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GENERICITY OF CLASSES OF OBSERVER-BASED COMPENSATORS*

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Abstract. We investigate basic properties of the set of observer-based state-space compensators (OBCs). In particular, we are interested in how much of this set is occupied by its OBC members. We examine both classical OBCs and generalized OBCs, as developed in [D. Cobb, *SIAM J. Control Optim.,* 50 (2012), pp. 1921–1949]. We perform our analysis for both real and complex compensators. In every case, we show that the set of OBCs is not open but has dense interior.

Key words. observer-based compensators, separation principle, singular systems

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1. Introduction.

The concept of an "observer-based compensator" (OBC) has been a cornerstone of feedback theory for many decades. In view of the celebrated "separation principle" in both the deterministic and stochastic settings, the idea of placing an observer in feedback with the plant enjoys certain theoretical advantages over more general compensation schemes. In spite of this fact, apparently little work has been done exploring the topological properties of the set of OBCs. In particular, one might ask the following: "Within the set of all compensators, how big is the set of OBCs?" In this paper, we provide a satisfying answer to this question by examining the elementary topological issues of openness and density. In addition, we are able to contrast our results with those of other authors (e.g., [5, sections 6.4.1–6.4.3]) who have previously addressed the nature of this set.

In [2, section 4], we introduced the notion of an observer for a singular plant. Although such an observer is necessarily a singular system, it is not the appropriate construction for this paper, since here we are interested in OBCs for ordinary (nonsingular) state-space systems. Instead, our present work is based largely on [1], which provides a more appropriate extension of OBCs and the separation principle. Reference [1] also provides the pertinent background in singular system theory.

We begin by summarizing the most relevant conclusions from [1]. Consider a state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1.1}$$

where $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$ is minimal (i.e., controllable and observable). The system has rational transfer function matrix

$$\mathcal{G}_p(s) = C(sI - A)^{-1} B.$$

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Taking (1.1) to be the "plant" in a feedback loop, consider compensators

$$E\dot{z} = Fz + Gy, \quad u = Hz, \tag{1.2}$$

where (E, F, G, H) lies in

$$\Omega_{\mathbb{C}} = \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p} \times \mathbb{C}^{m\times n}.$$

We will also examine the more "practical" case of real compensators

$$\Omega_{\mathbb{R}} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{m\times n}.$$

If

$$\det(sE - F) \not\equiv 0,$$

we say the matrix pencil (E, F) is *regular*. In this case, the compensator has the transfer function matrix

$$\mathcal{G}_c(s) = H(sE - F)^{-1} G.$$

In the case of compensators with strictly proper $\mathcal{G}_c$, we may wish to fix $E = I$ and consider triples (F, G, H) in either

$$\Sigma_{\mathbb{C}} = \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p} \times \mathbb{C}^{m\times n}$$

or

$$\Sigma_{\mathbb{R}} = \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p} \times \mathbb{R}^{m\times n}.$$

In the absence of exogenous inputs, the closed-loop system is

$$\begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

Thus we define the *closed-loop pencil*

$$(E_{cl}, F_{cl}) = \left(\begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix}, \begin{bmatrix} A & BH \\ GC & F \end{bmatrix} \right),$$

which has characteristic polynomial

$$\Delta_{cl}(s) = \det(sE_{cl} - F_{cl}).$$

If $(E, F, G, H) \in \Omega_{\mathbb{R}}$, then $\mathcal{G}_c$ and Δ_{cl} have real coefficients.

As in [1], we say (1.2) is an OBC if

$$(E, F, G, H) = (X_o X_c, X_o A X_c - X_o B Y_c - Y_o C X_c, Y_o, -Y_c) \in \Omega_{\mathbb{C}} \tag{1.3}$$

for some

$$(X_c, Y_c, X_o, Y_o) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{m\times n} \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times p}.$$

For the plant (A, B, C), we denote the set of all OBCs by $OBC\,(A, B, C)$. If the plant is fixed throughout the discussion, we simply write OBC. Assuming regularity, we have

$$\mathcal{G}_c\,(s) = -Y_c\,(sX_oX_c - X_oAX_c + X_oBY_c + Y_oCX_c)^{-1}\,Y_o.$$

For fixed nonsingular M and N, the mapping of parameters

$$(X_c, Y_c, X_o, Y_o) \longmapsto (X_cN, Y_cN, MX_o, MY_o)$$

is equivalent to the group action

$$(E, F, G, H) \to (MEN, MFN, MG, HN) \tag{1.4}$$

on OBC. Thus OBC is closed under system equivalence (1.4). The closed-loop pencil becomes

$$(E_{cl}, F_{cl}) = \left(\begin{bmatrix} I & 0 \\ 0 & X_oX_c \end{bmatrix}, \begin{bmatrix} A & -BY_c \\ Y_oC & X_oAX_c - X_oBY_c - Y_oCX_c \end{bmatrix}\right). \tag{1.5}$$

We showed in [1] that (1.3) leads to an extension of the classical "separation principle." Indeed, defining the nonsingular transformations

$$M_{cl} = \begin{bmatrix} I & 0 \\ X_o & -I \end{bmatrix}, \quad N_{cl} = \begin{bmatrix} X_c & I \\ I & 0 \end{bmatrix} \tag{1.6}$$

yields

$$(M_{cl}E_{cl}N_{cl}, M_{cl}F_{cl}N_{cl}) = \left(\begin{bmatrix} X_c & I \\ 0 & X_o \end{bmatrix}, \begin{bmatrix} AX_c - BY_c & A \\ 0 & X_oA - Y_oC \end{bmatrix}\right).$$

In this case, $\Delta_{cl} = \Delta_c\Delta_o$, where

$$\Delta_c\,(s) = \det\left(sX_c - (AX_c - BY_c)\right),$$
$$\Delta_o\,(s) = \det\left(sX_o - (X_oA - Y_oC)\right).$$

An OBC is regular iff both Δ_c and Δ_o are nontrivial.

Note that the form (1.3) includes the classical notion of an OBC, obtained by setting

$$(X_c, Y_c, X_o, Y_o) = (I, K, I, L)$$

to yield the compensator

$$(E, F, G, H) = (I, A - BK - LC, L, -K)\,. \tag{1.7}$$

Here the closed-loop pencil is

$$(E_{cl}, F_{cl}) = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}\right).$$

The transformations (1.6) reduce to

$$M_{cl} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}, \quad N_{cl} = \begin{bmatrix} I & I \\ I & 0 \end{bmatrix},$$

yielding the separation structure

$$(M_{cl}E_{cl}N_{cl}, M_{cl}F_{cl}N_{cl}) = \left(\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, \begin{bmatrix} A - BK & A \\ 0 & A - LC \end{bmatrix} \right). \tag{1.8}$$

This is essentially the classical result, requiring only right multiplication by

$$\widetilde{N} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

to achieve the familiar form

$$\left(M_{cl}E_{cl}N_{cl}\widetilde{N}, M_{cl}F_{cl}N_{cl}\widetilde{N} \right) = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right).$$

More generally, setting $(X_c, Y_c) = (I, K)$ in (1.3) yields the structure

$$(E, F, G, H) = (X_o, X_oA - X_oBK - Y_oC, Y_o, -K). \tag{1.9}$$

In [3] we showed that (1.9) accounts not only for all OBCs based on (1) full-order observers ($X_o = I$), but also those based on (2) $(n-p)$th-order Luenberger observers ($\operatorname{rank} X_o = n - p$, and $X_oA - Y_oC = I$), and (3) successive differentiation observers (X_o nilpotent and $X_oA - Y_oC = I$). The duals of (1)–(3) may be obtained by interchanging the roles of (X_c, Y_c) and (X_o, Y_o). Furthermore, in [1] we showed that every *static* compensator may be expressed in the form (1.3). These facts demonstrate the comprehensive nature of (1.3) and constitute a primary motivation for studying the topology of OBC.

2. Genericity of complex OBCs. In this section, we show that complex OBCs (1.3) account for "almost all" points in $\Omega_{\mathbb{C}}$ and $\Sigma_{\mathbb{C}}$. Classical OBCs (1.7) must be closed under similarity to achieve the same result. In section 3 we will prove the analogous results for real compensators. For any topological space Ω, we say $\mathcal{G} \subset \Omega$ is *generic* if it contains a set that is open and dense in Ω. Genericity and properness of subsets formalize the main idea.

We begin with a basic result on genericity that will be useful throughout this section. A modified version will be proven in section 3.

LEMMA 2.1. *Let $f : \mathbb{C}^j \to \mathbb{C}^{l \times l}$ be a polynomial, let $R, S \subset \mathbb{C}^l$ be subspaces, and let $q \in \{0, \dots, l\}$. Consider the set D of all $x \in \mathbb{C}^j$ such that*

(a) *$f(x)$ has l distinct eigenvalues,*

(b) *$f(x)$ has a q-dimensional invariant subspace $Q \subset \mathbb{C}^l$ with $Q \cap R = 0$ and $Q + S = \mathbb{C}^l$.*

Then D is open, and either $D = \phi$ or D is dense in $\mathbb{C}^j$.

Proof. If $D = \phi$, then D is open. Suppose $D \neq \phi$, and let $x \in D$ satisfy (a) and (b). Choose matrices $\widetilde{Q}$, $\widetilde{R}$, and $\widetilde{S}$ whose columns are bases for Q, R, and S, respectively. Then

$$\operatorname{rank} \begin{bmatrix} \widetilde{Q} & \widetilde{R} \end{bmatrix} = q + \dim R, \quad \operatorname{rank} \begin{bmatrix} \widetilde{Q} & \widetilde{S} \end{bmatrix} = l.$$

Let $x_k \in \mathbb{C}^j$ with $x_k \to x$. Since $f(x)$ has distinct eigenvalues and f is continuous, $f(x_k)$ has distinct eigenvalues for large k. Distinct eigenvalues in $f(x)$ also guarantee that Q is a stable invariant subspace. (See [6, Theorem 15.2.1].) Hence, each $f(x_k)$ has a q-dimensional invariant subspace $Q_k \subset \mathbb{C}^l$ such that $Q_k \to Q$ in the gap metric.

(See [6, Chapter 13 and section 15.2].) We may represent $Q_k = \operatorname{Im} \widetilde{Q}_k$ such that $\widetilde{Q}_k \to \widetilde{Q}$. For large k,

$$\operatorname{rank} \begin{bmatrix} \widetilde{Q}_k & \widetilde{R} \end{bmatrix} = q + \dim R, \quad \operatorname{rank} \begin{bmatrix} \widetilde{Q}_k & \widetilde{S} \end{bmatrix} = l,$$

so $Q_k \cap R = 0$ and $Q_k + S = \mathbb{C}^l$. Thus $x_k \in D$, proving that D is open.

To prove density, for each $x \in \mathbb{C}^j$ we consider the characteristic polynomial $\Delta(x, \cdot)$ of $f(x)$ and the resultant $r(x)$ of $\Delta(x, \cdot)$ and $\frac{\partial \Delta(x,\cdot)}{\partial s}$. Then r is a polynomial in x, and

$$D_a = \left\{ x \in \mathbb{C}^j \,\middle|\, f(x) \text{ has } l \text{ distinct eigenvalues} \right\}$$

is the complement of the algebraic variety $\{r(x) = 0\}$. But $D_a \supset D \neq \phi$, so D_a is dense in $\mathbb{C}^j$. It suffices to prove that D is dense in D_a.

Suppose $q = 0$. Then $Q = 0$ is the only admissible choice in (b), making (b) trivial with $S = \mathbb{C}^l$. Hence, $D = D_a$. Similarly, if $q = l$, then $Q = \mathbb{C}^l$ and $R = 0$, yielding $D = D_a$. Suppose $0 < q < l$, and let $x_0 \in D_a$, $x_1 \in D$, and

$$x(z) = (1 - z)\, x_0 + z x_1$$

for all $z \in \mathbb{C}$. Since $x(\cdot)$ is a polynomial, so are $f(x(\cdot))$ and $r(x(\cdot))$. It follows that the eigenvalues of $f(x(z))$ are algebraic functions of z. Let V be the variety $\{r(x(z)) = 0\}$. In particular, $x(0) = x_0$ and $x(1) = x_1$, so we have $0, 1 \notin V$. Thus V is a proper variety in one variable, so it is finite and there exists a simply connected region $\Gamma \subset \mathbb{C} - V$ such that $0, 1 \in \Gamma$. From [4, Theorem 12-3], the eigenvalues of $f(x(z))$ form l analytic functions $\lambda_1, \ldots, \lambda_l : \Gamma \to \mathbb{C}$. From [7, Theorem 4], there exist analytic functions $w_1, \ldots, w_l : \Gamma \to \mathbb{C}^l$ such that the $w_j(z)$ are linearly independent eigenvectors of $f(x(z))$, with w_j corresponding to λ_j. (The statement of Theorem 4 in [7] assumes compactness of Γ, but the proof rests merely on the fact that $r(x(\cdot))$ has finitely many zeros.) Since $x_1 \in D$, there exists Q_1 satisfying (b). From invariance, there are indices $j_1, \ldots, j_q$ such that

$$Q_1 = \operatorname{Im} \begin{bmatrix} w_{j_1}(1) & \cdots & w_{j_q}(1) \end{bmatrix}.$$

Let

$$\widetilde{Q}(z) = \begin{bmatrix} w_{j_1}(z) & \cdots & w_{j_q}(z) \end{bmatrix}$$

for all $z \in \Gamma$. Also, $x_1 \in D$ implies that $\begin{bmatrix} \widetilde{Q}(1) & \widetilde{R} \end{bmatrix}$ has a $(q + \dim R)$th-order minor $m_{R1} \neq 0$ and $\begin{bmatrix} \widetilde{Q}(1) & \widetilde{S} \end{bmatrix}$ has an lth-order minor $m_{S1} \neq 0$. Using the same rows and columns, we may define minors $m_R(z)$ and $m_S(z)$ of $\begin{bmatrix} \widetilde{Q}(z) & \widetilde{R} \end{bmatrix}$ and $\begin{bmatrix} \widetilde{Q}(z) & \widetilde{S} \end{bmatrix}$. Then $m_{RS} = m_R m_S$ is analytic with $m_{RS}(1) = m_{R1} m_{S1} \neq 0$, making m_{RS} nontrivial. Choose any neighborhood $U \subset \Gamma$ of 0. Then m_{RS} has countably many zeros in U and there exists $z_k \in U$ with $z_k \to 0$ such that $m_{RS}(z_k) \neq 0$ for every k. This implies $x(z_k) \in D$ and $x(z_k) \to x_0$, proving density of D in D_a. ∎

Before addressing genericity of OBC, we consider classical OBCs of the form (1.7). Actually, systems (1.7) may be viewed as points

$$(F, G, H) = (A - BK - LC, L, -K) \in \Sigma_{\mathbb{C}}, \tag{2.1}$$

suppressing the leading I to put the discussion in line with ordinary state-space theory. Simple examples show that the set of OBCs (2.1) is generally not closed under similarity. Furthermore, such OBCs do not form a dense set, since (2.1) parametrizes a proper affine set in $\Sigma_{\mathbb{C}}$.

A more fruitful approach is to consider the set

$$OBC_c = \left\{ \begin{array}{c} \left(T^{-1}\left(A - BK - LC\right)T, T^{-1}L, -KT\right) \in \Sigma_{\mathbb{C}} \\ T \in \mathbb{C}^{n\times n},\ \det T \neq 0,, K \in \mathbb{C}^{m\times n},\ L \in \mathbb{C}^{n\times p} \end{array} \right| \left. \vphantom{\begin{array}{c} a \\ b \end{array}} \right\}, \tag{2.2}$$

consisting of the union of all similarity orbits of points (2.1). As in [1, Theorem 2], a compensator $(F, G, H) \in OBC_c$ iff there exists a nonsingular T such that

$$TGCT - AT + TF - BH = 0. \tag{2.3}$$

The following example describes OBC_c for the simplest case.

EXAMPLE 2.2. *Let $n = m = p = 1$. From (2.3), $(F, G, H) \in OBC_c$ iff the equation*

$$GCx^2 + (F - A)x - BH = 0$$

has a nonzero root. Thus the complement $OBC_c^c = \mathbb{C}^3 - OBC_c$ is encompassed by three cases:

(1) *$F \neq A$, $G = H = 0$ (single root at $x = 0$).*
(2) *$F = A$, $G \neq 0$, $H = 0$ (double root at $x = 0$).*
(3) *$F = A$, $G = 0$, $H \neq 0$ (no root).*

Let e_j be the jth coordinate axis in $\mathbb{C}^3$. We may combine (1)–(3) and write

$$OBC_c^c = (A, 0, 0) + \left(\bigcup_{j=1}^{3} e_j - \{0\}\right).$$

In other words, OBC_c^c is a translation of the 3-axis, excluding their intersection. Note that OBC_c is generic and that it is not only proper, but also fails to be open.

The next two results establish basic topological properties of OBC_c. Here we remind the reader that (A, B, C) is a minimal, but otherwise arbitrary, state-space system.

PROPOSITION 2.3. *OBC_c is not open in $\Sigma_{\mathbb{C}}$.*

Proof. Choose $\lambda \in \mathbb{C} - \sigma(A)$ and consider systems

$$f(x) = (xA + (1 - x)\lambda I, 0, 0)$$

with $x \in [0, 1]$. Applying (2.3), $f(x) \in OBC_c$ iff

$$x(A - \lambda I) = T^{-1}(A - \lambda I)T$$

for some nonsingular $T \in \mathbb{C}^{n\times n}$. For $x < 1$, nonsingularity of $A - \lambda I$ guarantees

$$\sigma(x(A - \lambda I)) = x\sigma(A - \lambda I) \neq \sigma(A - \lambda I),$$

so $f(x) \notin OBC_c$. For $x = 1$, setting $T = I$ proves $f(1) \in OBC_c$. Continuity of f implies that $f(1)$ is a boundary point of OBC_c, violating openness. ∎

In particular, since $\Sigma_{\mathbb{C}}$ itself is open, OBC_c is a proper subset of $\Sigma_{\mathbb{C}}$.

THEOREM 2.4. *OBC_c is generic in $\Sigma_{\mathbb{C}}$.*

Proof. Let $f : \Sigma_{\mathbb{C}} \to \mathbb{C}^{2n\times 2n}$, and let complex subspaces R and S be defined by

$$f(F,G,H) = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix},$$

$$R = \operatorname{Im}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \subset \mathbb{C}^{2n}, \quad S = \operatorname{Im}\begin{bmatrix} 0 \\ I_n \end{bmatrix} \subset \mathbb{C}^{2n}.$$

Let $q = n$ and D be the set of all $(F,G,H) \in \Sigma_{\mathbb{C}}$ satisfying (a)–(b) in Lemma 2.1. From [1, Theorem 13], $(F,G,H) \in OBC_c$ iff there exists an $f(F,G,H)$-invariant subspace $Q \subset \mathbb{C}^{2n}$ such that

$$\dim Q = n, \quad Q \cap R = Q \cap S = 0.$$

But this is equivalent to

$$\dim Q = n, \quad Q \cap R = 0, \quad Q + S = \mathbb{C}^{2n},$$

so D consists of all $x \in OBC_c$ where $f(x)$ has distinct eigenvalues. Lemma 2.1 guarantees that either OBC_c is generic or D is empty. From (1.8), the eigenvalues of (E_{cl}, F_{cl}) are those of $A - BK$ and $A - LC$. Controllability and observability of (A,B,C) guarantee that K and L can be chosen to make the eigenvalues of $f(A - BK - LC, L, -K)$ distinct. Hence, D is nonempty. ▯

A common claim made in the control theory literature is that "every stabilizing compensator is observer-based." This statement is sometimes justified by invoking the Q-parametrization. (See [5, sections 6.4.1–6.4.3].) Such analyses typically require either a Q-dependent dilation of the plant or the assignment of extra inputs and outputs to attach the Q parameter to a fixed OBC. We view these constructions as somewhat artificial, since they go beyond the traditional OBC structure. Furthermore, invoking the Q-parametrization restricts the analysis a priori to stabilizing compensators. Hence, for structural results obtained in this way, stabilization may not be an essential assumption but merely a method of proof.

The above claim can be examined in the context of our results. Proposition 2.3 tells us that for any plant there are compensators outside OBC_c, but it does not clarify which of these stabilize the plant. (By closed-loop stability, we mean that the pencil (1.5) has unit index with all eigenvalues λ satisfying $\operatorname{Re}\lambda < 0$.) Theorem 2.4 indicates that stabilization is generically irrelevant to the OBC structure. Hence, we counter with our own statement: "Almost every compensator is observer-based." In the strictest sense, OBC structure does not actually follow from stabilization. Indeed, in Example 2.2 we may set $(A,B,C) = (-1,1,1)$ and $(F,G,H) = (-1,1,0)$ to obtain $(F,G,H) \notin OBC_c$. The closed-loop system is governed by

$$\begin{bmatrix} A & BH \\ GC & F \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

which is continuous-time stable. One might argue that this example is weak, since $(-1,1,0)$ has transfer function $\mathcal{G}_c = 0$, which is also realized by $(-1,0,0) \in OBC$. A better example is obtained by performing a frequency shift $s \longmapsto s+1$ in [1, Example 4]. This yields $n = m = p = 2$,

$$(A,B,C) = (-I,I,I),$$

$$\mathcal{G}_c(s) = \begin{bmatrix} 0 & \frac{1}{s+1} \\ 0 & 0 \end{bmatrix}.$$

It is apparent from (2.2) that in general the frequency shift $(A,B,C) \longmapsto (A-I,B,C)$ on plants imposes the equivalent shift $OBC_c \longmapsto (-I,0,0)+OBC_c$ on OBCs. Thus we may conclude as in [1] that no realization (F,G,H) of $\mathcal{G}_c$ lies in OBC_c. As in [5, section 6.4.2], the closed-loop system is governed by

$$\begin{bmatrix} I & -\mathcal{G}_c(s) \\ -C(sI-A)^{-1}B & I \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{(s+1)^2} & 0 & \frac{1}{s+1} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{s+1} & \frac{1}{(s+1)^3} & 1 & -\frac{1}{(s+1)^2} \\ 0 & -\frac{1}{s+1} & 0 & 1 \end{bmatrix},$$

which is stable. The only weakness in this example is that the plant is already stable. This begs the question of whether there exist an unstable minimal (A,B,C) and a stabilizing $\mathcal{G}_c$ such that no realization (F,G,H) of $\mathcal{G}_c$ lies in OBC. To our knowledge, this is an open question, leaving one last conjecture along the same lines: "Every stabilizing compensator for an unstable plant is observer-based." The veracity of this statement is still a possibility.

Another interesting issue is the possibility of extending our results to nonminimal plants. Unfortunately, the situation turns out to be complicated in that genericity of OBC depends on (A,B,C). The following example illustrates the problem. Such an extension requires further research.

EXAMPLE 2.5. (1) *Let* $(A,B,C)=(0,0,0)\in\mathbb{C}^3$. *Then*

$$OBC_c = \left\{ \left(0, \frac{L}{T}, -KT\right) \in \mathbb{C}^3 \,\middle|\, T \neq 0 \right\},$$

which has empty interior.

(2) *Now consider* $(A,B,C)=(0,1,0)$. *Then*

$$OBC_c = \left\{ \left(-K, \frac{L}{T}, -KT\right) \in \mathbb{C}^3 \,\middle|\, T \neq 0 \right\}.$$

For any $(F,G,H)\in\mathbb{C}^3$ *with* $FH\neq 0$, *we may set*

$$K=-F, \quad L=\frac{GH}{F}, \quad T=\frac{H}{F}$$

to obtain $\left(-K,\frac{L}{T},-KT\right)=(F,G,H)$. *Hence,*

$$OBC_c \supset \left\{ (F,G,H) \in \mathbb{C}^3 \,\middle|\, FH \neq 0 \right\},$$

which is open and dense.

Returning to general OBCs (1.3), we may pursue an analysis similar to Example 2.2, Proposition 2.3, and Theorem 2.4.

EXAMPLE 2.6. *Let* $n=m=p=1$. *By* [1, proof of Theorem 13], *a compensator* $(E,F,G,H)\in\mathbb{C}^4$ *belongs to* OBC *iff the equations*

$$uv - E = 0,$$
$$Auv - GCu + BHv - F = 0$$

have a solution (u,v). *Thus* OBC^c *is the union of four cases:*

(1) $E=0$, $F\neq 0$, $G=H=0$.

(2) $E \neq 0$, $F \neq AE$, $G = H = 0$.
(3) $E \neq 0$, $F = AE$, $G \neq 0$, $H = 0$.
(4) $E \neq 0$, $F = AE$, $G = 0$, $H \neq 0$.
Defining the planes

$$P_0 = \{E = F = 0\}, \quad P_1 = \{F = AE,\ G = 0\},$$
$$P_2 = \{F = AE,\ H = 0\}, \quad P_3 = \{G = H = 0\},$$

it is easy to show that

$$OBC^c = \bigcup_{j=1}^{3} P_j - \bigcap_{j=1}^{3} P_j - P_0.$$

For fixed $E \neq 0$*, the corresponding slice of* OBC^c *is the same as in Example* 2.2*, except here we replace* A *by* AE*. For* $E = 0$*, the slice is just the* F*-axis, excluding* $(0, 0, 0)$.

For arbitrary minimal plants (A, B, C), we have the following results.

PROPOSITION 2.7. *OBC is not open in $\Omega_{\mathbb{C}}$.*

Proof. Choose $\lambda \in \mathbb{C} - \sigma(A)$ and consider systems

$$f(x) = (I, xA + (1 - x)\lambda I, 0, 0),$$

with $x \in [0, 1]$. From [1, proof of Theorem 13], $f(x) \in OBC$ iff there exist U and V such that

$$VU = I, \quad VAU - GCU + VBH - F = 0. \tag{2.4}$$

Applied to $f(x)$, (2.4) becomes

$$V = U^{-1}, \quad x(A - \lambda I) = U^{-1}(A - \lambda I)U.$$

From here, the proof is the same as in Proposition 2.3, revealing that $f(x) \notin OBC$ for $x < 1$ and $f(1) \in OBC$. Hence, $f(1)$ is a boundary point of OBC. ∎

THEOREM 2.8. *OBC is generic in $\Omega_{\mathbb{C}}$.*

Proof. Let

$$\Gamma = \left\{ (E, F, G, H) \in \Omega_{\mathbb{C}} \;\middle|\; \det E \neq 0 \right\}.$$

Since $\det E = 0$ determines a proper algebraic variety, Γ is dense in $\Omega_{\mathbb{C}}$. It suffices to prove that $OBC \cap \Gamma$ contains a set U that is open and dense in Γ. Consider the map $f : \Gamma \to \Sigma_{\mathbb{C}}$ given by

$$f(E, F, G, H) = \left(E^{-1}F, E^{-1}G, H\right).$$

If $(E, F, G, H) \in \Gamma$ satisfies

$$f(E, F, G, H) \in OBC_c,$$

then there exist T, K, and L such that

$$\left(E^{-1}F, E^{-1}G, H\right) = \left(T^{-1}(A - BK - LC)T, T^{-1}L, -KT\right).$$

Setting

$$X_c = T, \quad Y_c = KT, \quad X_o = ET^{-1}, \quad Y_o = ET^{-1}L$$

yields

$$\begin{gathered} E = \left(ET^{-1}\right) T = X_o X_c, \\ F = ET^{-1}\left(A - BK - LC\right) T = \left(ET^{-1}\right) AT - \left(ET^{-1}\right) B\left(KT\right) - \left(ET^{-1}L\right) CT \\ = X_o A X_c - X_o B Y_c - Y_o C X_c, \\ G = ET^{-1}L = Y_o, \\ H = -KT = -Y_c, \end{gathered}$$

so $(E, F, G, H) \in OBC$. Hence, $f^{-1}(OBC_c) \subset OBC$. From Theorem 2.4, there exists an open set $D \subset OBC_c$ that is dense in $\Sigma_{\mathbb{C}}$. Let $U = f^{-1}(D)$. Then $U \subset OBC \cap \Gamma$. Since f is continuous, U is open. For any $(E, F, G, H) \in \Gamma$, density of D guarantees the existence of $(F_k, G_k, H_k) \in D$ such that

$$(F_k, G_k, H_k) \to \left(E^{-1}F, E^{-1}G, H\right).$$

Then

$$f\left(E, EF_k, EG_k, H_k\right) = \left(F_k, G_k, H_k\right) \in D,$$

so

$$\begin{gathered} (E, EF_k, EG_k, H_k) \in f^{-1}(D) = U, \\ (E, EF_k, EG_k, H_k) \to (E, F, G, H), \end{gathered}$$

proving density of U. ∎

3. Genericity of real OBCs. It is possible to modify the analysis of section 2 to address OBCs in $OBC_c \cap \Sigma_{\mathbb{R}}$ and $OBC \cap \Omega_{\mathbb{R}}$. We begin with an alternate version of Lemma 2.1. Recall that the *complexification* $S^{\mathbb{C}}$ of a subspace $S \subset \mathbb{R}^n$ is

$$S^{\mathbb{C}} = \left\{ x + iy \,\middle|\, x, y \in S \right\}.$$

$S^{\mathbb{C}}$ is a subspace of $\mathbb{C}^n$ with $\dim S^{\mathbb{C}} = \dim S$.

LEMMA 3.1. *Let $f : \mathbb{C}^j \to \mathbb{C}^{l \times l}$ be a polynomial with $f\left(\mathbb{R}^j\right) \subset \mathbb{R}^{l \times l}$, let $R, S \subset \mathbb{R}^l$ be subspaces, and let $q \in \{0, \ldots, l\}$. Consider the set D of all $x \in \mathbb{R}^j$ such that*

(a) *$f(x)$ has l distinct eigenvalues, and*

(b) *$f(x)$ has a q-dimensional invariant subspace $Q \subset \mathbb{C}^l$ with $Q \cap R^{\mathbb{C}} = 0$ and $Q + S^{\mathbb{C}} = \mathbb{C}^l$.*

Then D is open, and either $D = \phi$ or D is dense in $\mathbb{R}^j$.

Proof. The proof is the same as that of Lemma 2.1, except for the following changes: Since $f\left(\mathbb{R}^j\right) \subset \mathbb{R}^{l \times l}$, the polynomials f and r have real coefficients, making D_a the complement of the real variety $\{r(x) = 0\}$. Thus D_a is dense in $\mathbb{R}^j$. Since U contains countably many zeros of m_{RS}, there exist $z_k \in U \cap \mathbb{R}$ with $z_k \to 0$ such that $m_{RS}(z_k) \neq 0$ for every k, proving density of D in D_a. ∎

The classical, real OBCs are

$$OBC_c \cap \Sigma_{\mathbb{R}} = \left\{ \begin{array}{c} \left(T^{-1}\left(A - BK - LC\right) T, T^{-1}L, -KT\right) \in \Sigma_{\mathbb{R}} \\ T \in \mathbb{C}^{n \times n}, \ \det T \neq 0, \ K \in \mathbb{C}^{m \times n}, \ L \in \mathbb{C}^{n \times p} \end{array} \right| \left. \vphantom{\begin{array}{c} a \\ b \end{array}} \right\}.$$

It is important to note that, although (E, F, G, H) must be real, T, K, *and* L *need not be real.* Thus $OBC_c \cap \Sigma_{\mathbb{R}}$ is still characterized by (2.3) with *complex* T. Although mathematically easier, restricting to real T, K, and L would lead to a simplistic theory with little practical value.

For $n = m = p = 1$, the construction in Example 2.2 carries over to $OBC_c \cap \Sigma_{\mathbb{R}}$ verbatim, except that $e_j \in \mathbb{R}^3$.

PROPOSITION 3.2. *$OBC_c \cap \Sigma_{\mathbb{R}}$ is not open in $\Sigma_{\mathbb{R}}$.*

Proof. The proof is the same as that for Proposition 2.3, except that we must choose $\lambda \in \mathbb{R} - \sigma(A)$. Considering systems

$$f(x) = (xA + (1-x)\lambda I, 0, 0)$$

and applying (2.3), $f(x) \in OBC_c \cap \Sigma_{\mathbb{R}}$ iff

$$x(A - \lambda I) = T^{-1}(A - \lambda I)T$$

for some nonsingular $T \in \mathbb{C}^{n \times n}$. The arguments in Proposition 2.3 reveal that $f(x) \notin OBC_c \cap \Sigma_{\mathbb{R}}$ for $x < 1$ and $f(1) \in OBC_c \cap \Sigma_{\mathbb{R}}$. Hence, $f(1)$ is a boundary point of $OBC_c \cap \Sigma_{\mathbb{R}}$. $\square$

THEOREM 3.3. *$OBC_c \cap \Sigma_{\mathbb{R}}$ is generic in $\Sigma_{\mathbb{R}}$.*

Proof. Let $f : \Sigma_{\mathbb{C}} \to \mathbb{C}^{2n \times 2n}$ and real subspaces R and S be defined by

$$f(F, G, H) = \begin{bmatrix} A & BH \\ GC & F \end{bmatrix},$$

$$R = \operatorname{Im}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \subset \mathbb{R}^{2n}, \quad S = \operatorname{Im}\begin{bmatrix} 0 \\ I_n \end{bmatrix} \subset \mathbb{R}^{2n}.$$

Let $q = n$ and D be the set of all $(F, G, H) \in \Sigma_{\mathbb{R}}$ satisfying (a)–(b) in Lemma 3.1. Then f is a polynomial with $f(\Sigma_{\mathbb{R}}) \subset \mathbb{R}^{2n \times 2n}$,

$$R^{\mathbb{C}} = \operatorname{Im}\begin{bmatrix} I_n \\ 0 \end{bmatrix} \subset \mathbb{C}^{2n}, \quad S^{\mathbb{C}} = \operatorname{Im}\begin{bmatrix} 0 \\ I_n \end{bmatrix} \subset \mathbb{C}^{2n}.$$

From [1, Theorem 13], $(F, G, H) \in OBC_c \cap \Sigma_{\mathbb{R}}$ iff there exists an $f(F, G, H)$-invariant subspace $Q \subset \mathbb{C}^{2n}$ such that

$$\dim Q = n, \quad Q \cap R^{\mathbb{C}} = Q \cap S^{\mathbb{C}} = 0.$$

But this is equivalent to

$$\dim Q = n, \quad Q \cap R^{\mathbb{C}} = 0, \quad Q + S^{\mathbb{C}} = \mathbb{C}^{2n},$$

so $D \subset OBC_c \cap \Sigma_{\mathbb{R}}$ consists of all points such that $f(F, G, H)$ is real and has distinct eigenvalues. Lemma 3.1 guarantees that either $OBC_c \cap \Sigma_{\mathbb{R}}$ is generic or D is empty. From (1.8), the eigenvalues of (E_{cl}, F_{cl}) are those of $A - BK$ and $A - LC$. Controllability and observability of (A, B, C) guarantee that $K \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{n \times p}$ can be chosen to make the eigenvalues of $f(A - BK - LC, L, -K)$ distinct. Hence, D is nonempty. $\square$

Finally, we observe that in general the real OBCs form the set

$$OBC \cap \Omega_{\mathbb{R}} = \left\{ \begin{array}{c|} (X_o X_c, X_o A X_c - X_o B Y_c - Y_o C X_c, Y_o, -Y_c) \\ (X_c, Y_c, X_o, Y_o) \in \mathbb{C}^{n \times n} \times \mathbb{R}^{m \times n} \times \mathbb{C}^{n \times n} \times \mathbb{R}^{n \times p} \end{array} \right\}.$$

Again, note that X_c and X_o (but not Y_c and Y_o) may be complex. An analysis similar to Proposition 2.7 and Theorem 2.8 is possible. For $n = m = p = 1$, Example 2.6 carries over by substituting $\mathbb{R}^4$ for $\mathbb{C}^4$.

PROPOSITION 3.4. *$OBC \cap \Omega_{\mathbb{R}}$ is not open in $\Omega_{\mathbb{R}}$.*

Proof. Choose $\lambda \in \mathbb{R} - \sigma(A)$ and consider the points

$$f(x) = (I, xA + (1-x)\lambda I, 0, 0).$$

Applying (2.4), $f(x) \in OBC \cap \Omega_{\mathbb{R}}$ iff

$$x(A - \lambda I) = U^{-1}(A - \lambda I)U$$

for some nonsingular $U \in \mathbb{C}^{n\times n}$. The arguments in Proposition 2.3 reveal that $f(x) \notin OBC \cap \Omega_{\mathbb{R}}$ for $x < 1$ and $f(1) \in OBC \cap \Omega_{\mathbb{R}}$. Hence, $f(1)$ is a boundary point of $OBC \cap \Omega_{\mathbb{R}}$. ∎

THEOREM 3.5. *$OBC \cap \Omega_{\mathbb{R}}$ is generic in $\Omega_{\mathbb{R}}$.*

Proof. The proof is the same as that for Theorem 2.8, replacing $\Omega_{\mathbb{C}}$ by $\Omega_{\mathbb{R}}$ and $\Sigma_{\mathbb{C}}$ by $\Sigma_{\mathbb{R}}$. ∎

4. Conclusions.

In the state-space context, OBCs account for "almost all" compensators. Preliminary work has demonstrated that this statement does in fact carry over to the space of compensator transfer functions. This will be the topic of a later paper. The analysis of transfer functions is more difficult, owing to the problem of topologizing the space of rational functions. In general, the question of determining when the set of OBC transfer functions is proper appears to be hard.

In the discussion following Theorem 2.4, we have addressed the interplay between stabilization and OBC structure. Most popular conjectures appear to miss the point that the OBCs are generic, without restricting to stabilization. Even in our context, however, we feel that the relevance of stabilization is not completely settled.

In our opinion, genericity of OBCs lends new impetus to the use of observers and the separation principle in control theory. In optimal control, since cost functionals are typically continuous, restricting to a generic set does not drastically change the problem, as long as one is content with an ε-optimal solution. Genericity may also be useful in other long-standing problems such as those associated with static output feedback and, more generally, output feedback with an a priori bound on compensator order.

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