau_1, \ldots, \tau_p) \) be any partition of \((\tau, \infty)\) such that
\[
q(t, \tau) = |g(t, \tau_p)| + \sum_{i=1}^{p-1} |g(t, \tau_i) - g(t, \tau_{i+1})| + |g(t, \tau) - g(t, \tau_1)| > v(t, \tau) - \frac{\varepsilon}{2}.
\]

Since \( g \) is left-continuous, there exists \( N < \infty \) such that \( \pi_n \) contains rationals \( r_{kj_1} \) with \( \tau < r_{kj_1} < \cdots < r_{kj_p} \) such that
\[
|g(t, r_{kj_i}) - g(t, \tau_i)| < \frac{\varepsilon}{4p}
\]
for every \( i \). Thus
\[
v_n(t, \tau) \geq |g(t, \tau_p)| + \sum_{i=1}^{p-1} |g(t, r_{kj_i}) - g(t, r_{kj_{i+1}})| + |g(t, \tau) - g(t, r_{kj_1})|
\]
\[
\geq |g(t, \tau_p)| \quad \geq |g(t, \tau_{kj_1}) - g(t, \tau_{kj_2})|
\]
\[
+ \sum_{i=1}^{p-1} |g(t, \tau_i) - g(t, \tau_{i+1})| - \sum_{i=1}^{p-1} |g(t, r_{kj_i}) - g(t, \tau_i)|
\]
\[
- \sum_{i=1}^{p-1} |g(t, r_{kj_{i+1}}) - g(t, \tau_{i+1})|
\]
\[
+ |g(t, \tau) - g(t, \tau_1)| - |g(t, r_{kj_1}) - g(t, \tau_1)|
\]
\[
\geq q(t, \tau) - \frac{\varepsilon}{2}
\]
\[
> v(t, \tau) - \varepsilon.
\]

So \( v_n \to v \) a.e., and \( v \) is Lebesgue measurable. \( \Box \)

In particular, if \( g \) satisfies the conditions of Lemma 2.1, the map \( t \to \text{var}_r g(t, \tau) \) is Lebesgue measurable. Hence, we may define \( UBV \) to be the set of functions \( g : \mathbb{R}^2 \to \mathbb{R} \) satisfying the following properties:

(UBV1) \( g \) is Lebesgue measurable.

(UBV2) \( g(t, \cdot) \in NBV \) for a.e. \( t \).

(UBV3) \( \text{ess sup}_t \text{var}_r g(t, \tau) < \infty \).

We refer to \( UBV \) as the functions of \textit{uniformly bounded variation}. Let \( \| g \|_{UBV} = \text{ess sup}_t \text{var}_r g(t, \tau) \). Each \( g \in UBV \) is bounded, since
\[
|g(t, \tau)| \leq \text{var}_r g(t, \tau) \leq \| g \|_{UBV}.
\]

Thus \( g \in K'_2 \), and we may also define the set of partial derivatives
\[
DUBV = \left\{ \frac{\partial g}{\partial \tau} \mid g \in UBV \right\}.
\]
It is routine to verify that $UBV$ and $DUBV$ are linear spaces and that $\|\cdot\|_{UBV}$ and
\[
\left\| \frac{\partial g}{\partial \tau} \right\|_{DUBV} = \|g\|_{UBV}
\]
are norms on $UBV$ and $DUBV$.

For $1 \leq p < \infty$, let $UL^p$ be the space of Lebesgue measurable functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying
\[
\text{ess sup}_{t} \int_{-\infty}^{\infty} |f(t, \tau)|^p \, d\tau < \infty.
\]

(Lebesgue measurability of the map $t \to \int_{-\infty}^{\infty} |f(t, \tau)|^p \, d\tau$ follows from [9, Theorem 7.8].) $UL^p$ is the set of uniformly $L^p$ functions and has norm
\[
\|f\|_{UL^p} = \text{ess sup}_{t} \left( \int_{-\infty}^{\infty} |f(t, \tau)|^p \, d\tau \right)^{\frac{1}{p}}.
\]

We may consider $UL^p \subset K_2$, since, from Holder’s inequality,
\[
\int_{-a}^{a} \int_{-a}^{a} |f(t, \tau)| \, d\tau \, dt \leq \int_{-a}^{a} \left( \text{ess sup}_{t} \int_{-a}^{a} |f(t, \tau)| \, d\tau \right) \, dt = 2a \text{ess sup}_{t} \int_{-a}^{a} |f(t, \tau)| \, d\tau \leq (2a)^{\frac{p}{2}} \text{ess sup}_{t} \left( \int_{-a}^{a} |f(t, \tau)|^p \, d\tau \right)^{\frac{1}{p}}.
\]

For $p = \infty$, we define $UL^\infty$ to be the same as $L^\infty$ on $\mathbb{R}^2$.

Support constraints may be placed on the spaces above by setting $L^p_2 = L^p \cap K^1_2$, $UBV_+ = UBV \cap K^1_2$, $DUBV_+ = DUBV \cap K^1_2$, and $UL^p_+ = UL^p \cap K^1_2$.

**Theorem 2.2.** (1) Let $g \in UBV$ and $g_t = g(t, \cdot)$. Then
\[
\left\langle \frac{\partial g}{\partial \tau}, \psi \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau) \, dg_t(\tau) \, dt
\]
for every $\psi \in K_2$.

(2) $UL^1 \subset DUBV$ with $\|f\|_{DUBV} = \|f\|_{UL^1}$ for every $f \in UL^1$.

(3) $DUBV$ is the dual of $(K_2, \|\cdot\|_{1\infty})$.

(4) $UBV$ and $DUBV$ are isometrically isomorphic.

**Proof.** (1) Integration by parts yields
\[
\left\langle \frac{\partial g}{\partial \tau}, \psi \right\rangle = - \left\langle g, \frac{\partial \psi}{\partial \tau} \right\rangle = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_t(\tau) \frac{\partial \psi(t, \tau)}{\partial t} \, d\tau \, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau) \, dg_t(\tau) \, dt
\]
for every $\psi$.

(2) If $f \in UL^1$, there exists a Lebesgue measurable $A \subset \mathbb{R}$ such that $f(t, \cdot) \in L^1$ for every $t \in A$. Let $g(t, \tau) = \int_{-\infty}^{\tau} f(t, \eta) \, d\eta$ for $t \in A$. Then $g$ is Lebesgue measurable, $g(t, \cdot)$ is absolutely continuous, and $f = \frac{\partial g}{\partial \tau}$. The result follows from
\[
\|f\|_{DUBV} = \text{ess sup}_{t} \text{var} \, |g(t, \tau)| = \text{ess sup}_{t} \int_{-\infty}^{\infty} |f(t, \tau)| \, d\tau = \|f\|_{UL^1} < \infty.
\]
(3) We first prove that $DUBV$ is contained in the dual of $K_2$. Let $g$ and $g_r$ be as in (1) and $f = \frac{g}{dt}$. We must show that

$$\sup_{\|\psi\|_{1\infty} = 1} |\langle f, \psi \rangle| = \|f\|_{DUBV}.$$

From (1),

$$|\langle f, \psi \rangle| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau) dg(t, \tau) dt \right|$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\psi(t, \tau)| |dg(t, \tau)| dt$$

$$\leq \int_{-\infty}^{\infty} \sup_{\tau} |\psi(t, \tau)| \text{var} g(t, \tau) dt$$

$$\leq (\text{ess sup}_{\tau} \text{var} g(t, \tau)) \int_{-\infty}^{\infty} \sup_{\tau} |\psi(t, \tau)| dt$$

$$= \|f\|_{DUBV} \|\psi\|_{1\infty}.$$

so

$$\sup_{\|\psi\|_{1\infty} = 1} |\langle f, \psi \rangle| \leq \|f\|_{DUBV}.$$

To establish the reverse inequality, observe that, for $\phi_1, \phi_2 \in K_1$, setting $\psi(t, \tau) = \phi_1(t)\phi_2(\tau)$ yields $\psi \in K_2$ and

$$\|\psi\|_{1\infty} = \int_{-\infty}^{\infty} \sup_{\tau} |\phi_1(t)\phi_2(\tau)| dt = \|\phi_1\|_1 \|\phi_2\|_{\infty}.$$

Thus

$$\sup_{\|\psi\|_{1\infty} = 1} |\langle f, \psi \rangle| = \sup_{\|\psi\|_{1\infty} = 1} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau) dg(t, \tau) dt \right|$$

$$\geq \sup_{\|\phi_1\|_{1\infty} = 1} \sup_{\|\phi_2\|_{\infty} = 1} \left| \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \phi_2(\tau) dg(t, \tau) \right) \phi_1(t) dt \right|$$

$$= \sup_{\|\phi_2\|_{\infty} = 1} \text{ess sup}_{\|\phi_1\|_{1\infty} = 1} \left| \int_{-\infty}^{\infty} \phi_2(\tau) d\gamma(t, \tau) \right|$$

$$= \text{ess sup}_{t} \sup_{\|\phi_2\|_{\infty} = 1} |\langle \gamma(t), \phi_2 \rangle|$$

$$= \text{ess sup}_{t} \|\gamma(t)\|_{DBV}$$

$$= \|f\|_{DUBV}.$$

Next we show that $DUBV$ contains the dual of $(K_2, \|\cdot\|_{1\infty})$. Let $f$ be any continuous linear functional on $(K_2, \|\cdot\|_{1\infty})$. $(f$ is also a distribution, since $\psi_k \rightarrow 0$ strongly in $K_2$ implies $\|\psi_k\|_{1\infty} \rightarrow 0$ and $\langle f, \psi_k \rangle \rightarrow 0$.) Let $\psi_1 \in K_1$. Then each $\psi_2 \in K_1$ determines $\psi \in K_2$ by $\psi(t, \tau) = \phi_1(t)\phi_2(\tau)$ (i.e., $\psi$ is the "direct product" $\psi = \phi_1 \times \phi_2$). The map $\phi_2 \mapsto \psi$ is continuous from $(K_1, \|\cdot\|_{\infty})$ into $(K_2, \|\cdot\|_{1\infty})$, so $\phi_2 \mapsto \langle f, \phi_1 \times \phi_2 \rangle$ is a continuous linear functional on $(K_1, \|\cdot\|_{\infty})$. Since $K_1$ is dense in $C_0$, there exists $G(\phi_1; \cdot) \in NBV$ such that

$$\langle f, \phi_1 \times \phi_2 \rangle = \int_{-\infty}^{\infty} \phi_2(\tau) dG(\phi_1; \tau)$$
for every \( \phi_2 \). It is routine to show that the operator \( \phi_1 \mapsto G(\phi_1, \cdot) \) is linear. Continuity also holds, since

\[
\sup_{\|\phi_1\|_1 = 1} \|G(\phi_1, \cdot)\|_{NBV} = \sup_{\|\phi_1\|_1 = 1} \sup_{\|\phi_2\|_\infty = 1} \left| \int_{-\infty}^{\infty} \phi_2(\tau) dG(\phi_1, \tau) \right|
\]

\[
= \sup_{\|\phi_1\|_1 = 1} \sup_{\|\phi_2\|_\infty = 1} |\langle f, \phi_1 \times \phi_2 \rangle|
\]

\[
\leq \sup_{\|\psi\|_\infty = 1} |\langle f, \psi \rangle|< \infty.
\]

Therefore, \( \phi_1 \mapsto G(\phi_1, \cdot) \) is a continuous linear operator from \( (K_1, \| \cdot \|_1) \) into \( NBV \). From [7, Theorem 2.3.1] and (2.2), there exists \( g \in UBV \) such that

\[
\langle f, \phi_1 \times \phi_2 \rangle = \int_{-\infty}^{\infty} \phi_2(\tau)d \left( \int_{-\infty}^{\infty} g(t, \tau) \phi_1(t)dt \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(t)\phi_2(\tau)dg_t(\tau)dt
\]

for every \( \phi_1, \phi_2 \in K_1 \). Let \( II = \{ \phi_1 \times \phi_2 \mid \phi_1, \phi_2 \in K_1 \} \). By linearity,

\[
(2.3) \quad \langle f, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau)dg_t(\tau)dt
\]

for every \( \psi \) in \( \text{span} \ II \). From [6, p. 65], \( \text{span} \ II \) is strongly dense in \( K_2 \), so it is also dense relative to \( \| \cdot \|_\infty \). By continuity, (2.3) holds for all \( \psi \in K_2 \). From part (1), \( f = \frac{\partial g}{\partial t} \in DUBV \).

(4) Note that the map \( g \mapsto \frac{\partial g}{\partial t} \) from \( UBV \) into \( DUBV \) is defined to be linear, onto, and norm-preserving. It remains to show that the map is one-to-one. From part (1), if \( \frac{\partial g}{\partial t} = 0 \),

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau)dg_t(\tau)dt = 0
\]

for all \( \psi \in K_2 \). Hence

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(t)\phi_2(\tau)dg_t(\tau)dt = 0
\]

for all \( \phi_1, \phi_2 \in K_1 \). Since

\[
\left| \int_{-\infty}^{\infty} \phi_1(t)dg_t(\tau) \right| \leq \| \phi_1 \|_\infty \text{var} g(t, \tau)
\]

for a.e. \( t \), the map \( t \mapsto \int_{-\infty}^{\infty} \phi_1(t)dg_t(\tau) \) may be viewed as a distribution \( T(\phi_1) \). But \( \langle T(\phi_1), \phi_2 \rangle = 0 \) for every \( \phi_2 \), so \( T(\phi_1) = 0 \) and

\[
\int_{-\infty}^{\infty} \phi_1(t)dg_t(\tau) = 0
\]

a.e. for every \( \phi_1 \). Since \( NBV \) is the dual of \( (K_1, \| \cdot \|_\infty) \), \( g(t, \cdot) = 0 \) for a.e. \( t \) and \( g = 0 \) a.e. \( \square \).
3. Families and integral operators. In order to generalize the concept of an integral operator as in (1.1), we must study collections of distributions indexed by a real parameter. Let \( \{ f_t \mid t \in \mathbb{R} \} \) be a collection of distributions in \( K'_1 \). Suppose that, for each \( a < \infty \), there exist an integer \( p \geq 0 \) and an \( L^1 \) function \( M : [-a, a] \to \mathbb{R} \) such that

\[
|\langle f_t, \phi \rangle| \leq M(t) \|\phi\|_{C^p}
\]

for every \( \phi \in K_1 \) with \( \text{supp} \phi \subset [-a, a] \). Then we say that \( \{ f_t \} \) is an \( L^1 \) family of distributions on \( \mathbb{R} \).

**Theorem 3.1.** (1) If \( \{ f_t \} \) is an \( L^1 \) family, then the map \( \psi \to \int_{-\infty}^{\infty} \langle f_t, \psi(t, \cdot) \rangle \, dt \)

is a distribution in \( K'_2 \).

(2) If \( f = \{ f_t \} \) is an \( L^1 \) family, then so is \( \{ f_t \} \), and \( \frac{\partial f}{\partial t} = \{ f_t \} \).

(3) If \( f : \mathbb{R}^2 \to \mathbb{R} \) is locally integrable, then \( \{ f(t, \cdot) \} \) is an \( L^1 \) family.

(4) If \( f \in \mathcal{D}' \), then \( \{ f(t, \cdot) \} \) is an \( L^1 \) family.

**Proof.** (1) Let \( \psi \in K_2 \). Then \( \|\psi(t, \cdot)\|_{C^p} \leq \|\psi\|_{C^p} \) for every \( t \), so

\[
\int_{-a}^{a} |\langle f_t, \psi(t, \cdot) \rangle| \, dt \leq \int_{-a}^{a} M(t) \|\psi(t, \cdot)\|_{C^p} \, dt \leq \int_{-a}^{a} M(t) \, dt \|\psi\|_{C^p}.
\]

Thus the map \( \psi \to \int_{-\infty}^{\infty} \langle f_t, \psi(t, \cdot) \rangle \, dt \) is well defined, linear, and, from [4, p. 34], continuous.

(2) There exist \( M, p \) such that, for any \( \phi \in K_1 \) with \( \text{supp} \phi \subset [-a, a] \),

\[
|\langle f_t, \phi \rangle| = |\langle f_t, \dot{\phi} \rangle| \leq M(t) \|\dot{\phi}\|_{C^p} \leq M(t) \|\phi\|_{C^{p+1}}.
\]

For any \( \psi \in K_2 \),

\[
\left\langle \frac{\partial \{ f_t \}}{\partial t}, \psi \right\rangle = -\left\langle \{ f_t \}, \frac{\partial \psi(t, \cdot)}{\partial t} \right\rangle
\]

\[
= -\int_{-a}^{a} \left\langle f_t, \frac{\partial \psi(t, \cdot)}{\partial t} \right\rangle \, dt
\]

\[
= \int_{-a}^{a} \left\langle f_t, \psi(t, \cdot) \right\rangle \, dt
\]

\[
= \left\langle \{ f_t \}, \psi \right\rangle.
\]

(3) Let \( M(t) = \int_{-a}^{a} |f(t, \tau)| \, d\tau \). By Fubini's theorem, \( M \) is \( L^1 \) on \([-a, a]\). For any \( \phi \in K_1 \),

\[
|\langle f(t, \cdot), \phi \rangle| - \int_{-a}^{a} f(t, \tau) \phi(\tau) \, d\tau \leq M(t) \|\phi\|_{C^0}.
\]

(4) This follows immediately from part (3) and the fact that \( f \) is bounded on \( \mathbb{R}^2 \).

A slight modification of the argument used in Theorem 3.1(1) shows that each \( L^1 \) family \( \{ f_t \} \) also determines a distribution according to \( \psi \to \int_{-\infty}^{\infty} \langle f_t, \psi(\cdot, \tau) \rangle \, d\tau \). We will rely on the notation \( \{ f_t \} \) and \( \{ f_t \} \) to distinguish these two cases.

Consider a collection of distributions \( \{ f_t \} \) in \( K'_1 \) with \( t \to \langle f_t, \phi \rangle \) continuous for every \( \phi \in K_1 \). Then we say that \( \{ f_t \} \) is a \( C^0 \) family.

**Theorem 3.2.** Every \( C^0 \) family is an \( L^1 \) family.
Proof. Let \( \{ f_\tau \} \) be a \( C^0 \) family. According to [4, p. 34], each \( f_\tau \) has finite order on each bounded interval \([-a,a]\); i.e., for every \( \tau \in [-a,a] \) there exist integers \( M_\tau, p_\tau < \infty \) such that

\[
|\langle f_\tau, \phi \rangle| \leq M_\tau \| \phi \|_{C^p},
\]

for every \( \phi \in K_1 \) with \( \text{supp} \phi \subset [-a,a] \). Suppose each \( M_\tau \) and \( p_\tau \) are chosen to minimize \( q_\tau = \max\{ M_\tau, p_\tau \} \). If the set \( \{ q_\tau \mid \tau \in \mathbb{R} \} \) is unbounded, there exist \( \eta_k, \eta \in [-a,a] \) such that \( \eta_k \to \eta \) and \( q_{\eta_k} \to \infty \). On the other hand, since \( \langle f_\tau, \phi \rangle \) is continuous, \( \langle f_{\eta_k}, \phi \rangle \to \langle f_\eta, \phi \rangle \) for every \( \phi \). From [4, p. 57], there exist \( M, p \) such that

\[
|\langle f_{\eta_k}, \phi \rangle| \leq M \| \phi \|_{C^p}
\]

for every \( k \), yielding a contradiction. Hence, \( \{ q_\tau \} \) is bounded, and there exist \( M, p \) such that

\[
|\langle f_\tau, \phi \rangle| \leq M \| \phi \|_{C^p}
\]

for every \( \tau, \phi \). \( \Box \)

In addition to continuity, we might also consider families \( \{ f_\tau \} \) which are differentiable in \( \tau \). Define the weak* derivative \( \frac{\partial f_\tau}{\partial \tau} \mid_{\tau_0} \in K_2^1 \) of \( \{ f_\tau \} \) at \( \tau_0 \) according to

\[
\left\langle \frac{\partial f_\tau}{\partial \tau} \mid_{\tau_0}, \phi \right\rangle = \frac{d}{d\tau} \langle f_\tau, \phi \rangle \mid_{\tau_0} = \lim_{\tau_n \to \tau_0} \left\langle \frac{f_{\tau_n} - f_{\tau_0}}{\tau_n - \tau_0}, \phi \right\rangle,
\]

whenever the limit exists for every \( \phi \in K_1 \). According to [3, p. 368], (3.2) determines a distribution in \( K_1' \) for each \( \tau_0 \). If \( \{ \frac{\partial f_\tau}{\partial \tau} \mid_{\tau_0} \mid \tau_0 \in \mathbb{R} \} \) is a \( C^0 \) family, we denote it by \( \frac{\partial f_\tau}{\partial \tau} \) and say that \( \{ f_\tau \} \) is a \( C^1 \) family. Continuing in this way, if \( \{ \frac{\partial f_\tau}{\partial \tau} \} \) is a \( C^{p-1} \) family, \( \{ f_\tau \} \) is a \( C^p \) family. Applying (3.2), \( \{ f_\tau \} \) is a \( C^p \) family if and only if \( \tau \to \langle f_\tau, \phi \rangle \) is a \( C^p \) function for each \( \phi \). We may interpret the latter statement as a definition for \( p = \infty \). Since \( \langle f_\tau, \phi \rangle = -\langle f_\tau, \bar{\phi} \rangle \), \( \{ f_\tau \} \) is a \( C^p \) family whenever \( \{ f_\tau \} \) is a \( C^p \) family.

Next we relate two notions of differentiation for \( C^1 \) families.

**Theorem 3.3.** Suppose \( g = \{ g_\tau \} \) is a \( C^1 \) family. Then \( \frac{\partial g_\tau}{\partial \tau} = \left\{ \frac{\partial g_\tau}{\partial \tau} \right\} \).

_Pf._ Suppose

\[
\left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0}, \psi(\cdot, \tau_0) \right\rangle \neq \left\langle \frac{\partial g_\tau}{\partial \tau} \mid_{\tau_0}, \psi(\cdot, \tau_0) \right\rangle
\]

as \( \tau \to \tau_0 \). Then there exist \( \tau_n \to \tau_0 \) and \( \varepsilon > 0 \) such that

\[
\sup_j \left| \left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0} - \frac{\partial g_\tau}{\partial \tau} \mid_{\tau_0}, \psi(\cdot, \tau_j) \right\rangle \right| \geq \left| \left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0} - \frac{\partial g_\tau}{\partial \tau} \mid_{\tau_n}, \psi(\cdot, \tau_n) \right\rangle \right| > \varepsilon
\]

for every \( n \). But \( \psi(\cdot, \tau_n) \to \psi(\cdot, \tau_0) \) in \( K_1 \), so, from [4, p. 31], \( \{ \psi(\cdot, \tau_n) \} \subset K_1 \) is a bounded set. Hence, from [4, p. 56], the left side of (3.4) must converge to 0. This yields a contradiction, so we have convergence in (3.3). Thus

\[
\frac{d}{d\tau} \langle g_\tau, \psi(\cdot, \tau) \rangle \mid_{\tau_0} = \lim_{\tau \to \tau_0} \frac{\langle g_{\tau_n}, \psi(\cdot, \tau) \rangle - \langle g_{\tau_0}, \psi(\cdot, \tau) \rangle}{\tau - \tau_0}
\]

\[
= \lim_{\tau \to \tau_0} \left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0}, \psi(\cdot, \tau) \right\rangle + \lim_{\tau \to \tau_0} \left\langle g_{\tau_0}, \psi(\cdot, \tau) - \psi(\cdot, \tau_0) \right\rangle
\]

\[
= \left\langle \frac{\partial g_\tau}{\partial \tau} \mid_{\tau_0}, \psi(\cdot, \tau_0) \right\rangle + \left\langle g_{\tau_0}, \frac{\partial \psi}{\partial \tau} \mid_{\tau_0} \right\rangle,
\]
and, for every $\psi \in K_2$, 
\[
\left\langle \frac{\partial g}{\partial \tau}, \psi \right\rangle = -\left\langle g, \frac{\partial \psi}{\partial \tau} \right\rangle 
= -\int_{-\infty}^{\infty} \left\langle g, \frac{\partial \psi}{\partial \tau} \right\rangle d\tau 
= -\int_{-\infty}^{\infty} \left( \frac{d}{d\tau} (g, \psi(\cdot, \tau)) - \left\langle \frac{\partial g}{\partial \tau}, \psi(\cdot, \tau) \right\rangle \right) d\tau 
= -\lim_{\tau \to \infty} \left\langle g, \psi(\cdot, \tau) \right\rangle \left\| \lim_{\tau \to -\infty} \left\langle g, \psi(\cdot, \tau) \right\rangle \right\| \int_{-\infty}^{\infty} \left\langle \frac{\partial g}{\partial \tau}, \psi(\cdot, \tau) \right\rangle d\tau 
= \int_{-\infty}^{\infty} \left\langle \frac{\partial g}{\partial \tau}, \psi(\cdot, \tau) \right\rangle d\tau.
\]

Hence $\frac{\partial g}{\partial \tau} = \left\{ \frac{\partial g}{\partial \tau} \right\}$. \(\Box\)

Note that
\[
\left\langle \frac{\partial \theta_\tau}{\partial \tau}, \phi \right\rangle = \frac{\partial}{\partial \tau} \int_{\tau}^{\infty} \phi(\eta) d\eta = -\phi(\tau) = (-\delta_\tau, \phi)
\]
for every $\phi \in K_1$, where $\theta_\tau$ is a translation of the unit step as in (2.1). In view of Theorem 3.3, $\frac{\partial}{\partial \tau} \{ \theta_\tau \} = \left\{ \frac{\partial \theta_\tau}{\partial \tau} \right\} = -\{\delta_\tau\}$. By a similar calculation, $\frac{\partial}{\partial \tau} \{ \xi_\tau^{(n-1)} \} = -\{\xi_\tau^{(n)}\}$.

Any $C^\infty$ family $\{h_\tau\}$ belonging to $K_2^+$ determines a linear operator on $K_1^+$ in the following way. For each $\phi \in K_1$, let $\xi(\tau) = \langle h_\tau, \phi \rangle$. Then $\xi(\tau)$ is $C^\infty$ with $\xi(\tau) = 0$ for large $\tau$. Suppose $u \in K_1^+$ with supp $u \subset [-a, a]$, and let $\xi \in K_1$ with $\xi(\tau) = \xi(\tau)$ for $\tau \geq a$. It is easy to show that $y(\phi) = \langle u, \xi \rangle$ is independent of the choice of $\xi$. Indeed, let $\hat{\xi}_1$ and $\hat{\xi}_2$ be two such functions. Then $\langle \hat{\xi}_1, \hat{\xi}_2 \rangle \subset (\infty, a]$, and $\langle u, \xi_1 \rangle = \langle u, \xi_2 \rangle = \langle u, \xi_1 - \xi_2 \rangle = 0$.

**Theorem 3.4.** The map $u \mapsto y$ defines a linear operator $T : K_1^+ \to K_1^+$, where $T(\delta_{t_0}) = h_{t_0}$ for every $t_0$.

**Proof.** Linearity is obvious. Since $\{h_\tau\} \subset K_2^+$, there exists $b \in \mathbb{R}$ such that supp $\phi \subset (-a, b)$ implies supp $\xi \subset (-\infty, a)$. For any such $\phi$, $y(\phi) = 0$, so supp $y \subset [b, \infty)$. We must show that $y \in K_1$.

The topological space $K_1^+$ is not first-countable, so we must consider nets $\{\phi_\lambda\}$ in $K_1$. Suppose $\phi_\lambda \to 0$. Then there exists $b < \infty$ such that supp $\phi_\lambda \subset [-b, b]$ for every $\lambda$. Let $\xi_\lambda(\tau) = \langle h_\tau, \phi_\lambda \rangle$. Since $\{h_\tau\} \subset K_1^+$, there exists $c < \infty$ such that supp $\xi_\lambda(\tau) \subset (-\infty, c]$ for every $\lambda$. Arguing as in the proof of Theorem 3.2, there exist $M_n, n \in \mathbb{N}$ such that
\[
\left\langle \xi_\lambda^{(n)}(\tau) \right\rangle \leq M_n \|\phi\|_{C^n}
\]
for every $\tau \in [-a, c]$ and $\phi \in K_1$ with supp $\phi \subset [-b, b]$. Hence, $\xi_\lambda^{(n)} \to 0$ uniformly on $[-a, c]$ for every $n$. $\lambda \in K_1$ can be chosen so that $\xi_\lambda(\tau) = \xi_\lambda(\tau)$ on $[a, c]$ and each $\xi_\lambda^{(n)} \to 0$ uniformly on $[a, c]$. Thus $\xi_\lambda \to 0$ strongly in $K_1$, and $y(\phi_\lambda) = \langle u, \xi_\lambda \rangle \to 0$.

Finally, for $u = \delta_{t_0}$ and any $\phi \in K_1$, $T(\delta_{t_0}, \phi) = \xi(t_0) = \xi(t_0) = \langle h_{t_0}, \phi \rangle$. \(\Box\)

Suppose $h_\tau = h(\cdot, \tau)$ Then, under mild assumptions,
\[
\langle T(u), \phi \rangle = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(t, \tau) \phi(t) dt \right) u(\tau) d\tau = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(t, \tau) u(t) d\tau \right) \phi(t) dt,
\]
so the operator determined by Theorem 3.4 is "classical." For this reason, we call every operator of the type described by Theorem 3.4 a generalized integral operator. Suppose \( h_{r} \) is the \( \tau \)-translation of a fixed \( h \in K_{1}^{+} \). Then \( \{h_{r}\} \) is a \( C^{\infty} \) family and \( \{h_{r}\} \in K_{2}^{+} \), so \( \{h_{r}\} \) determines a generalized integral operator according to \( \langle T(u), \phi \rangle = (u, \xi) \), where \( \xi(\tau) = (h_{r}, \phi) = (h, \phi_{-r}) \). From [3, p. 103], constructing \( T(u) \) in this way yields the convolution operator \( u \to h \ast u \). This sets up the time-invariant analysis we carried out in [1].

We next address the issue of continuity of generalized integral operators.

**Theorem 3.5.** Let \( T : K_{1}^{+} \to K_{1}^{+} \) be any continuous linear operator, and suppose \( \{T(\delta_{r})\} \in K_{2}^{+} \) is a \( C^{\infty} \) family. Then \( T \) is a generalized integral operator.

**Proof.** Let \( \phi \in K_{1}^{+}, u \in K_{1}^{+}, \sup u \subset (a, \infty) \), \( \xi(\tau) = (T(\delta_{r}), \phi) \), and \( \xi \in K_{1}^{+} \) with \( \xi(\tau) = \xi(\tau) \) for \( \tau \geq a \). We must show that \( \langle T(u), \phi \rangle = (u, \xi) \). First, set \( u = \delta_{t_{0}}, \)

\[
\langle T(\delta_{t_{0}}), \phi \rangle = \xi(\tau) = (u, \xi) = \langle u, \xi \rangle.
\]

For arbitrary \( u \in K_{1}^{+}, \) [1, Lemma 2.2] shows that there exist \( t_{i,k} \geq a, \beta_{i,k} \in \mathbb{R}, \) and integers \( n_{k} > 0 \) such that

\[
n_{k} = \sum_{i=1}^{n_{k}} \beta_{i,k} \delta_{t_{i,k}} \to u,
\]

where the limit is weak*. For every \( k, \)

\[
\langle T(u_{k}), \phi \rangle = \sum_{i=1}^{n_{k}} \beta_{i,k} \langle T(\delta_{t_{i,k}}), \phi \rangle = \sum_{i=1}^{n_{k}} \beta_{i,k} \langle \delta_{t_{i,k}}, \xi \rangle = \langle u_{k}, \xi \rangle.
\]

Taking the limit yields

\[
\langle T(u), \phi \rangle = (u, \xi).
\]

The next result establishes conditions under which generalized integral operators are continuous. In particular, it shows that the converse to Theorem 3.5 is false.

**Theorem 3.6.** Let \( T : K_{1}^{+} \to K_{1}^{+} \) be a generalized integral operator.

(1) \( T \) is continuous if and only if there exists a function \( b : \mathbb{R} \to \mathbb{R} \) such that \( b(\tau) \to \infty \) as \( \tau \to -\infty \) and \( \sup \{T(\delta_{r})\} \subset \{ |t| \geq b(\tau) \} \).

(2) \( T \) is continuous on \( \{u \in K_{1}^{+} | \sup u \subset (a, \infty) \} \) for every \( a > -\infty \).

**Proof.** (1) (Sufficient) Consider any basic neighborhood \( Y = \{ y \in K_{1}^{+} | \langle (y, \phi) \rangle < \varepsilon; i = 1, \ldots, n \} \) of 0 in \( K_{1}^{+} \), and let \( \xi_{i}(\tau) = \langle T(\delta_{r}), \phi_{i} \rangle \). Since \( T \) is a generalized integral operator, each \( \xi_{i} \) is \( C^{\infty} \). \( \{h_{r}\} \in K_{2}^{+} \) guarantees that \( \sup \xi_{i} \) is right-bounded. The condition \( \sup \{T(\delta_{r})\} \subset \{ |t| \geq b(\tau) \} \) guarantees that \( \sup \xi_{i} \) is left-bounded. Hence, \( \xi_{i} \in K_{1} \), and \( \langle T(u), \phi_{i} \rangle = (u, \xi_{i}) \). It follows that the inverse image

\[
T^{-1}(Y) = \{ u \in K_{1}^{+} | \langle T(u), \phi_{i} \rangle < \varepsilon; i = 1, \ldots, n \} = \{ u \in K_{1}^{+} | \langle u, \xi_{i} \rangle < \varepsilon; i = 1, \ldots, n \}
\]

is open. Since \( \varepsilon, \phi_{1}, \ldots, \phi_{n} \) are arbitrary, \( T \) is continuous.

(Necessary) Suppose no such function \( b \) exists. Then there exists a sequence \( \tau_{k} \to -\infty \) and \( c < \infty \) such that \( \sup \{T(\delta_{r})\} \cap [-c, c] \) is nonempty for every \( k \). Hence, there exist \( \phi_{k} \in K_{1} \) such that \( \sup \phi_{k} \subset [-c, c] \) and \( \beta_{k} = \langle T(\delta_{r}), \phi_{k} \rangle \neq 0 \) for each \( k \). Let

\[
\xi_{k} = \frac{1}{k \|\phi_{k}\|_{C^{k}}} \phi_{k}.
\]
Then, for any integers \( p \geq 0 \) and \( k \geq p \),

\[
\|\xi_k\|_{C^p} = \frac{1}{k \|\phi_k\|_{C^k}} \|\phi_k\|_{C^p} \leq \frac{1}{k},
\]

so \( \xi_k \to 0 \). From [4, p. 31], \( \{\xi_k\} \) is a bounded set. Let

\[
u_k = \frac{k \|\phi_k\|_{C^k}}{\beta_k} \delta_{\tau_k}.
\]

Then \( \nu_k \to 0 \). On the other hand,

\[
\sup_m \langle T(u_k), \xi_m \rangle \geq \langle T(u_k), \xi_k \rangle
\]

\[
= \left\langle \frac{k \|\phi_k\|_{C^k}}{\beta_k} T(\delta_{\tau_k}), \xi_k \right\rangle
\]

\[
= \frac{1}{\beta_k} \langle T(\delta_{\tau_k}), k \|\phi_k\|_{C^k} \xi_k \rangle
\]

\[
= \frac{1}{\beta_k} \langle T(\delta_{\tau_k}), \phi_k \rangle = 1.
\]

From [4, p. 56], \( T(\nu_k) \not\to 0 \), which is a contradiction.

(2) Let \( u_\lambda \to 0 \) be a net with \( \text{supp} u_\lambda \subset [a, \infty) \), \( \phi \in K_1 \), and \( \xi(\tau) = \langle T(\delta_{\tau}), \phi \rangle \). Select \( \hat{\xi} \in K_1 \) with \( \hat{\xi}(\tau) = \xi(\tau) \) for \( \tau \geq a \). This gives \( \langle T(u_\lambda), \phi \rangle = \langle u_\lambda, \hat{\xi} \rangle \to 0 \), so \( T(u_\lambda) \to 0 \). \( \square \)

In view of Theorem 3.6, restricting attention to continuous linear operators on \( K_{1+} \) would be inadequate for developing a sufficiently comprehensive theory of stable linear systems. For example, even the “integrator system” \( T(\delta_{\tau}) = \theta_{\tau} \) is discontinuous. On the other hand, the class of all generalized integral operators on \( K_{1+} \) contains the full range of operators normally considered in linear system theory, so we will adopt these as our space of systems.

We end this section with a result relating impulse response and step response.

**Theorem 3.7.** Let \( T : K_{1+} \to K_{1+} \) be a generalized integral operator. Then \( \{T(\theta_{\tau})\} \) is a \( C^\infty \) family and \( T(\delta_{\tau}) = -\frac{\partial T(\theta_{\tau})}{\partial \tau} \).

**Proof.** From Theorem 3.6(2), \( T \) is continuous on \( \{u \in K_{1+} \mid \text{supp} u \subset [\tau_0-1, \infty)\} \) for any \( \tau_0 \). Hence, for any \( \phi \in K_1 \),

\[
\frac{\langle T(\theta_{\tau}), \phi \rangle - \langle T(\theta_{\tau_0}), \phi \rangle}{\tau - \tau_0} = \frac{\langle T \left( \frac{\theta_{\tau} - \theta_{\tau_0}}{\tau - \tau_0} \right), \phi \rangle}{\tau - \tau_0} \to \langle -T(\delta_{\tau_0}), \phi \rangle,
\]

\[
\frac{\langle T(\delta^{(n)}_{\tau}), \phi \rangle - \langle T(\delta^{(n)}_{\tau_0}), \phi \rangle}{\tau - \tau_0} = \frac{\langle T \left( \frac{\delta^{(n)}_{\tau} - \delta^{(n)}_{\tau_0}}{\tau - \tau_0} \right), \phi \rangle}{\tau - \tau_0} \to \langle -T(\delta^{(n+1)}_{\tau_0}), \phi \rangle
\]

as \( \tau \to \tau_0 \) for any \( n \), so \( \{T(\theta_{\tau})\} \) is \( C^\infty \). The first derivative is given by

\[
\frac{\partial T(\theta_{\tau})}{\partial \tau}, \phi = \frac{\partial}{\partial \tau} \langle T(\theta_{\tau}), \phi \rangle = \langle -(\delta_{\tau}), \phi \rangle. \quad \square
\]

**4. Extension of normed linear spaces.** In [1] we considered the problem of imbedding a normed linear space \( Y \) into a Hausdorff topological vector space \( X \) and extending the norm to a maximal linear subspace of \( X \). For example, we showed that
\(L^1\) can be imbedded in \(K_1^1\) and \(\|\cdot\|_1\) can be extended to all of \(DBV\). In particular, the extended \(L^1\) norm applied to the unit impulse evaluates to \(\|\delta\|_1^1 = 1\). We will again need these results to construct our time-varying theory.

Let \(\mathcal{I}\) be the topology on \(X_1\), \(\|\cdot\|\) the norm on \(Y \subset X_1\), and \(B(y, r) \subset Y\) the closed norm-ball about \(y \in Y\) with radius \(r\). We make the following assumptions on the 4-tuple \((X, \mathcal{I}, Y, \|\cdot\|)\):

T1. For every nonempty \(U \in \mathcal{I}\), \(U \cap Y = Y\) is nonempty.

T2. For every \(U \in \mathcal{I}\) and every \(y \in U \cap Y\), there exists \(r > 0\) such that \(B(y, r) \subset U\).

T3. There exists \(U \in \mathcal{I}\) such that \(U \cap Y = Y = B(0, 1)\).

Condition T1 says that \(Y\) is dense in \(X_1\). T2 requires that the norm topology on \(Y\) is at least as strong as the topology induced on \(Y\) by \(X_1\). T3 says that \(B(0, 1)\) in \(Y\) is closed relative to \(X_1\). Thus T2 and T3 give bounds on the topology induced on \(Y\) by \(X_1\). Under assumptions T1–T3, there exists a natural extension \(\|\cdot\|^{\varepsilon}\) of \(\|\cdot\|\) to a subspace \(Y_{\varepsilon} \supset Y\) of \(X_1\). In particular, for any \(y \in Y_{\varepsilon}\), \(\|y\|^{\varepsilon}\) is equal to the minimum value of \(\|y_\lambda\|^{\varepsilon}\) over all \(\mathcal{I}\)-approximating nets \(y_\lambda \to y\), \(y_\lambda \in Y_{\varepsilon}\). (See [1, section 3] for details.)

As mentioned in section 1, \(h \in UL^1\) is the classical condition for BIBO stability. Therefore it makes sense to examine the extension \(UL_{\varepsilon}^1\) of \(\|\cdot\|_{\infty 1}\) in \(K_2^1\) and check whether \(UL_{\varepsilon}^1\) actually characterizes BIBO stability for generalized integral operators.

First we must establish whether \(K_2^1\) and \(UL^1\) satisfy T1–T3.

**Lemma 4.1.** (1) If \(Y_1\) is dense in \(X\) relative to \(\|\cdot\|\), then the 4-tuple \((X, \mathcal{I}, Y_1, \|\cdot\|)\) satisfies T1–T3 and \(Y_{\varepsilon} \supset Y_1\).

(2) Let \(Y \subset X_1 \subset X\) and \(\mathcal{I}_1\) be the relative topology on \(X_1\) induced by \(\mathcal{I}\). Then the 4-tuple \((X_1, \mathcal{I}_1, Y, \|\cdot\|)\) satisfies T1–T3 and the corresponding extension of \(Y\) is \(Y_{\varepsilon} \cap X_1\).

Proof. (1) From T1 and T2, \(Y_1\) is dense in \(X\) relative to \(\mathcal{I}\), so T1 holds for \(Y_1\). If \(U \in \mathcal{I}\) and \(y \in U \cap Y_1\), then \(y \in U \cap Y\), so there exists \(r > 0\) such that \(B(y, r) \subset U\); where \(B(y, r)\) is a norm-ball in \(Y\). The corresponding ball in \(Y_1\) is \(B(y, r) \cap Y_1 \subset U\), so T2 holds. Finally, if \(U\) satisfies T3 relative to \(Y\), then \(U\) also satisfies T3 relative to \(Y_1\), since

\[
U \cap Y_1 = (U \cap Y) \cap Y_1 = (Y - B(0, 1)) \cap Y_1 = Y_1 - (B(0, 1) \cap Y_1).
\]

Finally, we must show that \(\|\cdot\|^{\varepsilon}\) satisfies [1, Proposition 3.1(1)–(3)] using \(Y_1\) in place of \(Y\). To prove (1), note that since \(\|\cdot\|\) and \(\|\cdot\|^{\varepsilon}\) coincide on \(Y\), they must coincide on \(Y_1\). Condition (2) holds, since it does not involve \(Y\). To prove (3), let \(x \in X\) with \(\|x\|^{\varepsilon} < \infty, \epsilon > 0\), and let \(U\) be a \(\mathcal{I}\)-neighborhood of \(x\). Then (3) applied to \(Y\) guarantees that there exists \(y \in U \cap Y\) such that

\[
\|y\| < \|x\|^{\varepsilon} + \frac{\epsilon}{2}.
\]

Density of \(Y_1\) in \(Y\) relative to \(\|\cdot\|\) and T2 imply that there exists \(y_1 \in U \cap Y_1\) such that \(\|y_1 - y\| < \frac{\epsilon}{2}\). Then

\[
\|y_1\| < \|y\| + \frac{\epsilon}{2} < \|x\|^{\varepsilon} + \epsilon.
\]

(2) Restricting \(\mathcal{I}\) to \(X_1\), T1–T3 are obvious. Suppose \(Y_{\varepsilon}\) is the extension of \(Y\) using \(X_1\), and \(\|\cdot\|^{\varepsilon}\) is the corresponding norm. Let \(\|\cdot\|^{\varepsilon}_1\) be the restriction of \(\|\cdot\|^{\varepsilon}\) to \(X_1\). We must show that \(\|\cdot\|^{\varepsilon}_1\) satisfies [1, Proposition 3.1(1)–(3)], using \(X_1\). To prove (1) and (2), note that \(\|x\|^{\varepsilon}\) and \(\|x\|^{\varepsilon}_1\) coincide on \(X_1\); hence, \(\|y\|^{\varepsilon}_1 = \|y\|\) for
$y \in Y$ and $\| \cdot \|_1$ is lower semicontinuous. To establish (3), let $u \in \mathcal{X} x \in U \cap X_1$ with $\|x\|_1 < \infty$, and $\varepsilon > 0$. Then $\|x\|_\varepsilon < \infty$ and there exists $y \in U \cap Y$ such that

$$\|y\| < \|x\|_\varepsilon + \varepsilon = \|x\|_1 + \varepsilon.$$ 

But $Y \subset X_1$, so $y \in (U \cap X_1) \cap Y$. \hfill \Box

**Theorem 4.2.** Let $X = K_2'$ and $Y = UL^1$. Then T1–T3 are satisfied.

**Proof.** First we note that $K_2' \subset UL^1$. From [4, p. 118], $K_2'$ is dense in $K_2'$, and T1 follows. If T2 holds for $y = 0$, then it holds for all $y$, since $B(0, r) \subset U$ implies $B(y, r) \subset y + U$. Thus it suffices to prove that, for every $n, \varepsilon > 0$, and $\psi_1, \ldots, \psi_n \in K_2'$, there exists $r > 0$ such that $B(0, r) \subset U$, where

$$U = \{ f \in K_2' \mid |\langle f, \psi_i \rangle| < \varepsilon; i = 1, \ldots, n \}.$$ 

Let

$$r < \varepsilon \min \left\{ \frac{1}{\|\psi_i\|_\infty} \right\}$$

and $f \in B(0, r)$. From Theorem 2.2,

$$|\langle f, \psi_i \rangle| \leq \|f\|_{DUBV} \|\psi_i\|_{1\infty} - \|f\|_{\infty} \|\psi_i\|_{1\infty} \leq r \max\{\|\psi_i\|_{1\infty}\} < \varepsilon$$

for every $i$, so $f \in U$.

To prove T3, let $y \in UL^1$ with $\|y\|_{\infty} > 1$, and choose $\varepsilon < \frac{1}{2}(\|y\|_{\infty} - 1)$. From Theorem 2.2, we may select $\psi \in K_2'$ with $\|\psi\|_{1\infty} = 1$ such that $|\langle y, \psi \rangle| > \|y\|_{\infty} - \varepsilon$.

Let

$$U = \{ x \in K_2' \mid |\langle x - y, \psi \rangle| < \varepsilon \}.$$ 

Then $y \in U \subset \mathcal{X}$, and $f \in U \cap Y$ implies

$$\|f\|_{\infty} \geq |\langle f, \psi \rangle| \geq |\langle y, \psi \rangle| - |\langle f - y, \psi \rangle| > \|y\|_{\infty} - 2\varepsilon > 1,$$

so $U \cap Y \subset Y - B(0, 1)$. \hfill \Box

We are now in a position to characterize the extension of $UL^1$ into $K_2'$.

**Theorem 4.3.** $UL^1_{\infty} = DUBV$.

**Proof.** Let

$$\|f\|_{\infty} = \begin{cases} \|f\|_{DUBV}, & f \in DUBV, \\ \infty, & f \in K_2' - DUBV. \end{cases}$$

We must verify that $\| \cdot \|_{\infty}$ satisfies [1, Lemma 3.1(1)–(3)] relative to $\| \cdot \|_{1 \infty}$. Condition (1) says that $\| \cdot \|_{1 \infty}$ and $\| \cdot \|_{DUBV}$ coincide on $UL^1$. This was established in Theorem 2.2.

Condition (2) requires that $\| \cdot \|_{\infty}$ be lower semicontinuous on $K_2'$. Equivalently, we must show that the set $\Sigma_M = \{ f \in K_2' \mid \|f\|_{\infty} > M \}$ is open for each $M$. Suppose $\|f\|_{\infty} > M$. From Theorem 2.2(3), there exists $\psi \in K_2'$ such that $\|\psi\|_{1\infty} = 1$ and $|\langle f, \psi \rangle| > M$. Let $U = \{ g \in K_2' \mid |\langle g, \psi \rangle| < |\langle f, \psi \rangle| - M \}$. $f + U$ is open in $K_2'$, and

$$\|f + g\|_{\infty} \geq |\langle f + g, \psi \rangle| \geq |\langle f, \psi \rangle| - |\langle g, \psi \rangle| > M$$

for every $g \in U$. Hence, $\Sigma_M$ is open.
Finally, condition (3) says that, for any \( f \in DUBV \), \( \varepsilon > 0 \), and neighborhood \( U \) of \( f \), there exists \( y \in U \cap U_1 \) such that \( \|y\|_{\infty} < \|f\|_{\infty} + \varepsilon \). We accomplish this by constructing a sequence \( f_n \rightarrow f \) with \( f_n \in U_1 \) and \( \|f_n\|_{\infty} \leq \|f\|_{DUBV} \). Then, for large \( n \), \( f_n \in U \), and \( y = f_n \) satisfies the conditions.

Our construction of \( f_n \) proceeds as follows. Let \( \phi_1, \phi_2 \in K_1 \) with \( \phi_1(t) \geq 0 \) for all \( t \), \( \int_{-\infty}^{\infty} \phi_1(t)dt = 1 \), \( \phi_2(0) = 1 \), and \( \|\phi_2\|_{\infty} = 1 \). Set \( \psi_n(t, \tau) = n\phi_1(n(t-\tau))\phi_2(\frac{\tau}{n}) \). Then \( \psi_n \in C^\infty \). Suppose supp \( \phi_1, \) supp \( \phi_2 \subseteq [-a, a] \). If \( |\tau| \geq na \), then \( \frac{|\tau|}{n} \geq a \), so \( \phi_2(\frac{\tau}{n}) = 0 \) and \( \psi_n(t, \tau) = 0 \). If \( |\tau| < na \) and \( |t| \geq (n + \frac{1}{n})a \), then

\[
\begin{align*}
  n |t-\tau| & \geq n (|t| - |\tau|) > \left(n + \frac{1}{n}\right)a - n^2a = a,
\end{align*}
\]

so \( \phi_1(n(t-\tau)) = 0 \) and \( \psi_n(t, \tau) = 0 \). Hence, \( \psi_n \in K_2 \).

Let

\[
  f_n(t, \tau) = \int_{-\infty}^{\infty} \psi_n(\tau, \eta)dg_t(\eta)
\]

and \( f = \frac{\partial g}{\partial \tau} \). Then

\[
\begin{align*}
  \int_{-\infty}^{\infty} |f_n(t, \tau)|d\tau & \leq \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \psi_n(\tau, \eta) \right| |dg_t(\eta)|d\tau \\
  & = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \psi_n(\tau, \eta) \right| |dg_t(\eta)| \\
  & = \int_{-\infty}^{\infty} \left| \phi_2 \left( \frac{\eta}{n} \right) \right| \left( \int_{-\infty}^{\infty} \phi_1(n(\tau-\eta))d\tau \right) |dg_t(\eta)| \\
  & = \int_{-\infty}^{\infty} \left| \phi_2 \left( \frac{\eta}{n} \right) \right| \left( \int_{-\infty}^{\infty} \phi_1(x)dx \right) |dg_t(\eta)| \\
  & \leq \varliminf_{n} g(t, \eta),
\end{align*}
\]

\[
\|f_n\|_{\infty} \leq \text{ess sup}_{t} \varliminf_{\tau} g(t, \tau) = \|f\|_{DUBV}.
\]

To show \( \langle f_n, \psi \rangle \rightarrow \langle f, \psi \rangle \), note that, for every \( t, \eta, x \in \mathbb{R}, \psi(t, x, \eta, \eta) \rightarrow \psi(t, \eta) \) and

\[
\int_{-\infty}^{\infty} \left| \phi_1(x)\psi \left( t, \frac{x}{n} + \eta \right) \right| dx \leq \|\psi\|_{\infty}.
\]

By the dominated convergence theorem,

\[
\begin{align*}
  \int_{-\infty}^{\infty} \psi_n(\tau, \eta)\psi(t, \tau)d\tau & = n\phi_2 \left( \frac{\eta}{n} \right) \int_{-\infty}^{\infty} \phi_1(n(\tau-\eta))\psi(t, \tau)d\tau \\
  & = \phi_2 \left( \frac{\eta}{n} \right) \int_{-\infty}^{\infty} \phi_1(x)\psi \left( t, \frac{x}{n} + \eta \right) dx \\
  & \rightarrow \phi_2(0) \int_{-\infty}^{\infty} \phi_1(x)\psi(t, \eta)dx \\
  & = \psi(t, \eta)
\end{align*}
\]
for every $t, \eta$. Furthermore,

$$\left| \int_{-\infty}^{\infty} \psi_n(\tau, \eta) \psi(t, \tau) d\tau \right| \leq n \phi_2 \left( \frac{\eta}{n} \right) \left| \int_{-\infty}^{\infty} \phi_1(n(\tau - \eta)) \left| \psi(t, \tau) \right| d\tau \right|

= \phi_2 \left( \frac{\eta}{n} \right) \left| \int_{-\infty}^{\infty} \phi_1(x) \left| \psi \left( t, \frac{x}{n} + \eta \right) \right| dx \right|

\leq \|\psi(t, \cdot)\|_{\infty},$$

so, if $\text{supp} \psi \subset (-a, a)^2$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \psi(t, \cdot) \right| \left| \delta_{\eta} \right| dt \leq \int_{-a}^{a} \|\psi(t, \cdot)\|_{\infty} \text{var}_\eta g(t, \eta) dt \leq 2a \|\psi\|_{\infty} \|g\|_{UBV}.$$

Again, by the dominated convergence theorem,

$$\langle f_n, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_n(\tau, \eta) \psi(t, \tau) d\tau \delta_{\eta} dt \to \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\tau, \eta) \delta_{\eta} dt = \langle f, \psi \rangle.$$

\textbf{Corollary 4.4.} Let $X = K_{2+}^{t}$ and $Y = UL_{+}^{1}$. Then $X$ and $Y$ satisfy T1–T3 and $Y_\epsilon = DUBV_{\epsilon}$.

\textbf{Proof.} We have $K_2 \subset UL_{+}^{1} \subset UL_{1} \subset K_{2+}$ with $K_2$ dense in $K_{2+}$ (see [4, p. 118]). Hence, $UL_{+}^{1}$ is dense in $UL_{1}$. From Lemma 4.1(1) and Theorems 4.2 and 4.3, $(K_{2+}, \delta_{\eta}UL_{+}^{1}, \|\cdot\|_{\infty})$ satisfies T1–T3 and $UL_{+}^{1} = UL_{1} = DUBV$. Furthermore, $UL_{+}^{1} \subset K_{2+} \subset K_{2}$, so Lemma 4.1(2) implies that $(K_{2+}, \delta_{\eta}UL_{+}^{1}, \|\cdot\|_{\infty})$ satisfies T1–T3 and $UL_{+}^{1} = DUBV \cap K_{2+} = DUBV_{\epsilon}$. \quad \Box

\textbf{5. BIBO stability.} In this section, we consider stability of linear operators $T : K_{1+}^{t+} \to K_{1+}^{t}$ (we must restrict our attention to $K_{1+}$ in order to have a consistent definition of generalized integral operators.) As in [2, p. 109], we define a \textit{BIBO stable} linear operator $T$ to be one such that

(S1) $T(L_{+}^{\infty}) = L_{+}^{\infty}$,

(S2) $T$ is continuous on $L_{+}^{\infty}$ relative to $\|\cdot\|_{\infty}$.

We have shown in [1] that, for time-invariant (i.e., convolution) operators, (S2) follows automatically from (S1). Unfortunately, this result does not extend to the time-varying setting. For example, let $h_{\epsilon} = e^{-\epsilon t} \delta_{\tau}$. For any $u \in L_{+}^{\infty}$, $\phi \in K_{1}$,

$$\xi(\tau) = \langle h_{\epsilon}, \phi \rangle = e^{-\epsilon t} \langle \delta_{\tau}, \phi \rangle = e^{-\epsilon t} \phi(\tau),$$

$$\langle y, \phi \rangle = \langle u, \xi \rangle = \int_{-\infty}^{\infty} u(\tau) e^{-\epsilon t} \phi(\tau) d\tau.$$

Thus $y(t) = e^{-\epsilon t} u(t)$ and $y \in L_{+}^{\infty}$. However, the map $u \mapsto y$ is not continuous on $L_{+}^{\infty}$.

Indeed, let $u_{k}(t) = \frac{1}{k} \theta(t + k)$; then $\|u_{k}\|_{\infty} = \frac{1}{k} \to 0$ but $y_{k}(t) = \frac{1}{k} e^{-\epsilon t} \theta(t + k)$ and $\|y_{k}\|_{\infty} = \frac{\epsilon^{k}}{k} \to \infty$. Thus, as in [2], we adopt (S2) as an independent assumption.

Since classical integral operators satisfying $T(\delta_{\tau}) \in UL_{1}$ are known to be BIBO stable, a natural conjecture is that a generalized integral operator is BIBO stable if and only if $\{T(\delta_{\tau})\} \in UL_{1}^{\epsilon+} (-DUBV_{\perp} \text{ by Corollary 4.4})$. The following example lends support to this idea. Let $T(\delta_{\tau}) = \delta^{(n)}_{\tau}$. Then $\langle T(u), \phi \rangle = \langle u, \xi \rangle$, where $\xi(\tau) = \langle \delta^{(n)}_{\tau}, \phi \rangle = (-1)^{n} \phi^{(n)}(\tau)$. Hence $\langle T(u), \phi \rangle = \langle u, (-1)^{n} \phi^{(n)} \rangle = \langle u^{(n)}, \phi \rangle$, and $T(u) = \delta^{(n)}_{\tau}$.
In view of (S1), the n-times differentiator is BIBO stable if and only if \( n = 0 \), since \( T(\theta) = \delta^{(n-1)} \notin L^\infty \) for \( n > 0 \). On the other hand, \( \{\delta_\tau\} = \{-\frac{\partial}{\partial \tau} \theta_\tau\} \), and \( \{\theta_\tau\} \in UBV_+ \), so \( \{\delta_\tau\} \in UL^1_{\mathcal{L}_{\mathcal{L}_+}} = UBV_+ \). But \( \{\delta^{(n)}_\tau\} = \{-\frac{\partial}{\partial \tau} \delta^{(n-1)}_\tau\} \) for \( n > 0 \), and \( \delta^{(n-1)}_\tau \notin UBV_+ \), so \( \{\delta^{(n)}_\tau\} \notin UL^1_{\mathcal{L}_{\mathcal{L}_+}} \).

Corresponding to each generalized integral operator \( T : K_1 \to C^\infty \) defined by

\[
\tilde{T}(\phi)(\tau) = \langle T(\delta_\tau), \phi \rangle.
\]

Let \( \phi \in K_1, u \in L^\infty_1, \xi = \tilde{T}(\phi) \). Then

\[
\langle T(u), \phi \rangle = \langle u, \xi \rangle = \int_{-\infty}^{\infty} u(\tau)\tilde{T}(\phi)(\tau)d\tau.
\]

This suggests an adjoint relationship between \( T \) and \( \tilde{T} \), which we will explore further in Theorem 5.4. First we need a result which shows that stability of \( T \) can be characterized in terms of \( \tilde{T} \).

**Lemma 5.1.** \( T \) is BIBO stable if and only if

\[
\sup_{\|\phi\|_1 \leq 1} \int_{-\infty}^{\infty} \left| \tilde{T}(\phi)(\tau) \right| d\tau < \infty.
\]

**Proof.** The proof is identical to the proof of [1, Lemma 4.1(2)], replacing the phrase “convolution operator” by “generalized integral operator.” \( \square \)

If \( T \) is BIBO stable, Lemma 5.1 indicates that \( \tilde{T} \) is a continuous linear operator from \( \langle K_1, \| \cdot \|_1 \rangle \) into \( \langle L^1, \| \cdot \|_{L^1} \rangle \). Since \( \{K_1, \| \cdot \|_1\} \) is dense in \( L^1, \tilde{T} \) extends uniquely to a continuous linear operator \( \tilde{T}_1 : L^1 \to L^1 \).

**Lemma 5.2.** Let \( T : K'_{\mathcal{L}} \to K'_{\mathcal{L}_+} \) be a BIBO stable generalized integral operator, \( s(\cdot, \tau) = T(\theta_\tau), s_1(t) = s(t, \tau), \phi \in L^1, u \in L^\infty \). Then

1. \( \tau \mapsto \int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt \) is absolutely continuous,
2. \( \tilde{T}_1(\phi)(\tau) = -\frac{d}{d\tau} \int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt \) for every \( \tau \in \mathbb{R} \),
3. \( \int_{-\infty}^{\infty} u(\tau)d\int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t)u(\tau)d\tilde{s}_1(\tau)dt \).

**Proof.** From [7, Theorem 2.3.9], there exists \( g : \mathbb{R}^2 \to \mathbb{R} \) such that \( g(\cdot, \cdot) - g(\cdot, \infty) \in UBV_+ \), the map \( \tau \mapsto \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt \) is absolutely continuous, and

\[
\tilde{T}_1(\phi)(\tau) = \frac{d}{d\tau} \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt
\]

for each \( \phi \in L^1 \). Let \( c = \int_{-\infty}^{\infty} g(t, \infty)\phi(t)dt \). Then

\[
\int_{-\infty}^{\infty} s(t, t_0)\phi(t)dt = \int_{-\infty}^{\infty} \theta_{t_0}(\tau)\tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \frac{d}{d\tau} \left( \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt \right)d\tau
\]

\[
= \int_{-\infty}^{\infty} (g(t, \infty) - g(t, t_0))\phi(t)dt
\]

\[
= c - \int_{-\infty}^{\infty} g(t, t_0)\phi(t)dt,
\]

for each \( \phi \in L^1 \). Let \( c = \int_{-\infty}^{\infty} g(t, \infty)\phi(t)dt \). Then

\[
\int_{-\infty}^{\infty} s(t, t_0)\phi(t)dt = \int_{-\infty}^{\infty} \theta_{t_0}(\tau)\tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \frac{d}{d\tau} \left( \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt \right)d\tau
\]

\[
= \int_{-\infty}^{\infty} (g(t, \infty) - g(t, t_0))\phi(t)dt
\]

\[
= c - \int_{-\infty}^{\infty} g(t, t_0)\phi(t)dt,
\]

for each \( \phi \in L^1 \). Let \( c = \int_{-\infty}^{\infty} g(t, \infty)\phi(t)dt \). Then

\[
\int_{-\infty}^{\infty} s(t, t_0)\phi(t)dt = \int_{-\infty}^{\infty} \theta_{t_0}(\tau)\tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \tilde{T}(\phi)(\tau)d\tau
\]

\[
= \int_{t_0}^{\infty} \frac{d}{d\tau} \left( \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt \right)d\tau
\]

\[
= \int_{-\infty}^{\infty} (g(t, \infty) - g(t, t_0))\phi(t)dt
\]

\[
= c - \int_{-\infty}^{\infty} g(t, t_0)\phi(t)dt,
\]
from which (1) and (2) follow.

To prove (3), note that, for any \( \phi \in L^1, \; |\phi(t)| < \infty \) a.e. and

\[
\mu_t(-\infty, \tau) = \phi(t) s(t, \tau)
\]
determines a finite signed Borel measure on \( \mathbb{R} \) for a.e. \( t \) as does

\[
\mu(-\infty, \tau) = g(\tau) = \int_{-\infty}^{\infty} \mu_t(-\infty, \tau) dt.
\]  

(5.1)

Consider the family \( \mathcal{L} \) of sets \( A \subset \mathbb{R} \) such that the map \( t \to \mu_t(A) \) is Borel measurable for a.e. \( t \). Since

\[
\mu_t(\mathbb{R}) = \lim_{\tau \to \infty} s(t, \tau),
\]

\( \mathbb{R} \in \mathcal{L} \). If \( A, B \in \mathcal{L} \) with \( A \subset B \), then

\[
\mu_t(B - A) = \mu_t(B) - \mu_t(A),
\]

so \( B - A \in \mathcal{L} \). If \( A_n \in \mathcal{L} \) with \( A_n \uparrow A \), then

\[
\mu_t(A) = \mu_t(A_1) + \sum_{n} \mu_t(A_{n+1} - A_n),
\]

so \( A \in \mathcal{L} \). From the \( \pi - \lambda \) theorem (see [8, Theorem 4.2]), every Borel set in \( \mathbb{R} \) belongs to \( \mathcal{L} \). Hence, \( t \to \mu_t(A) \) is Borel measurable for any Borel set \( A \) and

\[
\int_{-\infty}^{\infty} |\mu_t(A)| dt = \int_{-\infty}^{\infty} \left| \int_{A} \phi(t) d\tilde{s}_t(\tau) \right| dt \leq \int_{-\infty}^{\infty} \int_{A} |\phi(t)| |d\tilde{s}_t(\tau)| dt \leq \|s\|_{UBV} \|\phi\|_1.
\]

Also, if the \( A_n \) are pairwise disjoint,

\[
\int_{-\infty}^{\infty} \mu_t(\bigcup_{n} A_n) dt = \int_{-\infty}^{\infty} \sum_{n} \mu_t(A_n) dt = \sum_{n} \int_{-\infty}^{\infty} \mu_t(A_n) dt,
\]

so the map

\[
A \to \int_{-\infty}^{\infty} \mu_t(A) dt
\]

(5.2)

is a finite signed Borel measure. Since \( \mu \) and (5.2) have the same distribution function (5.1),

\[
\mu(A) = \int_{-\infty}^{\infty} \mu_t(A) dt
\]

for each \( A \).

Let \( I_A \) be the indicator function on \( A \). Then

\[
\int_{-\infty}^{\infty} I_A(\tau) d\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_A(\tau) d\mu_t dt,
\]

(5.3)

\[
\int_{-\infty}^{\infty} I_A(\tau) d \int_{-\infty}^{\infty} s(t, \tau) \phi(t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t) I_A(\tau) d\tilde{s}_t(\tau) dt.
\]
Both sides of (3) are bounded by \( \|u\|_{\infty} \|s\|_{UBV} \|\phi\|_1 \), so, as functions of \( u \), they both represent continuous linear functionals on \( L^\infty \). Since the span of the indicators \( I_A \) is dense in \( L^\infty \), (3) implies (3).

We are now in a position to prove our main result.

**Theorem 5.3.** Let \( T : K^*_1 \rightarrow K^*_1 \) be a generalized integral operator. The following statements are equivalent:

1. \( T \) is BIBO stable.
2. \( \{T(\delta_\tau)\} \in UL^\infty_{+e} \).
3. \( \{T(\theta_\tau)\} \in UBV^+ \).

**Proof.** From Theorem 3.7, \( \{T(\theta_\tau)\} \in UBV^+ \) if and only if \( \{T(\delta_\tau)\} \in DUBV^+ \). From Corollary 4.4, \( DUBV^+ = UL^\infty_{+e} \). Thus (2) and (3) are equivalent.

To prove that (3) implies (1), let \( u \in L^\infty_+ \), \( \phi \in K_1 \), \( s_\tau = T(\theta_\tau) \), \( s(t, \tau) = s_\tau(t) \), and note that

\[
\var \{s_\tau, \phi\} = \var \int_{-\infty}^{\infty} s(t, \tau) \phi(t) \, dt \leq \int_{-\infty}^{\infty} \left( \var \{s(t, \tau)\} \right) |\phi(t)| \, dt \leq \|s\|_{UBV} \|\phi\|_1.
\]

From Theorem 3.7 and Lemma 5.2,

\[
\langle T(u), \phi \rangle = -\int_{-\infty}^{\infty} u(\tau) \langle T(\delta_\tau), \phi \rangle \, d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau) \left( \frac{\partial s_\tau}{\partial \tau}, \phi \right) \, d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau) \left( \frac{\partial}{\partial \tau} (s_\tau, \phi) \right) \, d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau) d(s_\tau, \phi),
\]

\[
|\langle T(u), \phi \rangle| \leq \|u\|_{\infty} \var \{s_\tau, \phi\} \leq \|u\|_{\infty} \|s\|_{UBV} \|\phi\|_1.
\]

Therefore,

\[
\|T(u)\|_{\infty} = \sup_{\phi \in K^*_1 \atop \|\phi\| = 1} |\langle T(u), \phi \rangle| \leq \|u\|_{\infty} \|s\|_{UBV}.
\]

Finally, we prove that (1) implies (2). Let \( \phi_1, \phi_2 \in K_1 \), and let \( \psi \in K_2 \) be given by \( \psi(t, \tau) = \phi_1(t) \phi_2(\tau) \). From Lemma 5.2,

\[
\{\{T(\delta_\tau)\}, \psi\} = \int_{-\infty}^{\infty} \langle T(\delta_\tau), \phi_1 \rangle \phi_2(\tau) \, d\tau
\]

\[
= \int_{-\infty}^{\infty} \frac{d}{d\tau} \int_{-\infty}^{\infty} s(t, \tau) \phi_1(t) \, dt \phi_2(\tau) \, d\tau
\]

\[
= -\int_{-\infty}^{\infty} \phi_2(\tau) \frac{d}{d\tau} \int_{-\infty}^{\infty} s(t, \tau) \phi_1(t) \, dt \, d\tau
\]

\[
= -\int_{-\infty}^{\infty} \phi_1(t) \left( \int_{-\infty}^{\infty} \phi_2(\tau) \, ds_\tau(\tau) \right) \, dt
\]

\[
= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t, \tau) \, ds_\tau(\tau) \, dt.
\]
From linearity and continuity of the functionals in (5.4) and from [6, p. 65], (5.4) holds for any \( \psi \in K_2 \). Thus Theorem 2.2(1) implies

\[
\{T(\delta_r)\} = -\frac{\partial s}{\partial r} \in DUBV. \quad \Box
\]

We now examine a certain extension of the operator \( T \) and its relation to \( \tilde{T}_c \).

Consider the closure \( L_0^{\infty} \) of \( L_c^{\infty} \subset L^\infty \). It is easy to show that \( L_0^{\infty} \) is a closed proper subspace of \( L^\infty \) and

\[
L_0^{\infty} = \left\{ f \in L^\infty \mid \text{ess sup}_{t \in (-\infty, -n)} |f(t)| \to 0 \text{ as } n \to \infty \right\}.
\]

If \( T \) is stable, \( T \) extends uniquely to a continuous linear operator \( T_0 : L_0^{\infty} \to L_0^{\infty} \). This extension can be taken further.

**Theorem 5.4.** Suppose \( T : K_{1}^t \to K_{1}^t \) is a BIBO stable generalized integral operator, \( s(\cdot, \tau) = T(\theta_\tau) \), and \( \delta_t(\tau) = s(t, \tau) \). Let \( T_c : L^\infty \to L^\infty \) be the continuous linear operator defined by

\[
T_c(u)(t) = -\int_{-\infty}^{\infty} u(\tau)d\delta_t(\tau).
\]

Then

1. \( T_c(u) = T_0(u) \) for all \( u \in L_0^{\infty} \),
2. \( T_c \) is the adjoint of \( \tilde{T}_c \).

**Proof.** (1) Let \( u \in L_0^{\infty} \), \( \phi \in K_1 \). From Lemma 5.2,

\[
\langle T(u), \phi \rangle = \int_{-\infty}^{\infty} u(\tau)\tilde{T}(\phi)(\tau)\,d\tau
\]

\[
= \int_{-\infty}^{\infty} u(\tau)\langle T(\delta_\tau), \phi \rangle\,d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau)\left(\frac{\partial s_\tau}{\partial r}, \phi \right)\,d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau)\frac{\partial}{\partial r}(s_\tau, \phi)\,d\tau
\]

\[
= -\int_{-\infty}^{\infty} u(\tau)d(s_\tau, \phi)
\]

\[
= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau)\phi(t)d\delta_t(\tau)\,dt.
\]

Since \( \phi \) is arbitrary,

\[
T(u)(t) = -\int_{-\infty}^{\infty} u(\tau)d\delta_t(\tau)
\]

a.e. Since \( L_c^{\infty} \) is dense in \( L_0^{\infty} \), the result follows.
(2) Let $\phi \in L^1$ and $u \in L^\infty$. From Lemma 5.2,

$$
\langle T_\varepsilon(u), \phi \rangle = \int_{-\infty}^{\infty} T_\varepsilon(u)(t)\phi(t)\,dt \\
= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\tau)\phi(t)d\tilde{s}_\varepsilon(\tau)\,dt \\
= -\int_{-\infty}^{\infty} u(\tau)d\int_{-\infty}^{\infty} s(t, \tau)\phi(t)\,dt \\
= -\int_{-\infty}^{\infty} u(\tau)\left(\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t, \tau)\phi(t)\,dt\right)\,d\tau \\
= \int_{-\infty}^{\infty} u(\tau)\tilde{T}_\varepsilon(\phi)(\tau)\,d\tau \\
= \left\langle u, \tilde{T}_\varepsilon(\phi) \right\rangle.
$$

To conclude, we give several equivalent expressions for the "gain" of a BIBO stable linear operator.

**Theorem 5.5.** For any BIBO stable generalized integral operator $T : K'_+ \to K'_1$,

$$
\sup_{u \in L^\infty, \|u\|_{\infty} = 1} \|T(u)\|_{\infty} = \sup_{u \in L^\infty, \|u\|_{\infty} = 1} \|T_\varepsilon(u)\|_{\infty}
$$

$$
= \sup_{\phi \in L^1, \|\phi\|_{1} = 1} \left\|\tilde{T}_\varepsilon(\phi)\right\|_{1} - \|\{T(\theta)\}\|_{UHV} - \|\{T(\delta)\}\|_{U,L^1}.
$$

**Proof.** Let $v \in L^\infty$, $\|v\|_{\infty} = 1$, $v_n(\tau) = v(\tau)\theta(\tau + n)$, and $\varepsilon > 0$. From Theorem 5.4, for a.e. $t$ there exists $N < \infty$ such that $n > N$ implies

$$
|T_\varepsilon(v)(t)| - |T_\varepsilon(v_n)(t)| \leq |T_\varepsilon(v_n)(t)| - T_\varepsilon(v_n)(t) \\
= \left|\int_{-\infty}^{\infty} v(\tau)d\tilde{s}_\varepsilon(\tau)\right| \leq \int_{-\infty}^{\infty} |v(\tau)||d\tilde{s}_\varepsilon(\tau)| \leq \operatorname{var} s(t, \tau) < \varepsilon.
$$

Hence,

$$
\sup_{u \in L^\infty, \|u\|_{\infty} = 1} \|T(u)\|_{\infty} \geq \|T(v_n)\|_{\infty} \geq |T(v_n)(t)| > |T_\varepsilon(v)(t)| - \varepsilon.
$$

Since $v, t, \varepsilon$ are arbitrary,

$$
\sup_{u \in L^\infty, \|u\|_{\infty} = 1} \|T(u)\|_{\infty} \geq \sup_{\varepsilon > 0} \sup_{u \in L^\infty, \|u\|_{\infty} = 1} \sup_{t} (|T_\varepsilon(u)(t)| - \varepsilon) = \sup_{u \in L^\infty, \|u\|_{\infty} = 1} \|T_\varepsilon(u)\|_{\infty}.
$$

The second equality in (5.5) follows from Theorem 5.4(2). For the third equality, set

$$
s(t, \tau) = T(\theta)(t).
$$
From Lemma 5.2,

\[ \tilde{T}_e(\phi)(\tau) = - \frac{d}{d\tau} \int_{-\infty}^{\infty} s(t, \tau) \phi(t) dt \]

for every \( \tau \), and

\[ \left\| \tilde{T}_e(\phi) \right\|_1 = \int_{-\infty}^{\infty} \left| \frac{d}{d\tau} \int_{-\infty}^{\infty} s(t, \tau) \phi(t) dt \right| dt = \text{var} \int_{-\infty}^{\infty} s(t, \tau) \phi(t) dt. \]

From [7, Theorem 2.3.9],

\[ \sup_{\phi \in L^1} \left\| \tilde{T}_e(\phi) \right\|_1 = \| s \|_{UBV}. \]

The last equality in (5.5) follows from Theorems 3.7 and 4.3. □

REFERENCES