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A CHARACTERIZATION OF BOUNDED-INPUT BOUNDED-OUTPUT STABILITY FOR LINEAR TIME-VARYING SYSTEMS WITH DISTRIBUTIONAL INPUTS*

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Abstract. We consider the problem of extending the concept of bounded-input bounded-output stability to linear time-varying systems with distributional inputs. In particular, the notion of impulse response is examined in a functional analytic setting. This requires that we first extend the classical notion of an integral operator to distribution space. Duality theory for several key normed spaces is then examined. Next, the adjoint operator corresponding to the given system is studied. Finally, necessary and sufficient conditions for stability are established, along with several expressions for the "gain" of the system.

Key words. bounded-input bounded-output stability, distributions, time-varying systems

AMS subject classification. 93D25

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1. Introduction. The concept of impulse response has traditionally played a central role in linear system theory. In spite of this fact, certain fundamental system-theoretic ideas have apparently not been developed on a mathematically rigorous level for systems with arbitrary distributional inputs and outputs. In a previous paper we addressed the problem of characterizing bounded-input bounded-output (BIBO) stability in the time-invariant case. In this paper we extend the theory to include time-varying linear systems.

To frame the problem, recall that in classical system theory a "system" is typically viewed as an integral operator

$$y(t) = \int_{-\infty}^{\infty} h(t,\tau)u(\tau)d\tau. \tag{1.1}$$

It can be shown (e.g., see [2, p. 109]) that (1.1) determines a bounded linear operator on L^{∞} if and only if

$$\sup_t \int_{-\infty}^{\infty} |h(t,\tau)|\, d\tau < \infty. \tag{1.2}$$

Such a characterization is inadequate, however, for studying systems with distributional inputs u, since the integral (1.1) is not defined. In spite of this fact, the kernel $h(t,\tau)$ is often referred to as the system "impulse response." Furthermore, there are many common systems where h itself is a distribution. For example, consider the "time-varying gain"

$$y(t) = \beta(t)u(t).$$

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Based on formal manipulations,

$$h(t,\tau) = \beta(t)\delta(t-\tau),$$

where δ is the unit impulse; condition (1.2) cannot be applied directly to distributions. Our goal is to develop a more general theory that characterizes stability for systems with distributional inputs and distributional impulse responses.

In section 2, we present basic analytic results that will be required in subsequent sections. In section 3, we study families of distributions in one variable satisfying certain smoothness properties in the index. These are then interpreted as distributions in two variables and used to generalize the notion of an integral operator. Section 4 applies the theory of normed-space extensions, developed in [1], to distributions in two variables. It is shown in Theorem 4.3 that the space of BIBO stable kernels (i.e., functions satisfying (1.2)) extends naturally to the space of distributions which are derivatives of functions of uniformly bounded variation $DUBV$. Section 5 contains the main results of the paper. Theorem 5.3 states that BIBO stable linear systems are precisely those with $DUBV$ kernels, Theorem 5.4 gives an expression for the adjoint system, and Theorem 5.5 establishes several equivalent representations of the system gain.

2. Preliminaries. First we present some pertinent facts concerning the theory of distributions. See [3], [4], and [5] for more detail. If $\phi : \mathbb{R}^n \to \mathbb{R}$, define the *support* of ϕ (denoted $\operatorname{supp}\phi$) to be the closure of the set $\{(t_1,\ldots,t_n) \in \mathbb{R}^n | \phi(t_1,\ldots,t_n) \neq 0\}$. Let K_n be the space of C^∞ functions $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{supp}\phi$ bounded. Convergence in K_n can be defined in several ways. When we assign a norm $\|\cdot\|$ to K_n we will refer to the pair $(K_n, \|\cdot\|)$. For example, $K_1 \subset L^p$, $1 \leq p \leq \infty$, so we may consider $(K_1, \|\cdot\|_p)$. Also,

$$\|\psi\|_{p\infty} = \left(\int_{-\infty}^{\infty} \sup_{\tau} |\psi(t,\tau)|^p \, dt\right)^{\frac{1}{p}} < \infty$$

for $\psi \in K_2$, so $(K_2, \|\cdot\|_{p\infty})$ is well defined for $1 \leq p < \infty$. *Strong convergence* $\phi_k \to 0$ in K_n means that there exists $a < \infty$ such that $\operatorname{supp}\phi_k \subset [-a,a]$ for every k and $\|\phi_k\|_{C^p} \to 0$ for every integer $p \geq 0$, where

$$\|\phi\|_{C^p} = \max\left\{ \left| \frac{\partial^{i_1+\cdots+i_n}\phi(t_1,\ldots,t_n)}{\partial t_1^{i_1}\cdots\partial t_n^{i_n}} \right| \;\middle|\; 0 \leq i_1+\cdots+i_n \leq p; t_1,\ldots,t_n \in \mathbb{R} \right\}.$$

A *distribution* f is an element of K_n', the dual space of K_n under strong convergence. For $f \in K_n'$, $\operatorname{supp} f$ is defined to be the complement of the largest open set $U \subset \mathbb{R}^n$ such that $\operatorname{supp}\phi \subset U$ implies $\langle f, \phi\rangle = 0$. Let K_{1+}' be the set of all $f \in K_1'$ such that there exists $a \in \mathbb{R}$ with $\operatorname{supp} f \subset [a,\infty)$. Also let K_{2+}' be the set of all $f \in K_2'$ such that there exists a function $a : \mathbb{R} \to \mathbb{R}$ with $a(\tau) \to \infty$ as $\tau \to \infty$ and $\operatorname{supp} f \subset \{t \geq a(\tau)\}$. Note that K_{2+}' is a subspace of K_2'. *Weak* convergence* $f_k \to 0$ in K_n' means that $\langle f_k, \phi\rangle \to 0$ for every $\phi \in K_n$. One basis of *weak** neighborhoods of 0 in K_n' consists of all sets of the form $\{f \mid |\langle f, \phi_i\rangle| < \varepsilon; i = 1,\ldots,m\}$, where $\varepsilon > 0$ and $\phi_1,\ldots,\phi_m \in K_n$ are arbitrary.

The *partial derivative* of $f \in K_n'$ with respect to t_i is defined by $\langle \frac{\partial f}{\partial t_i}, \phi\rangle = -\langle f, \frac{\partial \phi}{\partial t_i}\rangle$ for $\phi \in K_n$. It follows that the differentiation operator $f \to \frac{\partial f}{\partial t_i}$ is (*weak**) continuous. In case $n = 1$, we denote a derivative by $\frac{df}{dt}$ or by an overdot $\dot{f}$; the kth derivative will denoted $\frac{d^k f}{dt^k}$ or $f^{(k)}$. For any $f \in K_1'$, define the t_0-*translation* $\Delta_{t_0} f$ by $\langle \Delta_{t_0} f, \phi \rangle = \langle f, \phi_{-t_0} \rangle$, where $\phi_{-t_0}(t) = \phi(t + t_0)$. By a routine calculation, $\frac{d}{dt}\Delta_{t_0} f = \Delta_{t_0} \dot{f}$. Multiplication of $f \in K_1'$ by a C^∞ function γ is defined by $\langle f\gamma, \phi \rangle = \langle f, \gamma\phi \rangle$.

A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is *locally integrable* if $\int_A |f| < \infty$ for every bounded interval $A \subset \mathbb{R}^n$. Every locally integrable f determines a distribution according to

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n)\phi(t_1, \ldots, t_n) dt_1 \cdots dt_n.$$

Note that the *unit step function*

$$\theta(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

may be considered a distribution in K_1'; the *unit impulse* $\delta \in K_1'$ is defined by $\langle \delta, \phi \rangle = \phi(0)$. Translations of θ and δ will be denoted by θ_{t_0} and δ_{t_0}, respectively. It is easily verified that

$$\theta_{t_0}(t) = \theta(t - t_0), \qquad \langle \delta_{t_0}, \phi \rangle = \phi(t_0). \tag{2.1}$$

If f is an absolutely continuous function, f and its classical derivative $\dot{f}$ are locally integrable. In this case, the classical and distributional derivatives of f coincide, since

$$\left\langle \dot{f}, \phi \right\rangle = \int_{-\infty}^{\infty} \dot{f}(t)\,\phi(t) dt = -\int_{-\infty}^{\infty} f(t)\,\dot{\phi}(t) dt = \left\langle f, \dot{\phi} \right\rangle.$$

Consider the spaces

$$BV = \{g : \mathbb{R} \to \mathbb{R} \mid \operatorname*{var}_t g(t) < \infty\},$$
$$NBV = \{g \in BV \mid g \text{ is left-continuous and } g(\infty) = 0\}$$

with norm $\|g\|_{NBV} = \operatorname{var}_t g(t)$. (Note that we are deviating slightly from the conventional definition of NBV, as in [9, p. 171].)

In [1] we also considered the space $DBV = \{\dot{g} \mid g \in NBV\}$ with norm $\|\dot{g}\|_{DBV} = \|g\|_{NBV}$. We showed in [1, p. 989] that DBV is isometrically isomorphic to the dual space of $(K_1, \|\cdot\|_\infty)$. Furthermore, for any $g \in NBV$ and $\phi \in K_1$, $\langle \dot{g}, \phi \rangle = \int_{-\infty}^{\infty} \phi(t) dg(t)$. We need to generalize these ideas to distributions on $\mathbb{R}^2$. The appropriate construction requires a preliminary result.

LEMMA 2.1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be Lebesgue measurable and $g(t, \cdot) \in NBV$ for a.e. t. Then the function*

$$v(t, \tau) = \operatorname*{var}_{\eta \geq \tau} g(t, \eta)$$

is Lebesgue measurable on $\mathbb{R}^2$.

Proof. Enumerate the rationals $\{r_n\}$, and consider the partition $\pi_n = (r_{k_1}, \ldots, r_{k_n})$ of $\mathbb{R}$, where $\{k_1, \ldots, k_n\} = \{1, \ldots, n\}$ and

$$r_{k_1} < r_{k_2} < \cdots < r_{k_n}.$$

Let $A = \{t \mid g(t,\cdot) \in NBV\}$. For each n , $j = 1, \dots, n-1$, and $(t,\tau) \in A \times (r_{k_j}, r_{k_{j+1}}]$, define

$$v_n(t,\tau) = |g(t, r_{k_n})| + \sum_{i=j+1}^{n} \left|g(t, r_{k_i}) - g\left(t, r_{k_{i+1}}\right)\right| + \left|g(t,\tau) - g\left(t, r_{k_{j+1}}\right)\right|.$$

For other (t,τ), set $v_n(t,\tau) = 0$. Each v_n is Lebesgue measurable. Let $\varepsilon > 0$, $(t,\tau) \in A \times \mathbb{R}$, and $\pi = (\tau_1, \dots, \tau_p)$ be any partition of (τ, ∞) such that

$$q(t,\tau) = |g(t,\tau_p)| + \sum_{i=1}^{p-1} |g(t,\tau_i) - g(t,\tau_{i+1})| + |g(t,\tau) - g(t,\tau_1)| > v(t,\tau) - \frac{\varepsilon}{2}.$$

Since g is left-continuous, there exists $N < \infty$ such that π_n contains rationals $r_{k_{j_i}}$ with $\tau < r_{k_{j_1}} < \cdots < r_{k_{j_p}}$ such that

$$\left|g\left(t, r_{k_{j_i}}\right) - g(t,\tau_i)\right| < \frac{\varepsilon}{4p}$$

for every i. Thus

$$\begin{aligned}
v_n(t,\tau) &\geq \left|g\left(t, r_{k_{j_p}}\right)\right| + \sum_{i=1}^{p-1} \left|g\left(t, r_{k_{j_i}}\right) - g\left(t, r_{k_{j_{i+1}}}\right)\right| + \left|g(t,\tau) - g\left(t, r_{k_{j_1}}\right)\right| \\
&\geq |g(t,\tau_p)| - \left|g\left(t, r_{k_{j_p}}\right) - g(t,\tau_p)\right| \\
&\quad + \sum_{i=1}^{p-1} |g(t,\tau_i) - g(t,\tau_{i+1})| - \sum_{i=1}^{p-1} \left|g\left(t, r_{k_{j_i}}\right) - g(t,\tau_i)\right| \\
&\quad - \sum_{i=1}^{p-1} \left|g\left(t, r_{k_{j_{i+1}}}\right) - g(t,\tau_{i+1})\right| \\
&\quad + |g(t,\tau) - g(t,\tau_1)| - \left|g\left(t, r_{k_{j_1}}\right) - g(t,\tau_1)\right| \\
&\geq q(t,\tau) - \frac{\varepsilon}{2} \\
&> v(t,\tau) - \varepsilon.
\end{aligned}$$

So $v_n \to v$ a.e., and v is Lebesgue measurable. □

In particular, if g satisfies the conditions of Lemma 2.1, the map $t \to \text{var}_\tau\, g(t,\tau)$ is Lebesgue measurable. Hence, we may define UBV to be the set of functions $g : \mathbb{R}^2 \to \mathbb{R}$ satisfying the following properties:

(UBV1) g is Lebesgue measurable.

(UBV2) $g(t,\cdot) \in NBV$ for a.e. t.

(UBV3) $\text{ess sup}_t\, \text{var}_\tau\, g(t,\tau) < \infty$.

We refer to UBV as the functions of *uniformly bounded variation.* Let $\|g\|_{UBV} = \text{ess sup}_t\, \text{var}_\tau\, g(t,\tau)$. Each $g \in UBV$ is bounded, since

$$|g(t,\tau)| \leq \operatorname*{var}_{\tau} g(t,\tau) \leq \|g\|_{UBV}.$$

Thus $g \in K_2'$, and we may also define the set of partial derivatives

$$DUBV = \left\{ \frac{\partial g}{\partial \tau} \mid g \in UBV \right\}.$$

It is routine to verify that UBV and $DUBV$ are linear spaces and that $\|\cdot\|_{UBV}$ and

$$\left\|\frac{\partial g}{\partial\tau}\right\|_{DUBV}=\|g\|_{UBV}$$

are norms on UBV and $DUBV$.

For $1\leq p<\infty$, let UL^p be the space of Lebesgue measurable functions $f:\mathbb{R}^2\to\mathbb{R}$ satisfying

$$\operatorname*{ess\,sup}_t\int_{-\infty}^{\infty}|f(t,\tau)|^p\,d\tau<\infty.$$

(Lebesgue measurability of the map $t\to\int_{-\infty}^{\infty}|f(t,\tau)|^p d\tau$ follows from [9, Theorem 7.8].) UL^p is the set of *uniformly* L^p functions and has norm

$$\|f\|_{\infty p}=\operatorname*{ess\,sup}_t\left(\int_{-\infty}^{\infty}|f(t,\tau)|^p\,d\tau\right)^{\frac{1}{p}}.$$

We may consider $UL^p\subset K_2'$, since, from Holder's inequality,

$$\begin{aligned}\int_{-a}^{a}\int_{-a}^{a}|f(t,\tau)|\,d\tau dt&\leq\int_{-a}^{a}\left(\operatorname*{ess\,sup}_t\int_{-a}^{a}|f(t,\tau)|\,d\tau\right)dt\\&=2a\operatorname*{ess\,sup}_t\int_{-a}^{a}|f(t,\tau)|\,d\tau\\&\leq(2a)^{2-\frac{1}{p}}\operatorname*{ess\,sup}_t\left(\int_{-a}^{a}|f(t,\tau)|^p\,d\tau\right)^{\frac{1}{p}}.\end{aligned}$$

For $p=\infty$, we define UL^∞ to be the same as L^∞ on $\mathbb{R}^2$.

Support constraints may be placed on the spaces above by setting $L_+^p=L^p\cap K_{1+}'$, $UBV_+=UBV\cap K_{2+}'$, $DUBV_+=DUBV\cap K_{2+}'$, and $UL^p{}_+=UL^p\cap K_{2+}'$.

THEOREM 2.2. (1) *Let* $g\in UBV$ *and* $g_t=g(t,\cdot)$. *Then*

$$\left\langle\frac{\partial g}{\partial\tau},\psi\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(t,\tau)dg_t(\tau)dt$$

for every $\psi\in K_2$.

(2) $UL^1\subset DUBV$ *with* $\|f\|_{DUBV}=\|f\|_{\infty 1}$ *for every* $f\in UL^1$.

(3) $DUBV$ *is the dual of* $(K_2,\|\cdot\|_{1\infty})$.

(4) UBV *and* $DUBV$ *are isometrically isomorphic.*

Proof. (1) Integration by parts yields

$$\left\langle\frac{\partial g}{\partial\tau},\psi\right\rangle=-\left\langle g,\frac{\partial\psi}{\partial\tau}\right\rangle=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_t(\tau)\frac{\partial\psi(t,\tau)}{\partial\tau}d\tau dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(t,\tau)dg_t(\tau)dt$$

for every ψ.

(2) If $f\in UL^1$, there exists a Lebesgue measurable $A\subset\mathbb{R}$ such that $f(t,\cdot)\in L^1$ for every $t\in A$. Let $g(t,\tau)=\int_{-\infty}^{\tau}f(t,\eta)d\eta$ for $t\in A$. Then g is Lebesgue measurable, $g(t,\cdot)$ is absolutely continuous, and $f=\frac{\partial g}{\partial\tau}$. The result follows from

$$\|f\|_{DUBV}=\operatorname*{ess\,sup}_t\operatorname*{var}_\tau g(t,\tau)=\operatorname*{ess\,sup}_t\int_{-\infty}^{\infty}|f(t,\tau)|\,d\tau=\|f\|_{\infty 1}<\infty.$$

(3) We first prove that $DUBV$ is contained in the dual of K_2. Let g and g_τ be as in (1) and $f = \frac{\partial g}{\partial \tau}$. We must show that

$$\sup_{\|\psi\|_{1\infty}=1} |\langle f, \psi\rangle| = \|f\|_{DUBV}\,.$$

From (1),

$$\begin{aligned}
|\langle f, \psi\rangle| &= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t,\tau)dg_t(\tau)dt\right| \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(t,\tau)|\,|dg_t(\tau)|\,dt \\
&\le \int_{-\infty}^{\infty} \sup_\tau |\psi(t,\tau)| \operatorname*{var}_\tau g(t,\tau)dt \\
&\le (\operatorname*{ess\,sup}_t \operatorname*{var}_\tau g(t,\tau)) \int_{-\infty}^{\infty} \sup_\tau |\psi(t,\tau)|\,dt \\
&= \|f\|_{DUBV}\,\|\psi\|_{1\infty},
\end{aligned}$$

so

$$\sup_{\|\psi\|_{1\infty}=1} |\langle f, \psi\rangle| \le \|f\|_{DUBV}\,.$$

To establish the reverse inequality, observe that, for $\phi_1, \phi_2 \in K_1$, setting $\psi(t,\tau) = \phi_1(t)\phi_2(\tau)$ yields $\psi \in K_2$ and

$$\|\psi\|_{1\infty} = \int_{-\infty}^{\infty} \sup_\tau |\phi_1(t)\phi_2(\tau)|\,dt = \|\phi_1\|_1\,\|\phi_2\|_\infty\,.$$

Thus

$$\begin{aligned}
\sup_{\|\psi\|_{1\infty}=1} |\langle f, \psi\rangle| &= \sup_{\|\psi\|_{1\infty}=1} \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t,\tau)dg_t(\tau)dt\right| \\
&\ge \sup_{\|\phi_2\|_\infty=1}\ \sup_{\|\phi_1\|_1=1} \left|\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \phi_2(\tau)dg_t(\tau)\right)\phi_1(t)dt\right| \\
&= \sup_{\|\phi_2\|_\infty=1}\ \operatorname*{ess\,sup}_t \left|\int_{-\infty}^{\infty} \phi_2(\tau)dg_t(\tau)\right| \\
&= \operatorname*{ess\,sup}_t \sup_{\|\phi_2\|_\infty=1} |\langle \dot{g}_t, \phi_2\rangle| \\
&= \operatorname*{ess\,sup}_t \|\dot{g}_t\|_{DBV} \\
&= \|f\|_{DUBV}\,.
\end{aligned}$$

Next we show that $DUBV$ contains the dual of $(K_2, \|\cdot\|_{1\infty})$. Let f be any continuous linear functional on $(K_2, \|\cdot\|_{1\infty})$. ($f$ is also a distribution, since $\psi_k \to 0$ strongly in K_2 implies $\|\psi_k\|_{1\infty} \to 0$ and $\langle f, \psi_k\rangle \to 0$.) Let $\phi_1 \in K_1$. Then each $\phi_2 \in K_1$ determines $\psi \in K_2$ by $\psi(t,\tau) = \phi_1(t)\phi_2(\tau)$ (i.e., ψ is the "direct product" $\psi = \phi_1 \times \phi_2$). The map $\phi_2 \longmapsto \psi$ is continuous from $(K_1, \|\cdot\|_\infty)$ into $(K_2, \|\cdot\|_{1\infty})$, so $\phi_2 \longmapsto \langle f, \phi_1 \times \phi_2\rangle$ is a continuous linear functional on $(K_1, \|\cdot\|_\infty)$. Since K_1 is dense in C_0, there exists $G(\phi_1; \cdot) \in NBV$ such that

$$\langle f, \phi_1 \times \phi_2\rangle = \int_{-\infty}^{\infty} \phi_2(\tau)dG(\phi_1;\tau) \tag{2.2}$$

for every ϕ_2. It is routine to show that the operator $\phi_1 \mapsto G(\phi_1;\cdot)$ is linear. Continuity also holds, since

$$\begin{aligned}\sup_{\|\phi_1\|_1=1} \|G(\phi_1;\cdot)\|_{NBV} &= \sup_{\|\phi_1\|_1=1} \sup_{\|\phi_2\|_\infty=1} \left| \int_{-\infty}^{\infty} \phi_2(\tau) dG(\phi_1;\tau) \right| \\ &= \sup_{\|\phi_1\|_1=1} \sup_{\|\phi_2\|_\infty=1} |\langle f, \phi_1 \times \phi_2 \rangle| \\ &\le \sup_{\|\psi\|_{1\infty}=1} |\langle f, \psi \rangle| \\ &< \infty.\end{aligned}$$

Therefore, $\phi_1 \mapsto G(\phi_1;\cdot)$ is a continuous linear operator from $(K_1, \|\cdot\|_1)$ into NBV. From [7, Theorem 2.3.1] and (2.2), there exists $g \in UBV$ such that

$$\langle f, \phi_1 \times \phi_2 \rangle = \int_{-\infty}^{\infty} \phi_2(\tau) d \left(\int_{-\infty}^{\infty} g(t,\tau)\phi_1(t) dt \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(t)\phi_2(\tau) dg_t(\tau) dt$$

for every $\phi_1, \phi_2 \in K_1$. Let $\Pi = \{\phi_1 \times \phi_2 \mid \phi_1, \phi_2 \in K_1\}$. By linearity,

$$\langle f, \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t,\tau) dg_t(\tau) dt \tag{2.3}$$

for every $\psi \in \operatorname{span}\Pi$. From [6, p. 65], $\operatorname{span}\Pi$ is strongly dense in K_2, so it is also dense relative to $\|\cdot\|_{1\infty}$. By continuity, (2.3) holds for all $\psi \in K_2$. From part (1), $f = \frac{\partial g}{\partial \tau} \in DUBV$.

(4) Note that the map $g \to \frac{\partial g}{\partial \tau}$ from UBV into $DUBV$ is defined to be linear, onto, and norm-preserving. It remains to show that the map is one-to-one. From part (1), if $\frac{\partial g}{\partial \tau} = 0$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(t,\tau) dg_t(\tau) dt = 0$$

for all $\psi \in K_2$. Hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(t)\phi_2(\tau) dg_t(\tau) dt = 0$$

for all $\phi_1, \phi_2 \in K_1$. Since

$$\left| \int_{-\infty}^{\infty} \phi_1(t) dg_t(\tau) \right| \le \|\phi_1\|_\infty \operatorname*{var}_{\tau} g(t,\tau)$$

for a.e. t, the map $t \to \int_{-\infty}^{\infty} \phi_1(t) dg_t(\tau)$ may be viewed as a distribution $T(\phi_1)$. But $\langle T(\phi_1), \phi_2 \rangle = 0$ for every ϕ_2, so $T(\phi_1) = 0$ and

$$\int_{-\infty}^{\infty} \phi_1(t) dg_t(\tau) = 0$$

a.e. for every ϕ_1. Since NBV is the dual of $(K_1, \|\cdot\|_\infty)$, $g(t,\cdot) = 0$ for a.e. t and $g = 0$ a.e. $\square$

3. Families and integral operators. In order to generalize the concept of an integral operator as in (1.1), we must study collections of distributions indexed by a real parameter. Let $\{f_t \mid t \in \mathbb{R}\}$ be a collection of distributions in K_1'. Suppose that, for each $a < \infty$, there exist an integer $p \geq 0$ and an L^1 function $M : [-a, a] \to \mathbb{R}$ such that

$$|\langle f_t, \phi \rangle| \leq M(t) \, \|\phi\|_{C^p}$$

for every $\phi \in K_1$ with $\operatorname{supp} \phi \subset [-a, a]$. Then we say that $\{f_t\}$ is an L^1 *family* of distributions on $\mathbb{R}$.

THEOREM 3.1. (1) *If $\{f_t\}$ is an L^1 family, then the map $\psi \to \int_{-\infty}^{\infty} \langle f_t, \psi(t, \cdot) \rangle dt$ is a distribution in K_2'.*

(2) *If $f = \{f_t\}$ is an L^1 family, then so is $\{\dot{f}_t\}$, and $\frac{\partial f}{\partial \tau} = \{\dot{f}_t\}$.*

(3) *If $f : \mathbb{R}^2 \to \mathbb{R}$ is locally integrable, then $\{f(t, \cdot)\}$ is an L^1 family.*

(4) *If $f \in UBV$, then $\{f(t, \cdot)\}$ is an L^1 family.*

Proof. (1) Let $\psi \in K_2$. Then $\|\psi(t, \cdot)\|_{C^p} \leq \|\psi\|_{C^p}$ for every t, so

$$\int_{-a}^{a} |\langle f_t, \psi(t, \cdot) \rangle| \, dt \leq \int_{-a}^{a} M(t) \, \|\psi(t, \cdot)\|_{C^p} \, dt \leq \int_{-a}^{a} M(t) dt \, \|\psi\|_{C^p} \, .$$

Thus the map $\psi \to \int_{-\infty}^{\infty} \langle f_t, \psi(t, \cdot) \rangle dt$ is well defined, linear, and, from [4, p. 34], continuous.

(2) There exist M, p such that, for any $\phi \in K_1$ with $\operatorname{supp} \phi \subset [-a, a]$,

$$\left| \left\langle \dot{f}_t, \phi \right\rangle \right| = \left| \left\langle f_t, \dot{\phi} \right\rangle \right| \leq M(t) \left\| \dot{\phi} \right\|_{C^p} \leq M(t) \, \|\phi\|_{C^{p+1}} \, .$$

For any $\psi \in K_2$,

$$\begin{aligned} \left\langle \frac{\partial \{f_t\}}{\partial \tau}, \psi \right\rangle &= - \left\langle \{f_t\}, \frac{\partial \psi(t, \tau)}{\partial \tau} \right\rangle \\ &= - \int_{-a}^{a} \left\langle f_t, \frac{\partial \psi(t, \tau)}{\partial \tau} \right\rangle dt \\ &= \int_{-a}^{a} \left\langle \dot{f}_t, \psi(t, \cdot) \right\rangle dt \\ &= \left\langle \left\{ \dot{f}_t \right\}, \psi \right\rangle . \end{aligned}$$

(3) Let $M(t) = \int_{-a}^{a} |f(t, \tau)| d\tau$. By Fubini's theorem, M is L^1 on $[-a, a]$. For any $\phi \in K_1$,

$$|\langle f(t, \cdot), \phi \rangle| = \left| \int_{-a}^{a} f(t, \tau) \phi(\tau) d\tau \right| \leq M(t) \, \|\phi\|_{C^0} \, .$$

(4) This follows immediately from part (3) and the fact that f is bounded on $\mathbb{R}^2$. $\square$

A slight modification of the argument used in Theorem 3.1(1) shows that each L^1 family $\{f_\tau\}$ also determines a distribution according to $\psi \to \int_{-\infty}^{\infty} \langle f_\tau, \psi(\cdot, \tau) \rangle d\tau$. We will rely on the notation $\{f_t\}$ and $\{f_\tau\}$ to distinguish these two cases.

Consider a collection of distributions $\{f_\tau\}$ in K_1' with $\tau \to \langle f_\tau, \phi \rangle$ continuous for every $\phi \in K_1$. Then we say that $\{f_\tau\}$ is a C^0 *family*.

THEOREM 3.2. *Every C^0 family is an L^1 family.*

Proof. Let $\{f_\tau\}$ be a C^0 family. According to [4, p. 34], each f_τ has finite order on each bounded interval $[-a,a]$; i.e., for every $\tau \in [-a,a]$ there exist integers M_τ, $p_\tau < \infty$ such that

$$|\langle f_\tau, \phi\rangle| \leq M_\tau \|\phi\|_{C^{p_\tau}} \tag{3.1}$$

for every $\phi \in K_1$ with $\operatorname{supp}\phi \subset [-a,a]$. Suppose each M_τ and p_τ are chosen to minimize $q_\tau = \max\{M_\tau, p_\tau\}$. If the set $\{q_\tau \mid |\tau| \leq a\}$ is unbounded, there exist $\eta_k, \eta \in [-a,a]$ such that $\eta_k \to \eta$ and $q_{\eta_k} \to \infty$. On the other hand, since $\langle f_\tau, \phi\rangle$ is continuous, $\langle f_{\eta_k}, \phi\rangle \to \langle f_\eta, \phi\rangle$ for every ϕ. From [4, p. 57], there exist M, p such that

$$|\langle f_{\eta_k}, \phi\rangle| \leq M \|\phi\|_{C^p}$$

for every k, yielding a contradiction. Hence, $\{q_\tau\}$ is bounded, and there exist M, p such that

$$|\langle f_\tau, \phi\rangle| \leq M \|\phi\|_{C^p}$$

for every τ, ϕ. □

In addition to continuity, we might also consider families $\{f_\tau\}$ which are differentiable in τ. Define the *weak* derivative* $\frac{\partial f_\tau}{\partial \tau}|_{\tau_0} \in K_2'$ of $\{f_\tau\}$ at τ_0 according to

$$\left\langle \frac{\partial f_\tau}{\partial \tau}|_{\tau_0}, \phi\right\rangle = \frac{d}{d\tau}\langle f_\tau, \phi\rangle|_{\tau_0} = \lim_{\tau_n \to \tau_0}\left\langle \frac{f_{\tau_n} - f_{\tau_0}}{\tau_n - \tau_0}, \phi\right\rangle, \tag{3.2}$$

whenever the limit exists for every $\phi \in K_1$. According to [3, p. 368], (3.2) determines a distribution in K_1' for each τ_0. If $\{\frac{\partial f_\tau}{\partial \tau}|_{\tau_0} \mid \tau_0 \in \mathbb{R}\}$ is a C^0 family, we denote it by $\frac{\partial f_\tau}{\partial \tau}$ and say that $\{f_\tau\}$ is a C^1 *family.* Continuing in this way, if $\{\frac{\partial f_\tau}{\partial \tau}\}$ is a C^{p-1} family, $\{f_\tau\}$ is a C^p *family.* Applying (3.2), $\{f_\tau\}$ is a C^p family if and only if $\tau \to \langle f_\tau, \phi\rangle$ is a C^p function for each ϕ. We may interpret the latter statement as a definition for $p = \infty$. Since $\langle \dot{f}_\tau, \phi\rangle = -\langle f_\tau, \dot{\phi}\rangle$, $\{\dot{f}_\tau\}$ is a C^p family whenever $\{f_\tau\}$ is a C^p family.

Next we relate two notions of differentiation for C^1 families.

THEOREM 3.3. *Suppose $g = \{g_\tau\}$ is a C^1 family. Then $\frac{\partial g}{\partial \tau} = \{\frac{\partial g_\tau}{\partial \tau}\}$.*

Proof. Suppose

$$\left\langle \frac{g_\tau - g_{\tau_0}}{\tau - \tau_0}, \psi(\cdot,\tau)\right\rangle \not\to \left\langle \frac{\partial g_\tau}{\partial \tau}|_{\tau_0}, \psi(\cdot,\tau_0)\right\rangle \tag{3.3}$$

as $\tau \to \tau_0$. Then there exist $\tau_n \to \tau_0$ and $\varepsilon > 0$ such that

$$\sup_j \left|\left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0} - \frac{\partial g_\tau}{\partial \tau}|_{\tau_n}, \psi(\cdot,\tau_j)\right\rangle\right| \geq \left|\left\langle \frac{g_{\tau_n} - g_{\tau_0}}{\tau_n - \tau_0} - \frac{\partial g_\tau}{\partial \tau}|_{\tau_n}, \psi(\cdot,\tau_n)\right\rangle\right| > \varepsilon \tag{3.4}$$

for every n. But $\psi(\cdot,\tau_n) \to \psi(\cdot,\tau_0)$ in K_1, so, from [4, p. 31], $\{\psi(\cdot,\tau_n)\} \subset K_1$ is a bounded set. Hence, from [4, p. 56], the left side of (3.4) must converge to 0. This yields a contradiction, so we have convergence in (3.3). Thus

$$\begin{aligned}\frac{d}{d\tau}\langle g_\tau, \psi(\cdot,\tau)\rangle|_{\tau_0} &= \lim_{\tau\to\tau_0}\frac{\langle g_\tau, \psi(\cdot,\tau)\rangle - \langle g_{\tau_0}, \psi(\cdot,\tau_0)\rangle}{\tau - \tau_0}\\ &= \lim_{\tau\to\tau_0}\left\langle \frac{g_\tau - g_{\tau_0}}{\tau - \tau_0}, \psi(\cdot,\tau)\right\rangle + \lim_{\tau\to\tau_0}\left\langle g_{\tau_0}, \frac{\psi(\cdot,\tau) - \psi(\cdot,\tau_0)}{\tau - \tau_0}\right\rangle\\ &= \left\langle \frac{\partial g_\tau}{\partial \tau}|_{\tau_0}, \psi(\cdot,\tau_0)\right\rangle + \left\langle g_{\tau_0}, \frac{\partial \psi}{\partial \tau}|_{\tau_0}\right\rangle,\end{aligned}$$

and, for every $\psi \in K_2$,

$$\begin{aligned}\left\langle \frac{\partial g}{\partial \tau}, \psi \right\rangle &= -\left\langle g, \frac{\partial \psi}{\partial \tau} \right\rangle \\ &= -\int_{-\infty}^{\infty} \left\langle g_\tau, \frac{\partial \psi}{\partial \tau} \right\rangle d\tau \\ &= -\int_{-\infty}^{\infty} \left(\frac{d}{d\tau} \langle g_\tau, \psi(\cdot,\tau)\rangle - \left\langle \frac{\partial g_\tau}{\partial \tau}, \psi(\cdot,\tau) \right\rangle \right) d\tau \\ &= -\lim_{\tau\to\infty} \langle g_\tau, \psi(\cdot,\tau)\rangle + \lim_{\tau\to-\infty} \langle g_\tau, \psi(\cdot,\tau)\rangle + \int_{-\infty}^{\infty} \left\langle \frac{\partial g_\tau}{\partial \tau}, \psi(\cdot,\tau) \right\rangle d\tau \\ &= \int_{-\infty}^{\infty} \left\langle \frac{\partial g_\tau}{\partial \tau}, \psi(\cdot,\tau) \right\rangle d\tau.\end{aligned}$$

Hence $\frac{\partial g}{\partial \tau} = \{\frac{\partial g_\tau}{\partial \tau}\}$. □

Note that

$$\left\langle \frac{\partial \theta_\tau}{\partial \tau}, \phi \right\rangle = \frac{\partial}{\partial \tau} \int_\tau^\infty \phi(\eta) d\eta = -\phi(\tau) = \langle -\delta_\tau, \phi \rangle$$

for every $\phi \in K_1$, where θ_τ is a translation of the unit step as in (2.1). In view of Theorem 3.3, $\frac{\partial}{\partial \tau}\{\theta_\tau\} = \{\frac{\partial \theta_\tau}{\partial \tau}\} = -\{\delta_\tau\}$. By a similar calculation, $\frac{\partial}{\partial \tau}\{\delta_\tau^{(n-1)}\} = -\{\delta_\tau^{(n)}\}$.

Any C^∞ family $\{h_\tau\}$ belonging to K'_{2+} determines a linear operator on K'_{1+} in the following way. For each $\phi \in K_1$, let $\xi(\tau) = \langle h_\tau, \phi \rangle$. Then $\xi(\tau)$ is C^∞ with $\xi(\tau) = 0$ for large τ. Suppose $u \in K'_{1+}$ with $\operatorname{supp} u \subset [a, \infty]$, and let $\bar{\xi} \in K_1$ with $\bar{\xi}(\tau) = \xi(\tau)$ for $\tau \geq a$. It is easy to show that $y(\phi) = \langle u, \bar{\xi} \rangle$ is independent of the choice of $\bar{\xi}$. Indeed, let $\bar{\xi}_1$ and $\bar{\xi}_2$ be two such functions. Then $\operatorname{supp}(\bar{\xi}_1 - \bar{\xi}_2) \subset (-\infty, a]$, and $\langle u, \bar{\xi}_1 \rangle - \langle u, \bar{\xi}_2 \rangle = \langle u, \bar{\xi}_1 - \bar{\xi}_2 \rangle = 0$.

THEOREM 3.4. *The map $u \longmapsto y$ defines a linear operator $T : K'_{1+} \to K'_{1+}$, where $T(\delta_{t_0}) = h_{t_0}$ for every t_0.*

Proof. Linearity is obvious. Since $\{h_\tau\} \in K'_{2+}$, there exists $b \in \mathbb{R}$ such that $\operatorname{supp} \phi \subset (-\infty, b]$ implies $\operatorname{supp} \xi \subset (-\infty, a)$. For any such ϕ, $y(\phi) = 0$, so $\operatorname{supp} y \subset [b, \infty)$. We must show that $y \in K'_1$.

The topological space K'_1 is not first-countable, so we must consider nets $\{\phi_\lambda\}$ in K_1. Suppose $\phi_\lambda \to 0$. Then there exists $b < \infty$ such that $\operatorname{supp} \phi_\lambda \subset [-b, b]$ for every λ. Let $\xi_\lambda(\tau) = \langle h_\tau, \phi_\lambda \rangle$. Since $\{h_\tau\} \in K'_{1+}$, there exists $c < \infty$ such that $\operatorname{supp} \xi_\lambda(\tau) \subset (-\infty, c]$ for every λ. Arguing as in the proof of Theorem 3.2, there exist $M_n, p_n < \infty$ such that

$$\left|\xi^{(n)}(\tau)\right| \leq M_n \|\phi\|_{C^{p_n}}$$

for every $\tau \in [a, c]$ and $\phi \in K_1$ with $\operatorname{supp} \phi \in [-b, b]$. Hence, $\xi_\lambda^{(n)} \to 0$ uniformly on $[a, c]$ for every n. $\bar{\xi}_\lambda \in K_1$ can be chosen so that $\bar{\xi}_\lambda(\tau) = \xi_\lambda(\tau)$ on $[a, \infty)$ and each $\bar{\xi}_\lambda^{(n)} \to 0$ uniformly on $[a, c]$. Thus $\bar{\xi}_\lambda \to 0$ strongly in K_1, and $y(\phi_\lambda) = \langle u, \bar{\xi}_\lambda \rangle \to 0$. Finally, for $u = \delta_{t_0}$ and any $\phi \in K_1$, $\langle T(\delta_{t_0}), \phi \rangle = \bar{\xi}(t_0) = \xi(t_0) = \langle h_{t_0}, \phi \rangle$. □

Suppose $h_\tau = h(\cdot, \tau)$. Then, under mild assumptions,

$$\langle T(u), \phi \rangle = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(t,\tau)\phi(t) dt \right) u(\tau) d\tau = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(t,\tau) u(\tau) d\tau \right) \phi(t) dt,$$

so the operator determined by Theorem 3.4 is "classical." For this reason, we call every operator of the type described by Theorem 3.4 a *generalized integral operator.* Suppose h_τ is the τ-translation of a fixed $h \in K'_{1+}$. Then $\{h_\tau\}$ is a C^∞ family and $\{h_\tau\} \in K'_{2+}$, so $\{h_\tau\}$ determines a generalized integral operator according to $\langle T(u), \phi\rangle = \langle u, \xi\rangle$, where $\xi(\tau) = \langle h_\tau, \phi\rangle = \langle h, \phi_{-\tau}\rangle$. From [3, p. 103], constructing $T(u)$ in this way yields the convolution operator $u \to h * u$. This sets up the time-invariant analysis we carried out in [1].

We next address the issue of continuity of generalized integral operators.

THEOREM 3.5. *Let $T : K'_{1+} \to K'_{1+}$ be any continuous linear operator, and suppose $\{T(\delta_\tau)\} \in K'_{2+}$ is a C^∞ family. Then T is a generalized integral operator.*

Proof. Let $\phi \in K_1$, $u \in K'_{1+}$, $\operatorname{supp} u \subset [a, \infty)$, $\xi(\tau) = \langle T(\delta_\tau), \phi\rangle$, and $\bar{\xi} \in K_1$ with $\bar{\xi}(\tau) = \xi(\tau)$ for $\tau \geq a$. We must show that $\langle T(u), \phi\rangle = \langle u, \bar{\xi}\rangle$. First, set $u = \delta_{t_0}$. Then

$$\langle T(\delta_{t_0}), \phi\rangle = \xi(t_0) = \bar{\xi}(t_0) = \langle \delta_{t_0}, \bar{\xi}\rangle .$$

For arbitrary $u \in K'_{1+}$, [1, Lemma 2.2] shows that there exist $t_{ik} \geq a$, $\beta_{ik} \in \mathbb{R}$, and integers $n_k > 0$ such that

$$u_k = \sum_{i=1}^{n_k} \beta_{ik} \delta_{t_{ik}} \to u,$$

where the limit is *weak**. For every k,

$$\langle T(u_k), \phi\rangle = \sum_{i=1}^{n_k} \beta_{ik} \langle T(\delta_{t_{ik}}), \phi\rangle = \sum_{i=1}^{n_k} \beta_{ik} \langle \delta_{t_{ik}}, \bar{\xi}\rangle = \langle u_k, \bar{\xi}\rangle .$$

Taking the limit yields

$$\langle T(u), \phi\rangle = \langle u, \bar{\xi}\rangle . \qquad \square$$

The next result establishes conditions under which generalized integral operators are continuous. In particular, it shows that the converse to Theorem 3.5 is false.

THEOREM 3.6. *Let $T : K'_{1+} \to K'_{1+}$ be a generalized integral operator.*

(1) *T is continuous if and only if there exists a function $b : \mathbb{R} \to \mathbb{R}$ such that $b(\tau) \to \infty$ as $\tau \to -\infty$ and* $\operatorname{supp}\{T(\delta_\tau)\} \subset \{|t| \geq b(\tau)\}$.

(2) *T is continuous on* $\{u \in K'_{1+} \mid \operatorname{supp} u \subset [a, \infty)\}$ *for every $a > -\infty$.*

Proof. (1) (Sufficient) Consider any basic neighborhood $Y = \{y \in K'_{1+} \mid |\langle y, \phi_i\rangle| < \varepsilon; i = 1, \dots, n\}$ of 0 in K'_{1+}, and let $\xi_i(\tau) = \langle T(\delta_\tau), \phi_i\rangle$. Since T is a generalized integral operator, each ξ_i is C^∞. $\{h_\tau\} \in K'_{1+}$ guarantees that $\operatorname{supp} \xi_i$ is right-bounded. The condition $\operatorname{supp}\{T(\delta_\tau)\} \subset \{|t| \geq b(\tau)\}$ guarantees that $\operatorname{supp} \xi_i$ is left-bounded. Hence, $\xi_i \in K_1$, and $\langle T(u), \phi_i\rangle = \langle u, \xi_i\rangle$. It follows that the inverse image

$$\begin{aligned} T^{-1}(Y) &= \{u \in K'_{1+} \mid |\langle T(u), \phi_i\rangle| < \varepsilon; i = 1, \dots, n\} \\ &= \{u \in K'_{1+} \mid |\langle u, \xi_i\rangle| < \varepsilon; i = 1, \dots, n\} \end{aligned}$$

is open. Since $\varepsilon, \phi_1, \dots, \phi_n$ are arbitrary, T is continuous.

(Necessary) Suppose no such function b exists. Then there exists a sequence $\tau_k \to -\infty$ and $c < \infty$ such that $\operatorname{supp} T(\delta_{\tau_k}) \cap [-c, c]$ is nonempty for every k. Hence, there exist $\phi_k \in K_1$ such that $\operatorname{supp} \phi_k \subset [-c, c]$ and $\beta_k = \langle T(\delta_{\tau_k}), \phi_k\rangle \neq 0$ for each k. Let

$$\xi_k = \frac{1}{k \, \|\phi_k\|_{C^k}} \phi_k.$$

Then, for any integers $p \geq 0$ and $k \geq p$,

$$\|\xi_k\|_{C^p} = \frac{1}{k\,\|\phi_k\|_{C^k}}\,\|\phi_k\|_{C^p} \leq \frac{1}{k},$$

so $\xi_k \to 0$. From [4, p. 31], $\{\xi_k\}$ is a bounded set. Let

$$u_k = \frac{k\,\|\phi_k\|_{C^k}}{\beta_k}\delta_{\tau_k}.$$

Then $u_k \to 0$. On the other hand,

$$\begin{aligned}\sup_m \langle T(u_k), \xi_m\rangle &\geq \langle T(u_k), \xi_k\rangle \\ &= \left\langle \frac{k\,\|\phi_k\|_{C^k}}{\beta_k} T(\delta_{\tau_k}), \xi_k \right\rangle \\ &= \frac{1}{\beta_k}\langle T(\delta_{\tau_k}), k\,\|\phi_k\|_{C^k}\,\xi_k\rangle \\ &= \frac{1}{\beta_k}\langle T(\delta_{\tau_k}), \phi_k\rangle = 1.\end{aligned}$$

From [4, p. 56], $T(u_k) \not\to 0$, which is a contradiction.

(2) Let $u_\lambda \to 0$ be a net with $\operatorname{supp} u_\lambda \subset [a,\infty)$, $\phi \in K_1$, and $\xi(\tau) = \langle T(\delta_\tau), \phi\rangle$. Select $\bar{\xi} \in K_1$ with $\bar{\xi}(\tau) = \xi(\tau)$ for $\tau \geq a$. This gives $\langle T(u_\lambda), \phi\rangle = \langle u_\lambda, \bar{\xi}\rangle \to 0$, so $T(u_\lambda) \to 0$. □

In view of Theorem 3.6, restricting attention to continuous linear operators on K'_{1+} would be inadequate for developing a sufficiently comprehensive theory of stable linear systems. For example, even the "integrator system" $T(\delta_\tau) = \theta_\tau$ is discontinuous. On the other hand, the class of all generalized integral operators on K'_{1+} contains the full range of operators normally considered in linear system theory, so we will adopt these as our space of systems.

We end this section with a result relating impulse response and step response.

THEOREM 3.7. *Let $T : K'_{1+} \to K'_{1+}$ be a generalized integral operator. Then $\{T(\theta_\tau)\}$ is a C^∞ family and $T(\delta_\tau) = -\frac{\partial T(\theta_\tau)}{\partial \tau}$.*

Proof. From Theorem 3.6(2), T is continuous on $\{u \in K'_{1+} \mid \operatorname{supp} u \subset [\tau_0-1,\infty)\}$ for any τ_0. Hence, for any $\phi \in K_1$,

$$\frac{\langle T(\theta_\tau), \phi\rangle - \langle T(\theta_{\tau_0}), \phi\rangle}{\tau - \tau_0} = \left\langle T\left(\frac{\theta_\tau - \theta_{\tau_0}}{\tau - \tau_0}\right), \phi\right\rangle \to \langle -T(\delta_{\tau_0}), \phi\rangle,$$

$$\frac{\left\langle T(\delta_\tau^{(n)}), \phi\right\rangle - \left\langle T(\delta_{\tau_0}^{(n)}), \phi\right\rangle}{\tau - \tau_0} = \left\langle T\left(\frac{\delta_\tau^{(n)} - \delta_{\tau_0}^{(n)}}{\tau - \tau_0}\right), \phi\right\rangle \to \left\langle -T(\delta_{\tau_0}^{(n+1)}), \phi\right\rangle$$

as $\tau \to \tau_0$ for any n, so $\{T(\theta_\tau)\}$ is C^∞. The first derivative is given by

$$\left\langle \frac{\partial T(\theta_\tau)}{\partial \tau}, \phi\right\rangle = \frac{\partial}{\partial \tau}\langle T(\theta_\tau), \phi\rangle = \langle -T(\delta_\tau), \phi\rangle. \qquad \square$$

4. Extension of normed linear spaces. In [1] we considered the problem of imbedding a normed linear space Y into a Hausdorff topological vector space X and extending the norm to a maximal linear subspace of X. For example, we showed that

L^1 can be imbedded in K_1' and $\|\cdot\|_1$ can be extended to all of DBV. In particular, the extended L^1 norm applied to the unit impulse evaluates to $\|\delta\|_1^e = 1$. We will again need these results to construct our time-varying theory.

Let $\mathfrak{T}$ be the topology on X, $\|\cdot\|$ the norm on $Y \subset X$, and $B(y,r) \subset Y$ the closed norm-ball about $y \in Y$ with radius r. We make the following assumptions on the 4-tuple $(X, \mathfrak{T}, Y, \|\cdot\|)$:

T1. For every nonempty $U \in \mathfrak{T}$, $U \cap Y$ is nonempty.

T2. For every $U \in \mathfrak{T}$ and every $y \in U \cap Y$, there exists $r > 0$ such that $B(y,r) \subset U$.

T3. There exists $U \in \mathfrak{T}$ such that $U \cap Y = Y - B(0,1)$.

Condition T1 says that Y is dense in X. T2 requires that the norm topology on Y is at least as strong as the topology induced on Y by X. T3 says that $B(0,1)$ in Y is closed relative to X. Thus T2 and T3 give bounds on the topology induced on Y by X. Under assumptions T1–T3, there exists a natural extension $\|\cdot\|^e$ of $\|\cdot\|$ to a subspace $Y_e \supset Y$ of X. In particular, for any $y \in Y_e$, $\|y\|^e$ is equal to the minimum value of $\lim \|y_\lambda\|$ over all $\mathfrak{T}$-approximating nets $y_\lambda \to y$, $y_\lambda \in Y$. (See [1, section 3] for details.)

As mentioned in section 1, $h \in UL^1$ is the classical condition for BIBO stability. Therefore it makes sense to examine the extension UL_e^1 of $\|\cdot\|_{\infty 1}$ in K_2' and check whether UL_e^1 actually characterizes BIBO stability for generalized integral operators. First we must establish whether K_2' and UL^1 satisfy T1–T3.

LEMMA 4.1. (1) *If Y_1 is dense in Y relative to $\|\cdot\|$, then the 4-tuple $(X, \mathfrak{T}, Y_1, \|\cdot\|)$ satisfies* T1–T3 *and $Y_{1e} = Y_e$.*

(2) *Let $Y \subset X_1 \subset X$ and $\mathfrak{T}_1$ be the relative topology on X_1 induced by $\mathfrak{T}$. Then the 4-tuple $(X_1, \mathfrak{T}_1, Y, \|\cdot\|)$ satisfies* T1–T3 *and the corresponding extension of Y is $Y_e \cap X_1$.*

Proof. (1) From T1 and T2, Y_1 is dense in X relative to $\mathfrak{T}$, so T1 holds for Y_1. If $U \in \mathfrak{T}$ and $y \in U \cap Y_1$, then $y \in U \cap Y$, so there exists $r > 0$ such that $B(y,r) \subset U$, where $B(y,r)$ is a norm-ball in Y. The corresponding ball in Y_1 is $B(y,r) \cap Y_1 \subset U$, so T2 holds. Finally, if U satisfies T3 relative to Y, then U also satisfies T3 relative to Y_1, since

$$U \cap Y_1 = (U \cap Y) \cap Y_1 = (Y - B(0,1)) \cap Y_1 = Y_1 - (B(0,1) \cap Y_1).$$

Finally, we must show that $\|\cdot\|^e$ satisfies [1, Proposition 3.1(1)–(3)] using Y_1 in place of Y. To prove (1), note that, since $\|\cdot\|$ and $\|\cdot\|^e$ coincide on Y, they must coincide on Y_1. Condition (2) holds, since it does not involve Y. To prove (3), let $x \in X$ with $\|x\|^e < \infty$, $\varepsilon > 0$, and let U be a $\mathfrak{T}$-neighborhood of x. Then (3) applied to Y guarantees that there exists $y \in U \cap Y$ such that

$$\|y\| < \|x\|^e + \frac{\varepsilon}{2}.$$

Density of Y_1 in Y relative to $\|\cdot\|$ and T2 imply that there exists $y_1 \in U \cap Y_1$ such that $\|y_1 - y\| < \frac{\varepsilon}{2}$. Then

$$\|y_1\| < \|y\| + \frac{\varepsilon}{2} < \|x\|^e + \varepsilon.$$

(2) Restricting $\mathfrak{T}$ to X_1, T1–T3 are obvious. Suppose Y_e is the extension of Y using X, and $\|\cdot\|^e$ is the corresponding norm. Let $\|\cdot\|^f$ be the restriction of $\|x\|^e$ to X_1. We must show that $\|\cdot\|^f$ satisfies [1, Proposition 3.1(1)–(3)], using X_1. To prove (1) and (2), note that $\|x\|^e$ and $\|x\|^f$ coincide on X_1; hence, $\|y\|^f = \|y\|$ for

$y \in Y$ and $\|\cdot\|^f$ is lower semicontinuous. To establish (3), let $U \in \mathfrak{T}$ $x \in U \cap X_1$ with $\|x\|^f < \infty$, and $\varepsilon > 0$. Then $\|x\|^e < \infty$ and there exists $y \in U \cap Y$ such that

$$\|y\| < \|x\|^e + \varepsilon = \|x\|^f + \varepsilon.$$

But $Y \subset X_1$, so $y \in (U \cap X_1) \cap Y$. □

THEOREM 4.2. *Let* $X = K_2'$ *and* $Y = UL^1$. *Then* T1–T3 *are satisfied.*

Proof. First we note that $K_2 \subset UL^1$. From [4, p. 118], K_2 is dense in K_2', and T1 follows. If T2 holds for $y = 0$, then it holds for all y, since $B(0,r) \subset U$ implies $B(y,r) \subset y+U$. Thus it suffices to prove that, for every n, $\varepsilon > 0$, and $\psi_1, \ldots, \psi_n \in K_2$, there exists $r > 0$ such that $B(0,r) \subset U$, where

$$U = \{f \in K_2' \mid |\langle f, \psi_i\rangle| < \varepsilon; i = 1, \ldots, n\}.$$

Let

$$r < \varepsilon \min\left\{\frac{1}{\|\psi_i\|_{1\infty}}\right\}$$

and $f \in B(0,r)$. From Theorem 2.2,

$$|\langle f, \psi_i\rangle| \le \|f\|_{DUBV}\, \|\psi_i\|_{1\infty} = \|f\|_{\infty 1}\, \|\psi_i\|_{1\infty} \le r \max\{\|\psi_j\|_{1\infty}\} < \varepsilon$$

for every i, so $f \in U$.

To prove T3, let $y \in UL^1$ with $\|y\|_{\infty 1} > 1$, and choose $\varepsilon < \frac{1}{2}(\|y\|_{\infty 1} - 1)$. From Theorem 2.2, we may select $\psi \in K_2$ with $\|\psi\|_{1\infty} = 1$ such that $|\langle y, \psi\rangle| > \|y\|_{\infty 1} - \varepsilon$. Let

$$U = \{x \in K_2' \mid |\langle x - y, \psi\rangle| < \varepsilon\}.$$

Then $y \in U \in \mathfrak{T}$, and $f \in U \cap Y$ implies

$$\|f\|_{\infty 1} \ge |\langle f, \psi\rangle| \ge |\langle y, \psi\rangle| - |\langle f - y, \psi\rangle| > \|y\|_{\infty 1} - 2\varepsilon > 1,$$

so $U \cap Y \subset Y - B(0,1)$. □

We are now in a position to characterize the extension of UL^1 into K_2'.

THEOREM 4.3. $UL^1_e = DUBV$.

Proof. Let

$$\|f\|^e = \begin{cases} \|f\|_{DUBV}, & f \in DUBV, \\ \infty, & f \in K_2' - DUBV. \end{cases}$$

We must verify that $\|\cdot\|^e$ satisfies [1, Lemma 3.1(1)–(3)] relative to $\|\cdot\|_{\infty 1}$. Condition (1) says that $\|\cdot\|_{\infty 1}$ and $\|\cdot\|_{DUBV}$ coincide on UL^1. This was established in Theorem 2.2.

Condition (2) requires that $\|\cdot\|^e$ be lower semicontinuous on K_2'. Equivalently, we must show that the set $\Sigma_M = \{f \in K_2' \mid \|f\|^e > M\}$ is open for each M. Suppose $\|f\|^e > M$. From Theorem 2.2(3), there exists $\psi \in K_2$ such that $\|\psi\|_{1\infty} = 1$ and $|\langle f, \psi\rangle| > M$. Let $U = \{g \in K_2' \mid |\langle g, \psi\rangle| < |\langle f, \psi\rangle| - M\}$. $f + U$ is open in K_2', and

$$\|f + g\|^e \ge |\langle f + g, \psi\rangle| \ge |\langle f, \psi\rangle| - |\langle g, \psi\rangle| > M$$

for every $g \in U$. Hence, Σ_M is open.

Finally, condition (3) says that, for any $f \in DUBV$, $\varepsilon > 0$, and neighborhood U of f, there exists $y \in U \cap UL^1$ such that $\|y\|_{\infty 1} < \|f\|^e + \varepsilon$. We accomplish this by constructing a sequence $f_n \to f$ with $f_n \in UL^1$ and $\|f_n\|_{\infty 1} \leq \|f\|_{DUBV}$. Then, for large n, $f_n \in U$, and $y = f_n$ satisfies the conditions.

Our construction of f_n proceeds as follows. Let $\phi_1, \phi_2 \in K_1$ with $\phi_1(t) \geq 0$ for all t, $\int_{-\infty}^{\infty} \phi_1(t)dt = 1$, $\phi_2(0) = 1$, and $\|\phi_2\|_\infty = 1$. Set $\psi_n(t,\tau) = n\phi_1(n(t-\tau))\phi_2(\frac{\tau}{n})$. Then $\psi_n \in C^\infty$. Suppose $\operatorname{supp}\phi_1, \operatorname{supp}\phi_2 \subset [-a,a]$. If $|\tau| \geq na$, then $|\frac{\tau}{n}| \geq a$, so $\phi_2(\frac{\tau}{n}) = 0$ and $\psi(t,\tau) = 0$. If $|\tau| < na$ and $|t| \geq (n + \frac{1}{n})a$, then

$$n\,|t-\tau| \geq n\,(|t| - |\tau|) > n\left(n + \frac{1}{n}\right)a - n^2 a = a,$$

so $\phi_1(n(t-\tau)) = 0$ and $\psi_n(t,\tau) = 0$. Hence, $\psi_n \in K_2$.

Let

$$f_n(t,\tau) = \int_{-\infty}^{\infty} \psi_n(\tau,\eta)dg_t(\eta)$$

and $f = \frac{\partial g}{\partial \tau}$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} |f_n(t,\tau)|\,d\tau &\leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_n(\tau,\eta)|\,|dg_t(\eta)|\,d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi_n(\tau,\eta)|\,d\tau\,|dg_t(\eta)| \\ &= n\int_{-\infty}^{\infty} \left|\phi_2\left(\frac{\eta}{n}\right)\right| \left(\int_{-\infty}^{\infty} \phi_1(n(\tau-\eta))d\tau\right) |dg_t(\eta)| \\ &= \int_{-\infty}^{\infty} \left|\phi_2\left(\frac{\eta}{n}\right)\right| \left(\int_{-\infty}^{\infty} \phi_1(x)dx\right) |dg_t(\eta)| \\ &\leq \operatorname*{var}_{\eta} g(t,\eta),\end{aligned}$$

$$\|f_n\|_{\infty 1} \leq \operatorname*{ess\,sup}_{t} \operatorname*{var}_{\tau} g(t,\tau) = \|f\|_{DUBV}\,.$$

To show $\langle f_n, \psi\rangle \to \langle f, \psi\rangle$, note that, for every $t, \eta, x \in \mathbb{R}$, $\psi(t, \frac{x}{n} + \eta) \to \psi(t,\eta)$ and

$$\int_{-\infty}^{\infty} \left|\phi_1(x)\psi\left(t, \frac{x}{n} + \eta\right)\right| dx \leq \|\psi\|_\infty\,.$$

By the dominated convergence theorem,

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_n(\tau,\eta)\psi(t,\tau)d\tau &= n\phi_2\left(\frac{\eta}{n}\right)\int_{-\infty}^{\infty} \phi_1(n(\tau-\eta))\psi(t,\tau)d\tau \\ &= \phi_2\left(\frac{\eta}{n}\right)\int_{-\infty}^{\infty} \phi_1(x)\psi\left(t, \frac{x}{n} + \eta\right)dx \\ &\to \phi_2(0)\int_{-\infty}^{\infty} \phi_1(x)\psi(t,\eta)dx \\ &= \psi(t,\eta)\end{aligned}$$

for every t, η. Furthermore,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \psi_n(\tau,\eta)\psi(t,\tau)d\tau\right| &\le n\left|\phi_2\left(\frac{\eta}{n}\right)\right|\int_{-\infty}^{\infty}\phi_1(n(\tau-\eta))\left|\psi(t,\tau)\right|d\tau \\ &= \left|\phi_2\left(\frac{\eta}{n}\right)\right|\int_{-\infty}^{\infty}\phi_1(x)\left|\psi\left(t,\frac{x}{n}+\eta\right)\right|dx \\ &\le \left\|\psi(t,\cdot)\right\|_{\infty},\end{aligned}$$

so, if $\operatorname{supp}\psi \subset [-a,a]^2$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\|\psi(t,\cdot)\right\|_{\infty}\left|dg_t(\eta)\right|dt \le \int_{-a}^{a}\left\|\psi(t,\cdot)\right\|_{\infty}\operatorname*{var}_{\eta} g(t,\eta)dt \le 2a\left\|\psi\right\|_{\infty}\left\|g\right\|_{UBV}.$$

Again, by the dominated convergence theorem,

$$\begin{aligned}\langle f_n,\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_n(\tau,\eta)\psi(t,\tau)d\tau dg_t(\eta)dt \\ &\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(\tau,\eta)dg_t(\eta)dt = \langle f,\psi\rangle. \qquad \square\end{aligned}$$

COROLLARY 4.4. *Let* $X = K'_{2+}$ *and* $Y = UL^1_+$. *Then* X *and* Y *satisfy* T1–T3 *and* $Y_e = DUBV_+$.

Proof. We have $K_2 \subset UL^1_+ \subset UL^1 \subset K'_2$ with K_2 dense in K'_2 (see [4, p. 118]). Hence, UL^1_+ is dense in UL^1. From Lemma 4.1(1) and Theorems 4.2 and 4.3, $(K'_2, \mathfrak{T}, UL^1_+, \|\cdot\|_{\infty 1})$ satisfies T1–T3 and $UL^1_{+e} = UL^1_e = DUBV$. Furthermore, $UL^1_+ \subset K'_{2+} \subset K'_2$, so Lemma 4.1(2) implies that $(K'_{2+}, \mathfrak{T}_1, UL^1_+, \|\cdot\|_{\infty 1})$ satisfies T1–T3 and $UL^1_{+e} = DUBV \cap K'_{2+} = DUBV_+$. $\square$

5. BIBO stability. In this section, we consider stability of linear operators $T : K'_{1+} \to K'_{1+}$. (We must restrict our attention to K'_{1+} in order to have a consistent definition of generalized integral operators.) As in [2, p. 109], we define a *BIBO stable* linear operator T to be one such that

(S1) $T(L^\infty_+) = L^\infty_+$,

(S2) T is continuous on L^∞_+ relative to $\|\cdot\|_\infty$.

We have shown in [1] that, for time-invariant (i.e., convolution) operators, (S2) follows automatically from (S1). Unfortunately, this result does not extend to the time-varying setting. For example, let $h_\tau = e^{-\tau}\delta_\tau$. For any $u \in L^\infty_+$, $\phi \in K_1$,

$$\begin{aligned}\xi(\tau) &= \langle h_\tau,\phi\rangle = e^{-\tau}\langle\delta_\tau,\phi\rangle = e^{-\tau}\phi(\tau), \\ \langle y,\phi\rangle &= \langle u,\xi\rangle = \int_{-\infty}^{\infty} u(\tau)e^{-\tau}\phi(\tau)d\tau.\end{aligned}$$

Thus $y(t) = e^{-t}u(t)$ and $y \in L^\infty_+$. However, the map $u \mapsto y$ is not continuous on L^∞_+. Indeed, let $u_k(t) = \frac{1}{k}\theta(t+k)$; then $\|u_k\|_\infty = \frac{1}{k} \to 0$. But $y_k(t) = \frac{1}{k}e^{-t}\theta(t+k)$ and $\|y_k\|_\infty = \frac{e^k}{k} \to \infty$. Thus, as in [2], we adopt (S2) as an independent assumption.

Since classical integral operators satisfying $T(\delta_\tau) \in UL^1$ are known to be BIBO stable, a natural conjecture is that a generalized integral operator is BIBO stable if and only if $\{T(\delta_\tau)\} \in UL^1_{e+}$ ($= DUBV_+$ by Corollary 4.4). The following example lends support to this idea. Let $T(\delta_\tau) = \delta^{(n)}_\tau$. Then $\langle T(u),\phi\rangle = \langle u,\xi\rangle$, where $\xi(\tau) = \langle\delta^{(n)}_\tau,\phi\rangle = (-1)^n\phi^{(n)}(\tau)$. Hence $\langle T(u),\phi\rangle = \langle u,(-1)^n\phi^{(n)}\rangle = \langle u^{(n)},\phi\rangle$, and $T(u) =$

$u^{(n)}$. In view of (S1), the n-times differentiator is BIBO stable if and only if $n = 0$, since $T(\theta) = \delta^{(n-1)} \notin L^\infty$ for $n > 0$. On the other hand, $\{\delta_\tau\} = -\frac{\partial}{\partial\tau}\{\theta_\tau\}$, and $\{\theta_\tau\} \in UBV_+$, so $\{\delta_\tau\} \in UL^1_{e+} = DUBV_+$. But $\{\delta^{(n)}_\tau\} = -\frac{\partial}{\partial\tau}\{\delta^{(n-1)}_\tau\}$ for $n > 0$, and $\delta^{(n-1)}_\tau \notin UBV$, so $\{\delta^{(n)}_\tau\} \notin UL^1_{e+}$.

Corresponding to each generalized integral operator T we may associate an operator $\tilde{T} : K_1 \to C^\infty$ defined by

$$\tilde{T}(\phi)(\tau) = \langle T(\delta_\tau), \phi\rangle .$$

Let $\phi \in K_1$, $u \in L^\infty_+$, $\xi = \tilde{T}(\phi)$. Then

$$\langle T(u), \phi\rangle = \langle u, \bar{\xi}\rangle = \int_{-\infty}^{\infty} u(\tau)\tilde{T}(\phi)(\tau)d\tau.$$

This suggests an adjoint relationship between T and $\tilde{T}$, which we will explore further in Theorem 5.4. First we need a result which shows that stability of T can be characterized in terms of $\tilde{T}$.

LEMMA 5.1. *T is BIBO stable if and only if*

$$\sup_{\substack{\phi\in K_1\\ \|\phi\|_1=1}} \int_{-\infty}^{\infty} \left|\tilde{T}(\phi)(\tau)\right| d\tau < \infty.$$

Proof. The proof is identical to the proof of [1, Lemma 4.1(2)], replacing the phrase "convolution operator" by "generalized integral operator." ☐

If T is BIBO stable, Lemma 5.1 indicates that $\tilde{T}$ is a continuous linear operator from $(K_1, \|\cdot\|_1)$ into $(L_1, \|\cdot\|_1)$. Since $(K_1, \|\cdot\|_1)$ is dense in $L^1, \tilde{T}$ extends uniquely to a continuous linear operator $\tilde{T}_e : L^1 \to L^1$.

LEMMA 5.2. *Let $T : K'_{1+} \to K'_{1+}$ be a BIBO stable generalized integral operator, $s(\cdot, \tau) = T(\theta_\tau)$, $s_\tau(t) = \widehat{s}_t(\tau) = s(t, \tau)$, $\phi \in L^1$, and $u \in L^\infty$. Then*

(1) *$\tau \to \int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt$ is absolutely continuous,*

(2) *$\tilde{T}_e(\phi)(\tau) = -\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt$ for every $\tau \in \mathbb{R}$,*

(3) *$\int_{-\infty}^{\infty} u(\tau)d\int_{-\infty}^{\infty} s(t, \tau)\phi(t)dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t)u(\tau)d\widehat{s}_t(\tau)dt$.*

Proof. From [7, Theorem 2.3.9], there exists $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g(\cdot,\cdot) - g(\cdot, \infty) \in UBV$, the map $\tau \to \int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt$ is absolutely continuous, and

$$\tilde{T}_e(\phi)(t) = \frac{d}{d\tau}\int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt$$

for each $\phi \in L^1$. Let $c = \int_{-\infty}^{\infty} g(t, \infty)\phi(t)dt$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} s(t, t_0)\, \phi(t)\, dt &= \int_{-\infty}^{\infty} \theta_{t_0}(\tau)\, \tilde{T}(\phi)(\tau)\, d\tau \\ &= \int_{t_0}^{\infty} \tilde{T}(\phi)(\tau)\, d\tau \\ &= \int_{t_0}^{\infty} \frac{d}{d\tau}\left(\int_{-\infty}^{\infty} g(t, \tau)\phi(t)dt\right) d\tau \\ &= \int_{-\infty}^{\infty} (g(t, \infty) - g(t, t_0))\, \phi(t)\, dt \\ &= c - \int_{-\infty}^{\infty} g(t, t_0)\phi(t)dt,\end{aligned}$$

from which (1) and (2) follow.

To prove (3), note that, for any $\phi \in L^1$, $|\phi(t)| < \infty$ a.e. and

$$\mu_t(-\infty, \tau] = \phi(t)s(t, \tau)$$

determines a finite signed Borel measure on $\mathbb{R}$ for a.e. t as does

$$\mu(-\infty, \tau] = g(\tau) = \int_{-\infty}^{\infty} \mu_t(-\infty, \tau] dt. \tag{5.1}$$

Consider the family $\mathcal{L}$ of sets $A \subset \mathbb{R}$ such that the map $t \to \mu_t(A)$ is Borel measurable for a.e. t. Since

$$\mu_t(\mathbb{R}) = \operatorname*{var}_{\tau} s(t, \tau),$$

$\mathbb{R} \in \mathcal{L}$. If $A, B \in \mathcal{L}$ with $A \subset B$, then

$$\mu_t(B - A) = \mu_t(B) - \mu_t(A),$$

so $B - A \in \mathcal{L}$. If $A_n \in \mathcal{L}$ with $A_n \uparrow A$, then

$$\mu_t(A) = \mu_t(A_1) + \sum_n \mu_t(A_{n+1} - A_n),$$

so $A \in \mathcal{L}$. From the $\pi - \lambda$ theorem (see [8, Theorem 4.2]), every Borel set in $\mathbb{R}$ belongs to $\mathcal{L}$. Hence, $t \to \mu_t(A)$ is Borel measurable for any Borel set A and

$$\int_{-\infty}^{\infty} |\mu_t(A)| \, dt = \int_{-\infty}^{\infty} \left| \int_A \phi(t) d\widehat{s}_t(\tau) \right| dt \leq \int_{-\infty}^{\infty} \int_A |\phi(t)| \, |d\widehat{s}_t(\tau)| \, dt \leq \|s\|_{UBV} \, \|\phi\|_1 .$$

Also, if the A_n are pairwise disjoint,

$$\int_{-\infty}^{\infty} \mu_t \left(\bigcup_n A_n \right) dt = \int_{-\infty}^{\infty} \sum_n \mu_t(A_n) dt = \sum_n \int_{-\infty}^{\infty} \mu_t(A_n) dt,$$

so the map

$$A \to \int_{-\infty}^{\infty} \mu_t(A) dt \tag{5.2}$$

is a finite signed Borel measure. Since μ and (5.2) have the same distribution function (5.1),

$$\mu(A) = \int_{-\infty}^{\infty} \mu_t(A) dt$$

for each A.

Let I_A be the indicator function on A. Then

$$\int_{-\infty}^{\infty} I_A(\tau) d\mu = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_A(\tau) d\mu_t dt,$$

$$\int_{-\infty}^{\infty} I_A(\tau) d \int_{-\infty}^{\infty} s(t, \tau)\phi(t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t) I_A(\tau) d\hat{s}_t(\tau) dt. \tag{5.3}$$

Both sides of (3) are bounded by $\|u\|_\infty \|s\|_{UBV} \|\phi\|_1$, so, as functions of u, they both represent continuous linear functionals on L^∞. Since the span of the indicators I_A is dense in L^∞, 5.3 implies (3). ∎

We are now in a position to prove our main result.

THEOREM 5.3. *Let $T : K'_{1+} \to K'_{1+}$ be a generalized integral operator. The following statements are equivalent:*

(1) *T is BIBO stable.*

(2) *$\{T(\delta_\tau)\} \in UL^1_{+e}$.*

(3) *$\{T(\theta_\tau)\} \in UBV_+$.*

Proof. From Theorem 3.7, $\{T(\theta_\tau)\} \in UBV_+$ if and only if $\{T(\delta_\tau)\} \in DUBV_+$. From Corollary 4.4, $DUBV_+ = UL^1_{+e}$. Thus (2) and (3) are equivalent.

To prove that (3) implies (1), let $u \in L^\infty_+$, $\phi \in K_1$, $s_\tau = T(\theta_\tau)$, $s(t,\tau) = s_\tau(t)$, and note that

$$\operatorname*{var}_\tau \langle s_\tau, \phi\rangle = \operatorname*{var}_\tau \int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt \le \int_{-\infty}^{\infty} \left(\operatorname*{var}_\tau s(t,\tau)\right) |\phi(t)|\, dt \le \|s\|_{UBV} \|\phi\|_1 .$$

From Theorem 3.7 and Lemma 5.2,

$$\begin{aligned}\langle T(u), \phi\rangle &= \int_{-\infty}^{\infty} u(\tau) \langle T(\delta_\tau), \phi\rangle\, d\tau \\ &= -\int_{-\infty}^{\infty} u(\tau) \left\langle \frac{\partial s_\tau}{\partial \tau}, \phi \right\rangle d\tau \\ &= -\int_{-\infty}^{\infty} u(\tau) \frac{\partial}{\partial \tau} \langle s_\tau, \phi\rangle\, d\tau \\ &= -\int_{-\infty}^{\infty} u(\tau) d\langle s_\tau, \phi\rangle ,\end{aligned}$$

$$|\langle T(u), \phi\rangle| \le \|u\|_\infty \operatorname*{var}_\tau \langle s_t, \phi\rangle \le \|u\|_\infty \|s\|_{UBV} \|\phi\|_1 .$$

Therefore,

$$\|T(u)\|_\infty = \sup_{\substack{\phi\in K_1 \\ \|\phi\|_1=1}} |\langle T(u), \phi\rangle| \le \|u\|_\infty \|s\|_{UBV} .$$

Finally, we prove that (1) implies (2). Let $\phi_1, \phi_2 \in K_1$, and let $\psi \in K_2$ be given by $\psi(t,\tau) = \phi_1(t)\phi_2(\tau)$. From Lemma 5.2,

$$\begin{aligned}\langle \{T(\delta_\tau)\}, \psi\rangle &= \int_{-\infty}^{\infty} \langle T(\delta_\tau), \phi_1\rangle\, \phi_2(\tau)d\tau \\ &= \int_{-\infty}^{\infty} \tilde{T}(\phi_1)(\tau)\phi_2(\tau)d\tau \\ &= -\int_{-\infty}^{\infty} \left(\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t,\tau)\phi_1(t)dt\right) \phi_2(\tau)d\tau \\ &= -\int_{-\infty}^{\infty} \phi_2(\tau)d\left(\int_{-\infty}^{\infty} s(t,\tau)\phi_1(t)dt\right) \\ &= -\int_{-\infty}^{\infty} \phi_1(t)\left(\int_{-\infty}^{\infty} \phi_2(\tau)ds_t(\tau)\right) dt \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(t,\tau)ds_t(\tau)dt.\end{aligned} \tag{5.4}$$

From linearity and continuity of the functionals in (5.4) and from [6, p. 65], (5.4) holds for any $\psi \in K_2$. Thus Theorem 2.2(1) implies

$$\{T(\delta_\tau)\} = -\frac{\partial s}{\partial \tau} \in DUBV. \qquad \square$$

We now examine a certain extension of the operator T and its relation to $\tilde{T}_e$. Consider the closure L_0^∞ of $L_+^\infty \subset L^\infty$. It is easy to show that L_0^∞ is a closed proper subspace of L^∞ and

$$L_0^\infty = \left\{ f \in L^\infty \mid \operatorname*{ess\,sup}_{t\in(-\infty,-n]} |f(t)| \to 0 \text{ as } n \to \infty \right\}.$$

If T is stable, T extends uniquely to a continuous linear operator $T_0 : L_0^\infty \to L_0^\infty$. This extension can be taken further.

THEOREM 5.4. *Suppose $T : K'_{1+} \to K'_{1+}$ is a BIBO stable generalized integral operator, $s(\cdot,\tau) = T(\theta_\tau)$, and $\hat{s}_t(\tau) = s(t,\tau)$. Let $T_e : L^\infty \to L^\infty$ be the continuous linear operator defined by*

$$T_e(u)(t) = -\int_{-\infty}^{\infty} u(\tau)d\hat{s}_t(\tau).$$

Then

(1) $T_e(u) = T_0(u)$ *for all* $u \in L_0^\infty$,

(2) T_e *is the adjoint of* $\tilde{T}_e$.

Proof. (1) Let $u \in L_+^\infty$, $\phi \in K_1$. From Lemma 5.2,

$$\begin{aligned}
\langle T(u), \phi\rangle &= \int_{-\infty}^{\infty} u(\tau)\tilde{T}(\phi)(\tau)d\tau \\
&= \int_{-\infty}^{\infty} u(\tau)\,\langle T(\delta_\tau), \phi\rangle\, d\tau \\
&= -\int_{-\infty}^{\infty} u(\tau)\left\langle \frac{\partial s_\tau}{\partial \tau}, \phi \right\rangle d\tau \\
&= -\int_{-\infty}^{\infty} u(\tau)\frac{\partial}{\partial \tau}\langle s_\tau, \phi\rangle\, d\tau \\
&= -\int_{-\infty}^{\infty} u(\tau)d\,\langle s_\tau, \phi\rangle \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)\phi(t)d\hat{s}_t(\tau)dt.
\end{aligned}$$

Since ϕ is arbitrary,

$$T(u)(t) = -\int_{-\infty}^{\infty} u(\tau)d\hat{s}_t(\tau)$$

a.e. Since L_+^∞ is dense in L_0^∞, the result follows.

(2) Let $\phi \in L^1$ and $u \in L^\infty$. From Lemma 5.2,

$$\begin{aligned}
\langle T_e(u), \phi \rangle &= \int_{-\infty}^{\infty} T_e(u)(t)\phi(t)dt \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\tau)\phi(t)d\hat{s}_t(\tau)dt \\
&= -\int_{-\infty}^{\infty} u(\tau)d\int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt \\
&= -\int_{-\infty}^{\infty} u(\tau)\left(\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt\right)d\tau \\
&= \int_{-\infty}^{\infty} u(\tau)\tilde{T}_e(\phi)(\tau)d\tau \\
&= \left\langle u, \tilde{T}_e(\phi)\right\rangle. \qquad \square
\end{aligned}$$

To conclude, we give several equivalent expressions for the "gain" of a BIBO stable linear operator.

THEOREM 5.5. *For any BIBO stable generalized integral operator* $T : K'_{1+} \to K'_{1+}$,

$$\begin{aligned}
\sup_{\substack{u\in L^\infty_+ \\ \|u\|_\infty=1}} \|T(u)\|_\infty &= \sup_{\substack{u\in L^\infty \\ \|u\|_\infty=1}} \|T_e(u)\|_\infty \\
&= \sup_{\substack{\phi\in L^1 \\ \|\phi\|_1=1}} \left\|\tilde{T}_e(\phi)\right\|_1 = \|\{T(\theta_\tau)\}\|_{UBV} = \|\{T(\delta_\tau)\}\|^e_{UL^1}.
\end{aligned} \tag{5.5}$$

Proof. Let $v \in L^\infty, \|v\|_\infty = 1, v_n(\tau) = v(\tau)\theta(\tau+n)$, and $\varepsilon > 0$. From Theorem 5.4, for a.e. t there exists $N < \infty$ such that $n > N$ implies

$$\begin{aligned}
|T_e(v)(t)| - |T(v_n)(t)| &\le |T_e(v)(t) - T(v_n)(t)| \\
&= \left|\int_{-\infty}^{-n} v(\tau)d\hat{s}_t(\tau)\right| \le \int_{-\infty}^{-n} |v(\tau)|\,|d\hat{s}_t(\tau)| \le \operatorname*{var}_{\tau\le -n} s(t,\tau) < \varepsilon.
\end{aligned}$$

Hence,

$$\sup_{\substack{u\in L^\infty_+ \\ \|u\|_\infty=1}} \|T(u)\|_\infty \ge \|T(v_n)\|_\infty \ge |T(v_n)(t)| > |T_e(v)(t)| - \varepsilon.$$

Since v, t, ε are arbitrary,

$$\sup_{\substack{u\in L^\infty_+ \\ \|u\|_\infty=1}} \|T(u)\|_\infty \ge \sup_{\substack{\varepsilon>0 \\ u\in L^\infty \\ \|u\|_\infty=1}} \operatorname*{ess\,sup}_t (|T_e(u)(t)| - \varepsilon) = \sup_{\substack{u\in L^\infty \\ \|u\|_\infty=1}} \|T_e(u)\|_\infty.$$

The second equality in (5.5) follows from Theorem 5.4(2). For the third equality, set

$$s(t,\tau) = T(\theta_\tau)(t).$$

From Lemma 5.2,

$$\tilde{T}_e(\phi)(\tau) = -\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt$$

for every τ, and

$$\left\|\tilde{T}_e(\phi)\right\|_1 = \int_{-\infty}^{\infty}\left|\frac{d}{d\tau}\int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt\right| d\tau = \operatorname*{var}_{\tau}\int_{-\infty}^{\infty} s(t,\tau)\phi(t)dt.$$

From [7, Theorem 2.3.9],

$$\sup_{\substack{\phi\in L^1\\ \|\phi\|_1=1}}\left\|\tilde{T}_e(\phi)\right\|_1 = \|s\|_{UBV}\,.$$

The last equality in (5.5) follows from Theorems 3.7 and 4.3. □

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