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EIGENVALUE CONDITIONS FOR CONVERGENCE OF SINGULARLY PERTURBED MATRIX EXPONENTIAL FUNCTIONS*

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Abstract. We investigate convergence of sequences of $n \times n$ matrix exponential functions $t \to e^{tA_k^{-1}}$ for $t > 0$, where $A_k \to A$, A_k is nonsingular and A is nilpotent. Specifically, we address pointwise convergence, almost uniform convergence, and, viewing the exponential as a Schwartz distribution, weak* convergence. We show that simple results can be obtained in terms of the eigenvalues of A_k^{-1} alone. In particular, a necessary and sufficient condition for weak* convergence in terms of eigenvalue behavior is attainable. We then apply our results to real-analytic matrices $A(\varepsilon)$ as $\varepsilon \to 0^+$. Our work is applicable to matrices over both $\mathbb{R}$ and $\mathbb{C}$.

Key words. singular perturbation, matrix exponential, sequential convergence, distribution theory

AMS subject classifications. 34, 93, 40

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1. Introduction.

The study of singularly perturbed differential equations has been a productive area of research for several decades, with applications to diverse fields such as control theory, fluid mechanics, and econometrics. For general background, see [1], [3], and [2]. A "singularly perturbed system" may be loosely described as a system of differential equations with "small" parameters multiplying some of the derivatives. For example, the seminal work of Vasil'eva [4], [5] investigates the behavior of the solutions of nonlinear systems of the form

$$\varepsilon \dot{x} = f(x, u)$$

as $\varepsilon \to 0^+$, where $x(t) \in \mathbb{R}^n$. Multiple parameter problems with $\varepsilon \in \mathbb{R}^p$ have also been considered, although typically with strict assumptions on the relative convergence rates of the parameters.

In [6], we initiated the study of linear singularly perturbed systems with a general perturbation structure. Specifically, we considered systems of the form

$$E_k \dot{x} = F_k x + G_k u, \tag{1}$$

where $E_k \to E \in \mathbb{R}^{n\times n}$, $F_k \to F \in \mathbb{R}^{n\times m}$, $G_k \to G \in \mathbb{R}^{p\times n}$, $\det E = 0$, and $\det(sE - F) \not\equiv 0$. Under these assumptions, we showed in [7] that the Weierstrass decomposition for matrix pencils may be extended to perturbation problems. (See [8, Chapter XII] for background on pencils.) Focusing on the unforced response of (1) and applying [7] to (1), there exist nonsingular $M_k \to M$ and $N_k \to N$ that achieve the sequential decomposition

$$M_k E_k N_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad M_k F_k N_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix},$$

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where $A_{sk} \to A_s$ and $A_{fk} \to A_f$ for some A_s and nilpotent A_f. Hence, a coordinate change in (1) yields the decoupled system

$$\begin{aligned} \dot{x}_s &= A_{sk} x_s, \\ A_{fk} \dot{x}_f &= x_f. \end{aligned} \tag{2}$$

In the present paper, we are interested exclusively in the unforced behavior of x_f as $k \to \infty$, under the assumption that A_{fk} is nonsingular for every k. Then

$$x_f(t) = e^{tA_{fk}^{-1}} x_f(0), \tag{3}$$

so it suffices to study the exponential function $t \longmapsto e^{tA_{fk}^{-1}}$ for $t > 0$. More advanced problems, such as the behavior of the forced response of (1) or the case where A_{fk} is singular for infinitely many k, also depend on convergence of the exponential function. However, we defer these issues to a later paper.

In section 2, we view the exponential as a Schwartz distribution and relate weak* (i.e., distributional) convergence to the behavior of the eigenvalues of A_{fk}. We show that a simple necessary and sufficient condition is achievable. In section 3, pointwise and almost uniform convergence are addressed. We demonstrate that eigenvalue behavior is again related to exponential convergence, although a condition that is both necessary and sufficient is not attainable. Although our development is couched in terms of complex matrices, all results specialize to the real case.

2. Weak* convergence. In this section, we interpret the exponential function as a Schwartz distribution. Dropping the subscript "f," we consider nonsingular matrix sequences $A_k \in \mathbb{C}^{n \times n}$, where $A_k \to A$ and A is nilpotent. Our goal is to study the behavior of $e^{tA_k^{-1}}$ as $k \to \infty$ relative to distributional convergence. The main result of this section establishes the surprising fact that convergence is completely determined by the asymptotic behavior of the eigenvalues of A_k.

Our work is based principally on the theory of distributions as outlined in [11] and [12]. Let $\mathcal{D}$ be the vector space of C^∞ "test functions" $\phi : \mathbb{R} \to \mathbb{C}$ with bounded support. A *distribution* is a member of the dual space $\mathcal{D}'$. In particular, any classical function $f : \mathbb{R} \to \mathbb{C}$ which is integrable on bounded intervals of $\mathbb{R}$ may be viewed as a distribution according to

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(t)\, \phi(t)\, dt.$$

Given two such functions f_1 and f_2, if $\langle f_1, \phi \rangle = \langle f_2, \phi \rangle$ for every ϕ, then $f_1(t) = f_2(t)$ a.e. (Lebesgue). In particular, if f_1 and f_2 are continuous, then $f_1(t) = f_2(t)$ everywhere, so the correspondence between continuous functions and their associated distributions is a bijection. Whether f refers to a function or its associated distribution will be clear from context. Of singular importance is the *unit impulse*

$$\langle \delta, \phi \rangle = \phi(0).$$

We say a sequence $f_k \in \mathcal{D}'$ *converges weak** to $f \in \mathcal{D}'$ if

$$\langle f_k, \phi \rangle \to \langle f, \phi \rangle \tag{4}$$

for every $\phi \in \mathcal{D}$. This definition may be extended to convergence on any open set $\Sigma \subset \mathbb{R}$ by requiring (4) merely for $\phi \in \mathcal{D}$ with $\operatorname{supp} \phi \subset \Sigma$.

For any $Q \in \mathbb{C}^{n\times n}$, consider the map $\exp(Q) : [0,\infty) \to \mathbb{C}^{n\times n}$ defined by

$$\exp(Q)(t) = e^{tQ}.$$

Since the function $\exp(Q)$ is continuous, we may define the (matrix-valued) distribution

$$\langle \exp(Q), \phi \rangle = \int_0^\infty e^{tQ}\phi(t)dt,$$

where ϕ ranges over $\mathcal{D}$. Suppose matrices Q_1 and Q_2 are given such that $e^{tQ_1} = e^{tQ_2}$ for all t. By differentiation, we obtain

$$Q_1 e^{tQ_1} = Q_2 e^{tQ_2}.$$

Taking the limit as $t \to 0^+$ yields $Q_1 = Q_2$. This proves that the map $Q \longmapsto \exp(Q)$ is $1-1$ for both interpretations $\exp(Q)$.

The (distributional) derivative of $f \in \mathcal{D}'$ is given by the formula

$$\left\langle \dot{f}, \phi \right\rangle = -\left\langle f, \dot{\phi} \right\rangle.$$

Hence, the pth derivative of $\exp(Q)$ is obtained through the integration by parts calculation

$$\left\langle \exp^{(p)}(Q), \phi \right\rangle = (-1)^p \left\langle \exp(Q), \phi^{(p)} \right\rangle = Q^p \left\langle \exp(Q), \phi \right\rangle + \sum_{m=0}^{p-1} (-1)^m \phi^{(m)}(0) Q^{p-m-1},$$

yielding

$$\exp^{(p)}(Q) = Q^p \exp(Q) + \sum_{m=0}^{p-1} \delta^{(m)} Q^{p-m-1}. \tag{5}$$

The (holomorphic) Fourier transform of $\exp(Q)$ is also relevant to our analysis. Recall that the classical transform $\mathcal{F}\{\exp(Q)\}$ is equal to the map $s \longmapsto (sI - Q)^{-1}$. The transform may be couched in terms of distributions as follows. (See [11, Chapter II] for details.) Let $\mathcal{Z}$ be the vector space of all entire functions $\psi : \mathbb{C} \to \mathbb{C}$ such that

$$|\psi(s)| < b_p \frac{e^{a|\mathrm{Re}\, s|}}{1 + |s|^p} \tag{6}$$

for some $a, b_0, b_1, b_2, \ldots > 0$. (The constants depend on ψ.) The Fourier transform

$$\psi(s) = \int_{-\infty}^{\infty} \phi(t) e^{st} dt$$

determines an isomorphism $\mathcal{D} \to \mathcal{Z}$. In turn, each $f \in \mathcal{D}'$ has a Fourier transform F defined by

$$\langle F, \psi \rangle = 2\pi i \langle f, \phi \rangle. \tag{7}$$

This determines the transform operator $\mathcal{F} : \mathcal{D}' \to \mathcal{Z}'$. Adopting weak* topology on $\mathcal{Z}'$, it can be shown that $\mathcal{F}$ is a homeomorphic isomorphism. Setting $f = \exp(Q)$ in (7) and choosing any $\xi \in \mathbb{C}$ such that $\xi > \operatorname{Re} \rho_j$ for every eigenvalue ρ_j of Q, we obtain

$$\int_{\xi - i\infty}^{\xi + i\infty} \psi(s)(sI - Q)^{-1} ds = 2\pi i \int_0^\infty \phi(t) e^{tQ} dt \tag{8}$$

for every $\phi \in \mathcal{D}$.

Our first result characterizes the limit(s) of $\exp(A_k^{-1})$ when a limit exists.

Lemma 1. *Let* $-\infty \leq a < b \leq \infty$ *and assume* $\exp(A_k^{-1}) \to W$ *weak* on* (a, b).

(i) *If* $n = 1$ *or* $0 \notin (a, b)$, *then* $W = 0$ *on* (a, b).

(ii) *If* $n > 1$ *and* $0 \in (a, b)$, *then*

$$W = -\sum_{p=1}^{n-1} \delta^{(p-1)} A^p$$

on (a, b).

(iii) *If* $n > 1$ *and* $(a, b) = \mathbb{R}$, *then*

$$\mathcal{F}\{W\} = A(sA - I)^{-1}. \tag{9}$$

Proof. From (5), taking the nth distributional derivative yields

$$\exp^{(n)}\left(A_k^{-1}\right) = A_k^{-n} \exp\left(A_k^{-1}\right) + \sum_{p=1}^{n} \delta^{(p-1)} A_k^{p-n},$$

$$\exp\left(A_k^{-1}\right) = A_k^n \exp^{(n)}\left(A_k^{-1}\right) - \sum_{p=1}^{n} \delta^{(p-1)} A_k^p.$$

Since $A^n = 0$,

$$\begin{aligned} \left\langle \exp\left(A_k^{-1}\right), \phi \right\rangle &= A_k^n \left\langle \exp^{(n)}\left(A_k^{-1}\right), \phi \right\rangle + \sum_{p=1}^{n} (-1)^p A_k^p \phi^{(p-1)}(0) \\ &= (-1)^n A_k^n \left\langle \exp\left(A_k^{-1}\right), \phi^{(n)} \right\rangle + \sum_{p=1}^{n} (-1)^p A_k^p \phi^{(p-1)}(0) \\ &\to \begin{cases} 0, & n = 1, \\ \displaystyle\sum_{p=1}^{n-1} (-1)^p A^p \phi^{(p-1)}(0), & n > 1, \end{cases} \\ &= \begin{cases} 0, & n = 1, \\ \left\langle -\displaystyle\sum_{p=1}^{n-1} \delta^{(p-1)} A^p, \phi \right\rangle, & n > 1 \end{cases} \end{aligned} \tag{10}$$

for any $\phi \in \mathcal{D}$. If $0 \notin (a, b)$, then $\delta = 0$ on (a, b), so $\langle \exp(A_k^{-1}), \phi \rangle = 0$ whenever $\operatorname{supp} \phi \subset (a, b)$. Part (iii) of Lemma 1 follows directly from the calculation

$$\mathcal{F}\{W\} = -\sum_{p=1}^{n-1} \mathcal{F}\left\{\delta^{(p-1)}\right\} A^p = -\sum_{p=1}^{n-1} s^{p-1} A^p = A(sA - I)^{-1}. \qquad \square$$

Lemma 1 has applications to the theory of singular differential equations. (See [10, Chapter 22], [6], [14], and [15].) It is easy to check that W from Lemma 1 satisfies the distributional differential equation

$$A\dot{W} = W + \delta A. \tag{11}$$

In fact, from [10, Chapter 22], W is the *unique* solution of (11) with $\operatorname{supp} W \subset [0, \infty)$. Multiplication of W by a "0^- initial condition" $x(0^-)$ yields the distribution $x = Wx(0^-)$, which solves the problem of "inconsistent" initial conditions in classical differential equations of the form

$$A\dot{x} = x.$$

The results of the present paper serve to place singular differential equations in the context of singular perturbations. However, a detailed discussion of this topic is beyond our scope.

For any $P \in \mathbb{C}^{n \times n}$, define the *inverse spectrum* of P as

$$\sigma^{-1}(P) = \left\{ \frac{1}{\eta_1}, \ldots, \frac{1}{\eta_p} \right\},$$

where $\{\eta_1, \ldots, \eta_p\}$ is the multiset of nonzero eigenvalues of P. Since A_k is nonsingular, we may write

$$\{\lambda_{1k}, \ldots, \lambda_{nk}\} = \sigma^{-1}(A_k)$$

with arbitrary indexing. Since A is nilpotent, $\lambda_{jk} \to \infty$ for every j. It follows from (8), (9), and Lemma 1 that $\exp(A_k^{-1})$ converges weak* iff

$$\int_{\xi_k - i\infty}^{\xi_k + i\infty} \psi(s) A_k (sA_k - I)^{-1} \, ds \to i \int_{-\infty}^{\infty} \psi(iy) A (iyA - I)^{-1} \, dy \tag{12}$$

for every $\psi \in \mathcal{Z}$, where $\xi_k > \max_j \operatorname{Re} \lambda_{jk}$.

The main results of this section will be stated in terms of the exponentially bounded regions

$$E^+(c) = \left\{ s \in \mathbb{C} \;\middle|\; \operatorname{Re} s < 0 \text{ or } |\operatorname{Im} s| > c^{\operatorname{Re} s} \right\},$$

where $c > 1$. Note that $c_2 < c_1$ implies $E^+(c_1) \subset E^+(c_2)$. It will prove useful to parametrize the boundary $\partial E^+(c)$ with the function

$$\pi_c^+(y) = \begin{cases} iy, & |y| \le 1, \\ \dfrac{\ln |y|}{\ln c} + iy, & |y| > 1. \end{cases} \tag{13}$$

We say that the eigenvalues λ_{jk} are *right exponentially bounded* if there exists $c > 1$ and $K < \infty$ such that $\lambda_{jk} \in E^+(c)$ for $k > K$ and all j.

For every $M \in \mathbb{N}$, let

$$U_M = E^+\left(1 + \frac{1}{M}\right) - \overline{D}(0, M),$$

where $D(0, M)$ is the open disk with radius M, centered at 0. For every strictly increasing sequence $\mathcal{M} = \{M_l\} \subset \mathbb{N}$, define the set

$$V_{\mathcal{M}} = \bigcup_l U_{M_l}.$$

The next lemma shows that the $V_{\mathcal{M}}$ act as "neighborhoods of ∞" relative to right exponential boundedness.

LEMMA 2. *$|\lambda_k| \to \infty$ and $\{\lambda_k\}$ is right exponentially bounded iff for every $\mathcal{M}$ there exists $K < \infty$ such that $\lambda_k \in V_{\mathcal{M}}$ for $k > K$.*

Proof (necessary). Let $c > 1$ and K_1 be such that $\lambda_k \in E^+(c)$ for $k > K_1$. Since $\mathcal{M}$ is monotonic, there exists $M_l \in \mathcal{M}$ such that

$$M_l > \frac{1}{c-1}.$$

Then

$$\lambda_k \in E^+\left(1 + \frac{1}{M_l}\right).$$

Choose $K > K_1$ such that $|\lambda_k| > M_l$ for $k > K$. Then $\lambda_k \in U_{M_l} \subset V_{\mathcal{M}}$.

Proof (sufficient). First, suppose $\{\lambda_k\}$ has a bounded subsequence $\{\lambda_{k_j}\}$ and choose any $\mathcal{M}$ with

$$M_1 > \sup_j \left|\lambda_{k_j}\right|.$$

Then $\lambda_{k_j} \notin V_{\mathcal{M}}$ for every j. Hence, $|\lambda_k| \to \infty$. Now suppose $\{\lambda_k\}$ is not right exponentially bounded. Then for every $c > 1$ there exist infinitely many λ_k such that $\lambda_k \notin E^+(c)$. Set $M_0 = 0$ and $k_1 = 1$. Choose $M_1 > |\lambda_1|$ and $k_2 > k_1$ such that

$$\lambda_{k_2} \notin E^+\left(1 + \frac{1}{M_1}\right).$$

Now suppose $M_{j-1} > \max\{|\lambda_{k_{j-1}}|, M_{j-2}\}$ and $k_j > k_{j-1}$ satisfies

$$\lambda_{k_j} \notin E^+\left(1 + \frac{1}{M_{j-1}}\right).$$

Let $M_j > \max\{|\lambda_{k_j}|, M_{j-1}\}$ and $k_{j+1} > k_j$ be such that

$$\lambda_{k_{j+1}} \notin E^+\left(1 + \frac{1}{M_j}\right).$$

By induction, there exist $M_j, k_j \uparrow \infty$ such that $|\lambda_{k_j}| < M_l$ for $l \geq j$ and

$$\lambda_{k_j} \notin E^+\left(1 + \frac{1}{M_l}\right)$$

for $l < j$. Hence, $\lambda_{k_j} \notin U_{M_l}$ for every l, so $\lambda_{k_j} \notin V_{\mathcal{M}}$ for every j. ∎

LEMMA 3. *Let $\phi \in \mathcal{D}$ be such that* $\operatorname{supp} \phi \subset [1, \infty)$, *$\phi(t) \in \mathbb{R}$ for all t, and $\int_{-\infty}^{\infty} \phi(t)dt \geq 1$. For every $\lambda \in \mathbb{C}$, let*

$$\phi_\lambda(t) = e^{-\lambda t}\phi(t).$$

For each $\mathcal{M}$ define

$$\Omega_{\mathcal{M}} = \left\{ \phi_\lambda \,\middle|\, \lambda \notin V_{\mathcal{M}} \right\},$$

$$W_{\mathcal{M}} = \left\{ f \in \mathcal{D}' \,\middle|\, \sup_{\phi \in \Omega_{\mathcal{M}}} |\langle f, \phi \rangle| < 1 \right\}.$$

Then $\Omega_{\mathcal{M}}$ is bounded, $W_{\mathcal{M}}$ is weak open, and $W_{\mathcal{M}} \cap \exp(\mathbb{C}) \subset \exp(V_{\mathcal{M}})$.*

Proof. Write $\lambda = a + ib$. Leibniz's formula yields

$$\begin{aligned} \left|\phi_\lambda^{(j)}(t)\right| &= \left|\sum_{l=0}^{j} \binom{j}{l} \left(\frac{d^l}{dt^l} e^{-\lambda t}\right) \phi^{(j-l)}(t)\right| \\ &\le \sum_{l=0}^{j} \binom{j}{l} |\lambda|^l e^{-at} \left\|\phi^{(j-l)}\right\|_\infty . \end{aligned}$$

Let

$$K_j = \sum_{l=0}^{j} \binom{j}{l} \left\|\phi^{(j-l)}\right\|_\infty$$

and choose $p_j \uparrow \infty$ such that

$$M_{p_j} > \frac{1}{e^{\frac{1}{j}} - 1}.$$

Then

$$\ln\left(1 + \frac{1}{M_{p_j}}\right) < \frac{1}{j}.$$

For $\lambda \notin E^+(1 + \frac{1}{M_{p_j}})$ and $t \ge 1$, we note that $a \ge 0$ and

$$|b| \le \left(1 + \frac{1}{M_{p_j}}\right)^a,$$

so

$$\begin{aligned} |\lambda|^{2j} e^{-2at} &\le |\lambda|^{2j} e^{-2a} \\ &= \left(\left(a^2 + b^2\right) e^{-2\frac{a}{j}}\right)^j \\ &\le \left(a^2 e^{-2\frac{a}{j}} + e^{2a\left(\ln\left(1+\frac{1}{M_{p_j}}\right) - \frac{1}{j}\right)}\right)^j \\ &< \left(e^{-2} j^2 + 1\right)^j . \end{aligned}$$

For $\lambda \in \overline{D}(0, M_{p_j})$,

$$|\lambda|^{2j} e^{-2at} < M_{p_j}^{2j}.$$

Hence, $\lambda \notin V_{\mathcal{M}}$ implies

$$\lambda \notin U_{M_{p_j}} = E^{+}\left(1 + \frac{1}{M_{p_j}}\right) - \overline{D}\left(0, M_{p_j}\right),$$

so

$$|\lambda|^{l} e^{-at} \leq \max\left\{\left(e^{-2}l^2+1\right)^{\frac{l}{2}}, M_{p_l}^{l}\right\} \leq \max\left\{\left(e^{-2}j^2+1\right)^{\frac{j}{2}}, M_{p_j}^{j}\right\}$$

for every $l \leq j$ and $t \geq 1$. Since $\operatorname{supp} \phi^{(j)} \subset [1, \infty)$,

$$\left\|\phi_{\lambda}^{(j)}\right\|_{\infty} \leq K_j \max\left\{\left(e^{-2}j^2+1\right)^{\frac{j}{2}}, M_{p_j}^{j}\right\}.$$

According to [12, p. 30], $\Omega_{\mathcal{M}}$ is a bounded set.

From [12, pp. 42, 56], boundedness of $\Omega_{\mathcal{M}}$ implies that $W_{\mathcal{M}}$ is weak* open. If $\lambda \notin V_{\mathcal{M}}$, then $\phi_{\lambda} \in \Omega_{\mathcal{M}}$ and

$$\sup_{\phi \in \Omega_{\mathcal{M}}} |\langle \exp(\lambda), \phi \rangle| \geq \left|\int_{-\infty}^{\infty} e^{\lambda t} \phi_{\lambda}(t) dt\right| = \left|\int_{-\infty}^{\infty} \phi(t) dt\right| \geq 1,$$

so $\exp(\lambda) \notin W_{\mathcal{M}}$. Hence, $\exp(\lambda) \in W_{\mathcal{M}}$ implies $\lambda \in V_{\mathcal{M}}$, so $\exp(\lambda) \subset \exp(V_{\mathcal{M}})$. ∎

We are now in a position to prove that right exponential boundedness is necessary for weak* convergence, at least in the case $n = 1$.

Theorem 4. *If $|\lambda_k| \to \infty$ and $\exp(\lambda_k)$ converges weak* on $\mathbb{R}$, then λ_k is right exponentially bounded.*

Proof. From Lemma 1, $\exp(\lambda_k) \to 0$. Choose any $\mathcal{M}$. Since $0 \in W_{\mathcal{M}}$, Lemma 3 guarantees there exists $K < \infty$ such that $\exp(\lambda_k) \in \exp(V_{\mathcal{M}})$ for $k > K$. Since exp is $1-1$, $\lambda_k \in V_{\mathcal{M}}$. From Lemma 2, λ_k is right exponentially bounded. ∎

In order to extend Theorem 4 to arbitrary n and prove sufficiency, we need several lemmas. Some of these will be stronger than necessary for subsequent results in this section. However, the full versions will be essential in section 3.

Lemma 5. *Let $\mu : (1, \infty) \to (1, \infty)$ be given by*

$$\mu(c) = \sqrt{1 + \frac{1}{\ln^2 c}}.$$

Then for every $c > 1$, $a \geq 0$, and $y \in \mathbb{R}$,

(i) $\sqrt{y^2 + a^2} \leq |\pi_c^{+}(y) + a| < \mu(c)|y| + a$,

(ii) $\left|\left(\pi_c^{+}\right)'(y)\right| < \mu(c)$,

(iii) $e^{a\left|\operatorname{Re} \pi_c^{+}(y)\right|} \leq 1 + |y|^{\frac{a}{\ln c}}$.

Proof. (i) For $|y| > 1$,

$$\left|\pi_c^{+}(y) + a\right|^2 = \frac{\ln^2|y|}{\ln^2 c} + 2a\frac{\ln|y|}{\ln c} + a^2 + y^2 > y^2 + a^2.$$

Then

$$\begin{aligned}\left|\pi_c^{+}(y) + a\right|^2 &< a^2 + 2\frac{a}{\ln c}|y| + \frac{y^2}{\ln^2 c} + y^2 \\ &< a^2 + 2\mu(c)\, a|y| + \mu^2(c)\, y^2 \\ &= (\mu(c)|y| + a)^2.\end{aligned}$$

For $|y| \leq 1$,

$$\left|\pi_c^+(y) + a\right|^2 = y^2 + a^2 < (\mu(c)|y| + a)^2 .$$

(ii) For $|y| > 1$,

$$\left(\pi_c^+\right)'(y) = \frac{1}{y \ln c} + i,$$

so

$$\left|\left(\pi_c^+\right)'(y)\right|^2 = \frac{1}{y^2 \ln^2 c} + 1 < \frac{1}{\ln^2 c} + 1.$$

For $|y| \leq 1$, $|(\pi_c^+)'(y)| = 1 < \mu(c)$.

(iii) For $|y| > 1$,

$$e^{a\left|\operatorname{Re} \pi_c^+(y)\right|} = e^{a \frac{\ln|y|}{\ln c}} = |y|^{\frac{a}{\ln c}}.$$

For $|y| \leq 1$, $\operatorname{Re} \pi_c^+(y) = 0$, so $e^{a|\operatorname{Re} \pi_c^+(y)|} = 1$. ∎

LEMMA 6. *Let $s, \eta \in \mathbb{C}$ with $|s - \eta| \geq r > 0$. Then*

$$\left|\frac{\eta}{s-\eta}\right| \leq \frac{|s|}{r} + 1.$$

Proof. If $|\eta| \leq |s| + r$,

$$\frac{|\eta|}{|s-\eta|} \leq \frac{|s| + r}{r}.$$

If $|\eta| > |s| + r$,

$$(|s| + r)(|\eta| - |s|) = |s|(|\eta| - |s| - r) + r|\eta| \geq r|\eta| .$$

Since $|\eta| - |s| > 0$,

$$\frac{|\eta|}{|s-\eta|} \leq \frac{|\eta|}{|\eta| - |s|} \leq \frac{|s| + r}{r}. \qquad \square$$

For any sets $\Gamma_1, \Gamma_2 \subset \mathbb{C}$, we invoke the distance functions

$$d(s, \Gamma_1) = \inf\left\{|s - s_1| \,\middle|\, s_1 \in \Gamma_1\right\},$$
$$d(\Gamma_1, \Gamma_2) = \inf\left\{d(s, \Gamma_2) \,\middle|\, s \in \Gamma_1\right\}.$$

LEMMA 7. *Let $B\langle\infty, r\rangle 0$, and $l \geq 0$. For each $\Gamma \subset \mathbb{C}$ define*

$$\mathcal{P}(\Gamma) = \Big\{ (P_1, P_2) \in \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \,\Big|\, \|P_1\| \leq B, \quad \|P_2\| \leq B, \quad d\left(\sigma^{-1}(P_1), \Gamma\right) \geq r, \quad d\left(\sigma^{-1}(P_2), \Gamma\right) \geq r \Big\}.$$

Then there exists $M < \infty$ such that

$$\left\|P_1^l (sP_1 - I)^{-1} - P_2^l (sP_2 - I)^{-1}\right\| \leq M \|P_1 - P_2\| \left(1 + |s|^{4n-1}\right)$$

for every Γ, $s \in \Gamma$, and $(P_1, P_2) \in \mathcal{P}(\Gamma)$.

Proof. Let

$$\left\{\nu_{1j},\ldots,\nu_{n_j j}\right\}=\sigma^{-1}\left(P_j\right),\quad j=1,2,$$
$$N\left(s\right)=\det\left(sP_2-I\right)P_1^l\operatorname{adj}\left(sP_1-I\right)-\det\left(sP_1-I\right)P_2^l\operatorname{adj}\left(sP_2-I\right),$$

and write

$$\begin{aligned}P_1^l\left(sP_1-I\right)^{-1}-P_2^l\left(sP_2-I\right)^{-1}&=\frac{1}{\det\left(sP_1-I\right)\det\left(sP_2-I\right)}N\left(s\right)\\&=\prod_{l=1}^{n_1}\left(\frac{\nu_{l1}}{s-\nu_{l1}}\right)\prod_{l=1}^{n_2}\left(\frac{\nu_{l2}}{s-\nu_{l2}}\right)N\left(s\right).\end{aligned}$$

From Lemma 6,

$$\left\|P_1^l\left(sP_1-I\right)^{-1}-P_2^l\left(sP_2-I\right)^{-1}\right\|\leq\left(\frac{|s|}{r}+1\right)^{2n}\left\|N\left(s\right)\right\|.$$

We will show that there exists $M_1<\infty$ such that

$$\left\|N\left(s\right)\right\|\leq M_1\left\|P_1-P_2\right\|\left(1+|s|^{2n-1}\right)$$

for all

$$\left(P_1,P_2\right)\in\mathcal{B}=\left\{\left\|P_j\right\|\leq B;\quad j=1,2\right\}$$

and all $s\in\mathbb{C}$. Then the result follows with M dependent on B, r, and l, but not Γ.

Each entry of N has the form

$$n_{ij}\left(s\right)=\sum_{l=0}^{2n-1}n_{ijl}s^l,$$

where the coefficients n_{ijl} are polynomial functions in the entries of P_1 and P_2. Since $N\equiv 0$ for $P_1=P_2$, $n_{ijl}\left(P_1,P_1\right)=0$ for all P_1. Setting $\Delta=P_1-P_2$, we write

$$n_{ijl}\left(P_1,P_2\right)=n_{ijl}\left(P_1,P_1-\Delta\right)=\sum_m\alpha_{ijlm}\left(P_1\right)\beta_{ijlm}\left(\Delta\right),$$

where each α_{ijlm} is a polynomial in P_1 and β_{ijlm} is a nonconstant monomial in Δ. Since $\mathcal{B}$ is bounded, each α_{ijlm} is bounded. Suppose δ is an entry of Δ that appears in β_{ijlm}. Then we may write $\beta_{ijlm}\left(\Delta\right)=\gamma\left(\Delta\right)\delta$ for some polynomial function γ. But γ is also bounded, so by elementary arguments there exist $M_1,\ldots,M_5<\infty$ such that for every i, j, l, and m

$$\begin{aligned}\left|\beta_{ijlm}(\Delta)\right|&\leq M_5\left|\delta\right|\leq M_4\left\|\Delta\right\|,\\\left|n_{ijl}\left(P_1,P_2\right)\right|&\leq M_3\left\|\Delta\right\|,\\\left|n_{ij}(s)\right|&\leq M_2\left\|\Delta\right\|\left(1+|s|^{2n-1}\right),\\\left\|N(s)\right\|&\leq M_1\left\|\Delta\right\|\left(1+|s|^{2n-1}\right).\qquad\square\end{aligned}$$

LEMMA 8. *If $c>1$, $\sigma^{-1}\left(P\right)\subset E^+\left(c\right)$, and $\psi\in\mathcal{Z}$, then*

$$i\int_{-\infty}^{\infty}\psi\left(iy\right)P\left(iyP-I\right)^{-1}dy=\int_{\partial E^+(c)}\psi(s)P\left(sP-I\right)^{-1}ds.\tag{14}$$

Proof. Since $E^+(c)$ is open, $\sigma^{-1}(P) \cap \partial E^+(c) = \phi$. Furthermore, $\sigma^{-1}(P)$ is finite and $\partial E^+(c)$ is closed, so $\varepsilon = d(\sigma^{-1}(P), \partial E^+(c)) > 0$. Setting $l = 1$, $\Gamma = \partial E^+(c)$, $P_1 = P$, and $P_2 = 0$ in Lemma 7 yields

$$\left\| P\left(sP - I\right)^{-1} \right\| \leq M_1 \left\| P \right\| \left(1 + |s|^{4n-1}\right)$$

for some M_1. For $x \in [0, \frac{\ln m}{\ln c}]$ and $m \in \mathbb{N}$, there exists M_2 such that

$$|x \pm im| \leq \left| \frac{\ln m}{\ln c} \pm im \right| < M_2 m,$$

so

$$\left\| P\left(\left(x \pm im\right)P - I\right)^{-1} \right\| < M_3 \left(1 + m^{4n-1}\right)$$

for some M_3. Setting $a = 0$ in Lemma 5, part (i), there exists M_4 such that

$$\left\| P\left(\pi_c^+(y)P - I\right)^{-1} \right\| < M_4 \left(1 + |y|^{4n-1}\right)$$

for every $y \in \mathbb{R}$. Let

$$\begin{aligned} F(y) &= \left(\pi_c^+\right)'(y)\psi\left(\pi_c^+(y)\right)P\left(\pi_c^+(y)P - I\right)^{-1}, \\ G(y) &= i\psi(iy)P(iyP - I)^{-1}. \end{aligned}$$

Then from Lemma 5 and (6), there exist $\alpha, b_0, b_1, \ldots$ such that

$$\begin{aligned} |F(y)| &< \mu(c) \frac{b_p\left(1 + |y|^{\frac{\alpha}{\ln c}}\right)}{1 + |y|^p} M_4 \left(1 + |y|^{4n-1}\right), \\ |G(y)| &< \frac{b_p}{1 + |y|^p} M_1 \left(1 + |y|^{4n-1}\right) \end{aligned}$$

for every p. Let $p > \frac{\alpha}{\ln c} + 4n$. Then $F, G \in L^1$, so both integrals in (14) exist. Let

$$I_m = \int_{-m}^{m} G(y)dy$$

and consider the curves

$$\begin{aligned} \gamma_{1,3}(x) &= x \mp im, \quad x \in \left[0, \frac{\ln m}{\ln c}\right], \\ \gamma_2(y) &= \pi_c^+(y), \quad |y| \leq m, \\ \gamma_4(y) &= y, \quad |y| \leq m. \end{aligned}$$

Since

$$\gamma_1\left(\frac{\ln m}{\ln c}\right) = \gamma_2(-m), \quad \gamma_2(m) = \gamma_3\left(\frac{\ln m}{\ln c}\right), \quad \gamma_3(0) = \gamma_4(m), \quad \gamma_4(-m) = \gamma_1(0),$$

we may combine the 4 segments into a single (counterclockwise) closed curve γ. Applying Cauchy's theorem around γ and canceling along γ_4, we obtain

$$\begin{aligned} I_m &= \int_{-m}^{m} G(y)dy + \int_\gamma \psi(s)P(sP - I)^{-1}ds \\ &= \int_0^{\frac{\ln m}{\ln c}} \psi(x - im)P((x - im)P - I)^{-1}dx \\ &\quad + \int_{-m}^{m} F(y)dy - \int_0^{\frac{\ln m}{\ln c}} \psi(x + im)P((x + im)P - I)^{-1}dx. \end{aligned}$$

But

$$\left|\int_0^{\frac{\ln m}{\ln c}} \psi(x \pm im)P((x \pm im)P - I)^{-1}dx\right|$$
$$< b_p \frac{m^{\frac{\alpha}{\ln c}}}{1 + (x^2 + m^2)^{\frac{p}{2}}} M_3 \left(1 + m^{4n-1}\right) \frac{\ln m}{\ln c} \to 0$$

as $m \to \infty$. Hence,

$$\begin{aligned} i\int_{-\infty}^{\infty} \psi(iy)P(iyP - I)^{-1}dy &= \int_{-\infty}^{\infty} G(y)dy = \lim_{m\to\infty} I_m \\ &= \int_{-\infty}^{\infty} F(y)dy = \int_{\partial E^+(c)} \psi(s)P(sP - I)^{-1}ds. \qquad \square \end{aligned}$$

Lemma 9. *Let $1 < c_2 < c_1$ and $a \geq 0$. Then*

$$\inf_{|u|\leq y} \inf_v \left|\frac{\ln y}{\ln c_2} + iu - \pi_{c_1}^+(v) - a\right| \to \infty$$

as $y \to \infty$.

Proof. From conjugate symmetry of $\pi_{c_1}^+$, it suffices to prove

$$\inf_{0\leq u\leq y} \inf_v \left|\frac{\ln y}{\ln c_2} + iu - \pi_{c_1}^+(v) - a\right| \to \infty.$$

Consider

$$f_y(u, v) = \left|\frac{\ln y}{\ln c_2} + iu - \pi_{c_1}^+(v) - a\right|^2.$$

For $|v| \leq 1$,

$$f_y(u, v) = \left(\frac{\ln y}{\ln c_2} - a\right)^2 + (u - v)^2 \geq \left(\frac{\ln y}{\ln c_2} - a\right)^2,$$

$$\inf_{0\leq u\leq y} \inf_{|v|\leq 1} f_y(u, v) \to \infty$$

as $y \to \infty$. It remains to show that

$$\inf_{0\leq u\leq y} \inf_{|v|>1} f_y(u, v) \to \infty.$$

For $|v| > 1$,

$$f_y(u, v) = \left(\frac{\ln y}{\ln c_2} - \frac{\ln|v|}{\ln c_1} - a\right)^2 + (u - v)^2.$$

If $v < -1$,

$$f_y(u, -v) = \left(\frac{\ln y}{\ln c_2} - \frac{\ln|v|}{\ln c_1} - a\right)^2 + (u + v)^2 \leq f_y(u, v),$$

so we need only consider $v > 1$ when taking the infimum. In this case, we note that $f_y(u,\cdot)$ is C^1 for every u, so the infimum with respect to v is achieved at either $v = 1$, $v = \infty$, or $v = v_0$, where

$$\frac{\partial f_y}{\partial v}(u, v_0) = 0.$$

But

$$\begin{aligned} f_y(u,1) &= \left(\frac{\ln y}{\ln c_2} - a\right)^2 + (u-1)^2 \to \infty, \\ \lim_{v\to\infty} f_y(u,v) &= \infty \to \infty, \end{aligned}$$

so the infimum is achieved at some critical point v_0 of $f_y(u,\cdot)$.

Differentiation yields

$$\frac{\partial f_y}{\partial v}(u, v_0) = -\frac{2}{v_0 \ln c_1}\left(\frac{\ln y}{\ln c_2} - \frac{\ln v_0}{\ln c_1} - a\right) - 2(u - v) = 0.$$

Suppose there exist $M < \infty$ and sequences $y_k \to \infty$, $u_k \in [0, y_k]$, and $v_{0k} < M$ such that

$$\frac{\partial f_{y_k}}{\partial v}(u_k, v_{0k}) = 0$$

for every k. For

$$y > c_2^a M^{\frac{\ln c_2}{\ln c_1}},$$

we obtain

$$\begin{aligned} \frac{\ln y}{\ln c_2} &> \frac{\ln M}{\ln c_1} + a, \\ \frac{\partial f_{y_k}}{\partial v}(u_k, v_{0k}) &< -\frac{2}{M \ln c_1}\left(\frac{\ln y_k}{\ln c_2} - \frac{\ln M}{\ln c_1} - a\right) + 2M \to -\infty, \end{aligned}$$

which is a contradiction. Hence, $v_0 \to \infty$ as $y \to \infty$, independent of $u \in [0, y]$.

We complete the proof by showing that

$$\inf_{0\le u\le y} f_y(u,v) \to \infty$$

as $y, v \to \infty$. Suppose there exist sequences $y_k, v_k \to \infty$ and $u_k \in [0, y_k]$ and $M < \infty$ such that $f_{y_k}(u_k, v_k) < M^2$. Then $|u_k - v_k| < M$ and

$$\begin{aligned} \frac{\ln y_k}{\ln c_2} - \frac{\ln v_k}{\ln c_1} - a &> \frac{\ln y_k}{\ln c_2} - \frac{\ln(u_k + M)}{\ln c_1} - a \\ &= \left(\frac{1}{\ln c_2} - \frac{1}{\ln c_1}\right)\ln y_k + \frac{1}{\ln c_1}\ln\left(\frac{y_k}{u_k + M}\right) \\ &\ge \left(\frac{1}{\ln c_2} - \frac{1}{\ln c_1}\right)\ln y_k + \frac{1}{\ln c_1}\ln\left(\frac{y_k}{y_k + M}\right) \\ &\to \infty, \end{aligned}$$

which is a contradiction. $\square$

LEMMA 10. *Let* $1 < c_2 < c_1$ *and* $r > 1$. *Then*

$$d\left(E^{+}\left(c_1\right) - D\left(0, r\right), \partial E^{+}\left(c_2\right)\right) > 0.$$

Proof. For each $R > r$, there exists a unique $y_R > 1$ such that

$$\frac{\ln^2 y_R}{\ln^2 c_2} + y_R^2 = \left|\pi_{c_2}^{+}\left(y_R\right)\right|^2 = R^2.$$

Obviously, $y_R \to \infty$ as $R \to \infty$. Consider the decomposition

$$\partial E^{+}\left(c_2\right) = \left(\partial E^{+}\left(c_2\right) \cap D\left(0, R\right)\right) \cup \left(\partial E^{+}\left(c_2\right) - D\left(0, R\right)\right).$$

Since the closure of $E^{+}\left(c_1\right) - D\left(0, r\right)$ and $\partial E^{+}\left(c_2\right)$ are disjoint,

$$f\left(R\right) = d\left(E^{+}\left(c_1\right) - D\left(0, r\right), \partial E^{+}\left(c_2\right) \cap D\left(0, R\right)\right) > 0.$$

But

$$\partial\left(E^{+}\left(c_1\right) - D\left(0, r\right)\right) \subset \partial D\left(0, r\right) \cup \partial E^{+}\left(c_1\right),$$

so setting $a = 0$ in Lemma 9 yields

$$\begin{aligned} g\left(R\right) &= d\left(E^{+}\left(c_1\right) - D\left(0, r\right), \partial E^{+}\left(c_2\right) - D\left(0, R\right)\right) \\ &\geq \min\left\{d\left(\partial D\left(0, r\right), \partial E^{+}\left(c_2\right) - D\left(0, R\right)\right), d\left(\partial E^{+}\left(c_1\right), \partial E^{+}\left(c_2\right) - D\left(0, R\right)\right)\right\} \\ &= \min\left\{R - r, \inf_{y \geq y_R} \inf_{v} \left|\pi_{c_2}^{+}(y) - \pi_{c_1}^{+}\left(v\right)\right|\right\} \\ &\geq \min\left\{R - r, \inf_{y \geq y_R} \inf_{|u| \leq y} \inf_{v} \left|\frac{\ln y}{\ln c_2} + iu - \pi_{c_1}^{+}\left(v\right)\right|\right\} \\ &\to \infty \end{aligned}$$

as $R \to \infty$. Hence, there exists R_1 such that $g\left(R_1\right) > 0$, from which we obtain

$$d\left(E^{+}\left(c_1\right) - D\left(0, r\right), \partial E^{+}\left(c_2\right)\right) = \min\left\{f\left(R_1\right), g\left(R_1\right)\right\} > 0. \qquad \square$$

We may now prove the main result of this section.

THEOREM 11. $\exp(A_k^{-1})$ *converges weak* on* $\mathbb{R}$ *iff the* λ_{jk} *are right exponentially bounded.*

Proof (necessary). For $n = 1$, Theorem 4 applies directly. Suppose $n > 1$ choose any $\phi \in \mathcal{D}$; and set

$$N_k = \int_0^\infty \phi(t) e^{tA_k^{-1}} dt, \quad N = \sum_{p=1}^{n-1} (-1)^p \phi^{(p-1)}\left(0\right) A^p.$$

The eigenvalues of N_k are

$$\rho_{jk} = \int_0^\infty \phi(t) e^{\lambda_{jk} t} dt.$$

Since A is nilpotent, so is N. Lemma 1 implies

$$N_k \to \left\langle -\sum_{p=1}^{n-1} \delta^{(p-1)} A^p, \phi \right\rangle = N, \tag{15}$$

so $\rho_{jk} \to 0$ for every j. Theorem 4 guarantees the existence of $c_j > 1$ and $K_j < \infty$ such that $\lambda_{jk} \in E^+(c_j)$ for $k > K_j$. Let $c = \min\{c_j\}$ and $K = \max\{K_j\}$. Then $\lambda_{jk} \in E^+(c)$ for $k > K$ and all j.

Proof (sufficient). It suffices to proves (12) for every $\psi \in \mathcal{Z}$ and some $\xi_k > \operatorname{Re}\lambda_{jk}$. Suppose $\lambda_{jk} \in E^+(c)$ for large k, and let $c_1 \in (0, c)$ and

$$F_k(y) = \left(\pi_{c_1}^+\right)'(y)\,\psi\left(\pi_{c_1}^+(y)\right)\left(A_k\left(\pi_{c_1}^+(y)A_k - I\right)^{-1} - A\left(\pi_{c_1}^+(y)A - I\right)^{-1}\right).$$

From Lemma 8,

$$\begin{aligned}\int_{\xi_k - i\infty}^{\xi_k + i\infty} &\psi(s)\left(A_k\left(sA_k - I\right)^{-1} - A\left(sA - I\right)^{-1}\right) ds \\ &= \int_{\partial E^+(c_1)} \psi(s)\left(A_k\left(sA_k - I\right)^{-1} - A\left(sA - I\right)^{-1}\right) ds \\ &= \int_{-\infty}^{\infty} F_k(y)dy.\end{aligned}$$

From Lemma 7, there exist $\varepsilon_k \to 0$ such that

$$\left\|A_k\left(sA_k - I\right)^{-1} - A\left(sA - I\right)^{-1}\right\| = \varepsilon_k\left(1 + |s|^{4n-1}\right).$$

From (6) and Lemmas 5 and 10, $|F_k(y)| < \varepsilon_k G_p(y)$, where

$$\begin{aligned}G_p(y) &= \mu(c_1)b_p \frac{e^{a\left|\operatorname{Re}\pi_{c_1}^+(y)\right|}}{1 + \left|\pi_{c_1}^+(y)\right|^p}\left(1 + \left|\pi_{c_1}^+(y)\right|^{4n-1}\right) \\ &< \mu(c_1)b_p \frac{1 + |y|^{\frac{a}{\ln c_1}}}{1 + |y|^p}\left(1 + \mu^{4n-1}(c_1)|y|^{4n-1}\right)\end{aligned}$$

for $p = 0, 1, 2, \ldots$. Setting

$$p > \frac{a}{\ln c_1} + 4n,$$

we obtain $G_p \in L^1$, so

$$\left|\int_{-\infty}^{\infty} F_k(y)dy\right| \le \int_{-\infty}^{\infty} |F_k(y)|\,dy < \varepsilon_k \int_{-\infty}^{\infty} G_p(y)dy \to 0. \qquad \square$$

The final result of this section is a simple theorem which further demonstrates the weakness of weak* convergence.

THEOREM 12. *If there exists* $t > 0$ *such that* $e^{tA_k^{-1}}$ *is bounded, then* $\exp(A_k^{-1})$ *converges weak**.

Proof. For any matrix norm, there exists $M_1 < \infty$ such that

$$e^{\operatorname{Re}\lambda_{jk}t} = \left|e^{\lambda_{jk}t}\right| < M_1\left\|e^{tA_k^{-1}}\right\|$$

for every j and k. Hence, there exists $M_2 < \infty$ such that $\operatorname{Re}\lambda_{jk} < M_2$ for every j and k. Choose any $c > 1$. Since $|\lambda_{jk}| \to \infty$, $\lambda_{jk} \in E^+(c)$ for large k, and the result follows from Theorem 11. $\square$

Example 13. The converse of Theorem 12 is false. Let $n = 1$, $c < e$, and

$$A_k = \frac{1}{k + ie^k}.$$

Then $\lambda_{1k} = k + ie^k \in E^+(c)$, so Lemma 1 and Theorem 11 indicate $\exp(A_k^{-1}) \to 0$ weak*. But $\|e^{tA_k^{-1}}\| = e^{kt} \to \infty$ for every $t > 0$.

3. Pointwise and almost uniform convergence. The main result of this section, Theorem 22, addresses almost uniform convergence of $\exp(A_k^{-1})$. However, certain statements can be made concerning mere pointwise convergence. Throughout the analysis, we maintain our assumptions that each $A_k \in \mathbb{C}^{n\times n}$ is nonsingular, A is nilpotent, and $A_k \to A$.

LEMMA 14. *If* $0 \le a < b$, $|\lambda_k| \to \infty$, *and* $\exp(\lambda_k)$ *converges a.e. (Lebesgue) on* (a,b), *then* $\operatorname{Re}\lambda_k \to -\infty$.

Proof. It suffices to prove the result for $b < \infty$. Suppose $e^{\lambda_k t} \to p(t)$ a.e. on (a,b), and let $\alpha_k + i\beta_k = \lambda_k$. Then there exists $t > 0$ such that $e^{\alpha_k t} = |e^{\lambda_k t}| \to |p(t)|$. Hence, either α_k converges to a finite value α or $\alpha_k \to -\infty$. If $\alpha_k \to \alpha$, then $e^{i\beta_k t} = e^{-\alpha_k t}e^{\lambda_k t} \to e^{-\alpha t}p(t)$. But $|e^{i\beta_k t}| = 1$, so $e^{-\alpha t}|p(t)| = 1$ a.e. on (a,b). Let $\delta > 0$. Denoting the Lebesgue measure by m, Egorov's theorem guarantees that there exists $\Omega \subset (a,b)$ such that $m\Omega > b - a - \delta$ and $e^{i\beta_k t} \to e^{-\alpha t}p(t)$ uniformly on Ω. Let $\phi \in \mathcal{D}$ with $\operatorname{supp}\phi \subset (a,b)$, and set $M = \max_t |\phi(t)|$. Then

$$\begin{aligned}
\left|\int_a^b \left(e^{i\beta_k t} - e^{-\alpha t}p(t)\right)\phi(t)dt\right| &\le \left|\int_\Omega \left(e^{i\beta_k t} - e^{-\alpha t}p(t)\right)\phi(t)dt\right| \\
&\quad + \left|\int_{(a,b)-\Omega} \left(e^{i\beta_k t} - e^{-\alpha t}p(t)\right)\phi(t)dt\right| \\
&\le \int_\Omega \left|e^{i\beta_k t} - e^{-\alpha t}p(t)\right| |\phi(t)|\, dt \\
&\quad + \int_{(a,b)-\Omega} \left(\left|e^{i\beta_k t}\right| + e^{-\alpha t}\,|p(t)|\right)|\phi(t)|\, dt \\
&< M\left(\int_\Omega \left|e^{i\beta_k t} - e^{-\alpha t}p\,(t)\right| dt + 2\delta\right) \\
&\to 2M\delta.
\end{aligned}$$

Hence $e^{i\beta_k t} \to e^{-\alpha t}p(t)$ weak* on (a,b). But Lemma 1 indicates $p(t) = 0$ a.e. on (a,b), which contradicts $e^{-\alpha t}|p(t)| = 1$ a.e. ∎

THEOREM 15. *If* $\exp(A_k^{-1})$ *converges pointwise on* $(0,t_0)$ *for some* $t_0 > 0$, *then* $\operatorname{Re}\lambda_{jk} \to -\infty$ *for every* j *and* $\exp(A_k^{-1}) \to 0$ *pointwise on* $(0,\infty)$.

Proof. Suppose $e^{tA_k^{-1}} \to L(t)$ on $(0,t_0)$. Then the eigenvalues $\eta_1(t),\ldots,\eta_n(t)$ of $L(t)$ may be indexed such that $e^{\lambda_{jk}t} \to \eta_j(t)$ on $(0,t_0)$. Since each map $t \longmapsto e^{\lambda_{jk}t}$ is measurable, so is η_j. By Lemma 14, $\operatorname{Re}\lambda_{jk} \to -\infty$. Hence, $\eta_j(t) = 0$ and $L(t)$ is nilpotent on $(0,t_0)$. For any $t > 0$, there exists $q \ge n$ such that $\frac{t}{q} < t_0$, so

$$e^{tA_k^{-1}} = \left(e^{\frac{t}{q}A_k^{-1}}\right)^q \to L^q(t) = 0. \qquad \square$$

Next, we present some counterexamples to possible converses of Theorem 15.

Example 16. The converse of Theorem 15 obviously holds. However, the condition $\operatorname{Re}\lambda_{jk} \to -\infty$ alone is not sufficient for pointwise convergence, even for one $t > 0$. Let $n = 2$, $\lambda_k = -k + ie^{k^2}$,

$$\begin{aligned}
\alpha_k &= \frac{e^{k^2}}{\lambda_k^2}, \\
A_k &= \begin{bmatrix} \frac{1}{\lambda_k} & \alpha_k \\ 0 & \frac{1}{\lambda_k} \end{bmatrix}.
\end{aligned}$$

Then A_k is nonsingular, $A_k \to 0$, and

$$e^{tA_k^{-1}} = e^{\lambda_k t} \begin{bmatrix} 1 & -\alpha_k \lambda_k^2 t \\ 0 & 1 \end{bmatrix}.$$

For an appropriate matrix norm,

$$\left\| e^{tA_k^{-1}} \right\| \geq \left| \alpha_k \lambda_k^2 t e^{\lambda_k t} \right| = e^{k^2} t e^{-kt} \to \infty$$

for every $t > 0$. It is worth noting that λ_{1k} and λ_{2k} are right exponentially bounded, so Theorem 11 does guarantee that $\exp\left(A_k^{-1}\right)$ converges weak*.

Example 17. A more refined example is obtained by choosing $t_0 > 0$ and substituting

$$\alpha_k = \frac{e^{kt_0}}{\lambda_k^2}$$

in Example 16. Then

$$\left| e^{\lambda_k t} \right| = e^{-kt},$$
$$\left| \alpha_k \lambda_k^2 t e^{\lambda_k t} \right| = t e^{-k(t-t_0)},$$

so $\exp(A_k^{-1})$ converges uniformly on $[t_0 + \varepsilon, \infty)$ for every $\varepsilon > 0$, but

$$\left\| e^{tA_k^{-1}} \right\| \to \infty$$

for $t \in (0, t_0)$.

Example 18. By doubling the dimension, we can achieve the same properties as in Examples 16 and 17, but with real matrices. Let

$$\widehat{A}_k = \begin{bmatrix} \operatorname{Re} A_k & \operatorname{Im} A_k \\ -\operatorname{Im} A_k & \operatorname{Re} A_k \end{bmatrix} \in \mathbb{R}^{2n \times 2n}.$$

Note that $\widehat{A}_k$ may be block-diagonalized by applying the similarity

$$T = \begin{bmatrix} I & I \\ iI & -iI \end{bmatrix}$$

to yield

$$\widehat{A}_k = T \begin{bmatrix} A_k & 0 \\ 0 & A_k^* \end{bmatrix} T^{-1}.$$

Then

$$\exp\left(\widehat{A}_k^{-1}\right) = T \begin{bmatrix} \exp\left(A_k^{-1}\right) & 0 \\ 0 & \exp\left(A_k^{-1}\right)^* \end{bmatrix} T^{-1},$$

so convergence of $\exp(\widehat{A}_k^{-1})$ is equivalent to convergence of $\exp(A_k^{-1})$ relative to any topological vector space of functions or distributions.

In view of Examples 16, 17, and 18, we need to assume more than $\operatorname{Re} \lambda_{jk} \to -\infty$ to guarantee convergence of $\exp(A_k^{-1})$ in any classical sense. Similar to the approach of section 2, many of our results will be stated in terms of the exponentially bounded region

$$E^{-}(c)=\left\{s\in\mathbb{C}\;\middle|\;\operatorname{Re}s<0\text{ and }|\operatorname{Im}s|<c^{-\operatorname{Re}s}\right\},$$

where $c>1$. Note that $c_2<c_1$ implies $E^{-}(c_1)\supset E^{-}(c_2)$. The boundary $\partial E^{-}(c)$ of $E^{-}(c)$ is parametrized by

$$\pi_c^{-}(y)=-\pi_c^{+*}(y)\begin{cases} iy, & |y|\le 1,\\ -\frac{\ln|y|}{\ln c}+iy, & |y|>1.\end{cases}\tag{16}$$

We say that the λ_{jk} are *left exponentially bounded* if for every $c>1$ there exists $K<\infty$ such that $\lambda_{jk}\in E^{-}(c)$ for $k>K$ and all j.

We need several lemmas.

LEMMA 19. *Let μ be as in Lemma 5. Then for every $c>1$, $a\ge 0$, $y\in\mathbb{R}$, and $t>0$,*

(i) $\sqrt{y^2+a^2}\le\left|\pi_c^{-}(y)-a\right|<\mu(c)\,|y|+a,$

(ii) $\left|\left(\pi_c^{-}\right)'(y)\right|<\mu(c),$

(iii) $e^{\operatorname{Re}\pi_c^{-}(y)t}\le\min\left\{1,|y|^{-\frac{t}{\ln c}}\right\}.$

Proof. Since

$$\left|\pi_c^{-}(y)-a\right|=\left|-\pi_c^{+*}(y)-a\right|=\left|\pi_c^{+}(y)+a\right|,$$
$$\left|\left(\pi_c^{-}\right)'(y)\right|=\left|\left(-\pi_c^{+*}\right)'(y)\right|=\left|\left(\pi_c^{+}\right)'(y)\right|,$$

(i) and (ii) follow from Lemma 5, parts (i) and (ii). To prove (iii) of Lemma 19, note that

$$e^{\operatorname{Re}\pi_c^{-}(y)t}=\begin{cases}|y|^{-\frac{t}{\ln c}}, & |y|>1,\\ 1, & |y|\le 1.\end{cases}\qquad\square$$

LEMMA 20. *Let $a\ge 0$, $p\ge 0$, $t_0>0$, and*

$$c_{\max}=\begin{cases}e^{\frac{t_0}{p}}, & p>0,\\ \infty, & p=0.\end{cases}$$

If $1<c<c_{\max}$ and $P\in\mathbb{C}^{n\times n}$ satisfies $\sigma^{-1}(P)\subset -a+E^{-}(c)$, then

$$i^{p+1}\frac{d^p}{dt^p}\lim_{m\to\infty}\int_{-m}^{m}e^{iyt}P^n\left(iyP-I\right)^{-1}dy=\int_{-a+\partial E^{-}(c)}s^pe^{st}P^n\left(sP-I\right)^{-1}ds\tag{17}$$

for $t>t_0$.

Proof. We first establish the result for $p=0$. Let T be nonsingular and such that

$$T^{-1}PT=\begin{bmatrix}C & 0\\ 0 & N\end{bmatrix},$$

where C is nonsingular and N is nilpotent. Then $C^n(sC-I)$ is strictly proper and $N^n(sN-I)^{-1}\equiv 0$, so $P^n(sP-I)^{-1}$ is strictly proper. Furthermore, $\sigma^{-1}(P)\subset -a+E^{-}(c)$ implies the existence of $M<\infty$ such that

$$\left\|P^n\left(sP-I\right)^{-1}\right\|<\frac{M}{1+|s|}$$

for $s \notin -a + E^{-}(c)$. Hence,

$$\left\| P^{n} \left(iyP - I \right)^{-1} \right\| < \frac{M}{1 + |y|},$$
$$\left\| P^{n} \left(\left(x \pm im \right) P - I \right)^{-1} \right\| < \frac{M}{1 + |x \pm im|} < \frac{M}{1 + m}$$

for all $x \geq -\frac{\ln |y|}{\ln c} - a$ and $m \in \mathbb{N}$. Also, from Lemma 19, part (i),

$$\left\| P^{n} \left(\left(\pi_{c}^{-}(y) - a \right) P - I \right)^{-1} \right\| < \frac{M}{1 + \left| \pi_{c}^{-}(y) - a \right|} \leq \frac{M}{1 + \sqrt{y^{2} + a^{2}}}.$$

Define

$$F(y,t) = \left(\pi_{c}^{-} \right)'(y) e^{\left(\pi_{c}^{-}(y) - a \right) t} P^{n} \left(\left(\pi_{c}^{-}(y) - a \right) P - I \right)^{-1}$$

on $\mathbb{R} \times (0, \infty)$. From Lemma 19, parts (ii) and (iii),

$$\|F(y,t)\| < \mu(c) e^{-at} \min \left\{ 1, |y|^{-\frac{t}{\ln c}} \right\} \frac{M}{1 + \sqrt{y^{2} + a^{2}}}. \tag{18}$$

Thus $F(\cdot, t) \in L^{1}$ for every $t > t_{0}$, so the second integral in (17) exists. Let

$$I_{m} = i \int_{-m}^{m} e^{iyt} P^{n} \left(iyP - I \right)^{-1} dy.$$

Applying Cauchy's theorem, we obtain

$$\begin{aligned} I_{m} = &- \int_{-\frac{\ln m}{\ln c} - a}^{0} e^{(x - im)t} P^{n} \left(\left(x - im \right) P - I \right)^{-1} dx + \int_{-m}^{m} F(y,t) dy \\ &+ \int_{-\frac{\ln m}{\ln c} - a}^{0} e^{(x + im)t} P^{n} \left(\left(x + im \right) P - I \right)^{-1} dx. \end{aligned}$$

But

$$\begin{aligned} \left\| \int_{-\frac{\ln m}{\ln c} - a}^{0} e^{(x \pm im)t} P^{n} \left(\left(x \pm im \right) P - I \right)^{-1} dx \right\| &< \frac{M}{1 + m} \int_{-\frac{\ln m}{\ln c} - a}^{0} e^{xt} dx \\ &< \frac{M}{1 + m} \int_{-\infty}^{0} e^{xt} dx \\ &= \frac{M}{(1 + m) t} \\ &\to 0 \end{aligned}$$

as $m \to \infty$, so

$$I_{m} \to \int_{-\infty}^{\infty} F(y,t) \, dy = \int_{-a + \partial E^{+}(c)} e^{st} P^{n} \left(sP - I \right)^{-1} ds.$$

For $p > 0$, we note that

$$\frac{\partial^{p} F}{\partial t^{p}}(y,t) = \left(\pi_{c}^{-}(y) - a \right)^{p} F(y,t).$$

Lemma 19, part (i), and (18) yield

$$\left\| \frac{\partial^q F}{\partial t^q}(y,t) \right\| < (\mu(c)|y|+a)^q \, \mu(c) e^{-at_0} \min\left\{1, |y|^{-\frac{t_0}{\ln c}}\right\} \frac{M}{1+\sqrt{y^2+a^2}} \tag{19}$$

for $q = 1, \ldots, p$. Since $\frac{t_0}{\ln c} > p \geq q$, the right side of (19) is L^1. From [9, Theorem 10.39],

$$\begin{aligned} \frac{d^p}{dt^p} \int_{-\infty}^{\infty} F(y,t)dy &= \int_{-\infty}^{\infty} \frac{\partial^p F}{\partial t^p}(y,t)dy = \int_{-\infty}^{\infty} \left(\pi_c^-(y) - a\right)^p F(y,t)dy \\ &= \int_{-a+\partial E^-(c)} s^p e^{st} P^n (sP - I)^{-1} ds. \qquad \square \end{aligned}$$

Lemma 21. *Let $1 < c_2 < c_1$ and $a \geq 0$. Then*

$$d\left(E^-(c_2) - D(0,R), -a + \partial E^-(c_1)\right) \to \infty$$

as $R \to \infty$.

Proof. For each $R > 1$ there exists a unique $y_R > 1$ such that

$$\frac{\ln^2 y_R}{\ln^2 c_2} + y_R^2 = \left|\pi_{c_2}^-(y_R)\right|^2 = R^2.$$

Obviously, $y_R \to \infty$ as $R \to \infty$. Consider the half-plane

$$H_R = \left\{ \operatorname{Re} s < -\frac{\ln y_R}{\ln c_2} \right\}$$

and the line segments

$$\Lambda(y) = \left\{ -\frac{\ln y}{\ln c_2} + iu \,\middle|\, |u| < y \right\}.$$

We may write

$$E^-(c_2) - D(0,R) \subset E^-(c_2) \cap H_R = \bigcup_{y > y_R} \Lambda(y).$$

Thus Lemma 9 yields

$$\begin{aligned} d\left(E^-(c_2) - D(0,R), -a + \partial E^-(c_1)\right) &\geq d\left(E^-(c_2) \cap H_R, -a + \partial E^-(c_1)\right) \\ &= \inf_{y > y_R} d\left(\Lambda(y), -a + \partial E^+(c_1)\right) \\ &= \inf_{y > y_R} \inf_{|u| \leq y} \inf_v \left| \frac{\ln y}{\ln c_2} - iu + \pi_{c_1}^-(v) - a \right| \\ &= \inf_{y > y_R} \inf_{|u| \leq y} \inf_v \left| \frac{\ln y}{\ln c_2} + iu - \pi_{c_1}^+(v) - a \right| \\ &\to \infty \end{aligned}$$

as $R \to \infty$. $\square$

Now we can prove the main result of this section.

THEOREM 22. *If the λ_{jk} are left exponentially bounded, then for every $M < \infty$, $p \geq 0$, and $t_0 > 0$ there exist $K < \infty$ such that $\|A_k^{-p} e^{tA_k^{-1}}\| < e^{-Mt}$ for every $t > t_0$ and $k > K$.*

Proof. For certain A_k, the Fourier transform $y \longmapsto A_k (iyA_k - I)^{-1}$ may not be L^1. In such cases, the Cauchy principal value may be invoked as in [10, Theorem 24.4] to yield

$$e^{tA_k^{-1}} = \frac{1}{2\pi} \lim_{m\to\infty} \int_{-m}^{m} e^{iyt} A_k (iyA_k - I)^{-1} \, dy.$$

Let $1 < c_2 < c_1 < e^{\frac{t_0}{p+5n-1}}$ and $B = \max_k \{\|A_k\|\}$. From Lemma 21, for every a there exists R_a such that

$$d\left(E^- (c_2) - D(0, R_a), -a + \partial E^- (c_1)\right) > 1.$$

Since the λ_{jk} are left exponentially bounded, there exists $K_a < \infty$ such that

$$\sigma^{-1} (A_k) \subset E^- (c_2) - D(0, R_a)$$

for $k > K_a$. Setting $\Gamma = -a + \partial E^- (c_1)$, $l = n$, $P_1 = A_k$, and $P_2 = 0$ in Lemma 7 yields

$$\left\| A_k^n \left(\left(\pi_{c_1}^-(y) - a \right) A_k - I \right)^{-1} \right\| \leq M_1 B \left(1 + \left| \pi_{c_1}^-(y) - a \right|^{4n-1} \right)$$

for $k > K_a$, where M_1 is independent of a. Define

$$g_q(w) = \begin{cases} \binom{p+n-1}{q} w^{p+n-q-1} + \binom{p+5n-2}{q} w^{p+5n-q-2}, & q = 0, \ldots, p+n-1, \\ \binom{p+5n-2}{q} w^{p+5n-q-2}, & q = p+n, \ldots, p+5n-2. \end{cases}$$

Then

$$y \longmapsto g_q(\mu(c_1)|y|) \min\left\{1, |y|^{-\frac{t}{\ln c_1}}\right\}$$

is L^1 for every q and $t > t_0$, so we may define functions

$$\beta_q(t) = \frac{M_1 B}{2\pi} \mu(c_1) \int_{-\infty}^{\infty} g_q(\mu(c_1)|y|) \min\left\{1, |y|^{-\frac{t}{\ln c_1}}\right\} dy.$$

Since

$$|y|^{-\frac{t}{\ln c_1}} < |y|^{-\frac{t_0}{\ln c_1}}$$

for $y \neq 0$ and $t > t_0$, $\beta_q(t) < \beta_q(t_0)$. At this stage, we fix $a > M$ such that

$$e^{-at_0} \sum_{q=0}^{p+5n-2} \beta_q(t_0) a^q < e^{-Mt_0}$$

and define

$$F_k(y,t) = \frac{1}{2\pi i^{p+n}} \left(\pi_{c_1}^-\right)'(y) \left(\pi_{c_1}^-(y) - a\right)^{p+n-1} e^{\left(\pi_{c_1}^-(y)-a\right)t} A_k^n \left(\left(\pi_{c_1}^-(y) - a\right) A_k - I \right)^{-1}.$$

From Lemma 20,

$$\begin{aligned}
A_k^{-p} e^{tA_k^{-1}} &= A_k^{n-1} \frac{d^{p+n-1}}{dt^{p+n-1}} e^{tA_k^{-1}} \\
&= \frac{1}{2\pi} \frac{d^{p+n-1}}{dt^{p+n-1}} \lim_{m\to\infty} \int_{-m}^{m} e^{iyt} A_k^n \left(iyA_k - I\right)^{-1} dy \\
&= \frac{1}{2\pi i^{p+n}} \int_{-a+\partial E^-(c_1)} s^{p+n-1} e^{st} A_k^n \left(sA_k - I\right)^{-1} ds \\
&= \int_{-\infty}^{\infty} F_k\left(y,t\right) dy.
\end{aligned}$$

By Lemma 19 and the binomial theorem,

$$\begin{aligned}
\left\| F_k\left(y,t\right)\right\| <& e^{-\alpha t} \frac{M_1 B}{2\pi} \mu\left(c_1\right) \left(\mu\left(c_1\right)|y| + a\right)^{p+n-1} \min\left\{1, |y|^{-\frac{t}{\ln c_1}}\right\} \\
& \left(1 + \left(\mu\left(c_1\right)|y| + a\right)^{4n-1}\right) \\
=& e^{-\alpha t} \frac{M_1 B}{2\pi} \mu\left(c_1\right) \min\left\{1, |y|^{-\frac{t}{\ln c_1}}\right\} \sum_{q=0}^{p+5n-2} g_q\left(\mu\left(c_1\right)|y|\right) a^q
\end{aligned}$$

for $t > t_0$ and $k > K_a$. Thus

$$\begin{aligned}
\left\| A_k^{-p} e^{tA_k^{-1}} \right\| &\leq \int_{-\infty}^{\infty} \left\| F_k\left(a,y\right)\right\| dy \\
&< e^{-at} \sum_{q=0}^{p+5n-2} \beta_q(t) a^q \\
&< e^{-a(t-t_0)} e^{-at_0} \sum_{q=0}^{p+5n-2} \beta_q\left(t_0\right) a^q \\
&< e^{-M(t-t_0)} e^{-Mt_0} \\
&= e^{-Mt}. \qquad \square
\end{aligned}$$

COROLLARY 23. *If the* λ_{jk} *are left exponentially bounded, then* $\exp(A_k^{-1}) \to 0$ *uniformly on* $[t_0, 0)$ *for every* $t_0 > 0$.

Proof. Let $t_0, \delta > 0$ and $M > -\frac{\ln \delta}{t_0}$. From Theorem 22, there exist $K < \infty$ such that

$$\left\| e^{tA_k^{-1}} \right\| < e^{-Mt} < \delta^{\frac{t}{t_0}} < \delta$$

for $t > t_0$ and $k > K$. $\square$

We note that Theorem 22 is considerably stronger than necessary to establish Corollary 23. However, Theorem 22 has applications in possible sequels to this paper. For example, one might examine the case where A_{fk} in (2) is singular for infinitely many k. Also, a forcing function $u(t)$ might be introduced in (1), requiring convergence of the convolution $A_k^{-1}\exp(A_k^{-1}) * u$. Both these problems involve the expressions $A_k^{-p}\exp(A_k^{-1})$.

Example 24. It is easy to see that the converse of Corollary 23 is false. Let $n = 1$ and

$$A_k = \frac{1}{-k + ie^{k^2}}.$$

Then

$$\left\| e^{tA_k^{-1}} \right\| = e^{-kt} \to 0$$

uniformly on $[t_0, 0)$ for $t_0 > 0$, but λ_k is not left exponentially bounded.

4. Analytic matrices. Our results in sections 3 and 4 will now be applied to the case of real analytic functions $A : (-\delta_1, \delta_1) \to \mathbb{C}^{n\times n}$. The appropriate assumptions for this section are that $A(\varepsilon)$ is nonsingular for small $\varepsilon \neq 0$ and that $A(0)$ is nilpotent. We are interested in the behavior of $\exp(A^{-1}(\varepsilon))$ as $\varepsilon \to 0^+$. Our two main results offer a dramatic improvement over both the weak* and sequential analyses of sections 2 and 3. In particular, pointwise and almost uniform convergence will be shown to be equivalent and completely characterized by the eigenvalues of A.

Setting $\varepsilon = \frac{1}{k}$ in Lemma 1 and Theorem 15 shows that the only possible weak* and a.e. limits of $\exp(A^{-1}(\varepsilon))$ are the same as in sections 2 and 3. Further analysis hinges on the Puiseux series. (See, e.g., [13, section 3.2].) For $\varepsilon > 0$, consider the eigenvalues $\lambda_j(\varepsilon) \in \sigma^{-1}(A(\varepsilon))$. By Puiseux's theorem, the $\lambda_j(\varepsilon)$ may be reindexed for each ε in some $(0, \delta_2)$ such that

$$\lambda_j(\varepsilon) = \varepsilon^{-p_j} f_j\left(\varepsilon^{\frac{1}{q_j}}\right), \quad j = 1, \ldots, n, \tag{20}$$

where $p_j, q_j > 0$ are integers and the $f_j : (-\delta_2, \delta_2) \to \mathbb{C}$ are (complex-valued) real analytic functions. The concept of exponential boundedness may be adapted to the analytic case in the obvious way: The λ_j are *right exponentially bounded* if there exists $c > 1$ and $\delta > 0$ such that $\lambda_j(\varepsilon) \in E^+(c)$ for all j and $\varepsilon \in (0, \delta)$. The λ_j are *left exponentially bounded* if for every $c > 1$ there exists $\delta > 0$ such that $\lambda_j(\varepsilon) \in E^-(c)$ for all j and $\varepsilon \in (0, \delta)$.

LEMMA 25. (i) *The λ_j are right exponentially bounded iff there exist $M < \infty$ and $\delta > 0$ such that* $\operatorname{Re}\lambda_j(\varepsilon) < M$ *for all j and $\varepsilon \in (0, \delta)$.*

(ii) *The λ_j are left exponentially bounded iff* $\operatorname{Re}\lambda_j(\varepsilon) \to -\infty$ *as $\varepsilon \to 0^+$ for every j.*

Proof. (i) Sufficiency is obvious. For necessity, it suffices to prove the result individually for each j, where λ_j has the form (20). We may alternatively write (20) as

$$\lambda_j(\varepsilon) = \varepsilon^{-\frac{l_j}{q_j}} g_j\left(\varepsilon^{\frac{1}{q_j}}\right) + i\varepsilon^{-\frac{m_j}{q_j}} h_j\left(\varepsilon^{\frac{1}{q_j}}\right), \tag{21}$$

where $l_j, m_j \geq 0$ and $q_j > 0$ are integers, g and h are real analytic, and $g_j(0), h_j(0) \neq 0$. If $g_j(0) < 0$, then $\operatorname{Re}\lambda_j(\varepsilon) < 0$ for small ε. If $l_j = 0$, then

$$\operatorname{Re}\lambda_j(\varepsilon) = g_j\left(\varepsilon^{\frac{1}{q_j}}\right) < 1 + g_j(0).$$

It remains to examine the case where $g_j(0), l_j > 0$. Then $\operatorname{Re}\lambda_j(\varepsilon) \to \infty$, so right exponential boundedness implies

$$\left|\operatorname{Im}\lambda_j(\varepsilon)\right| > c^{\operatorname{Re}\lambda_j(\varepsilon)} \tag{22}$$

for small $\varepsilon > 0$. Taking the logarithm of (22) and multiplying by $\varepsilon^{\frac{l_j}{q_j}}$ yields

$$\varepsilon^{\frac{l_j}{q_j}} \left(\frac{m_j}{q_j} \ln\left(\frac{1}{\varepsilon}\right) + \ln\left| h_j\left(\varepsilon^{\frac{1}{q_j}}\right)\right| \right) > g_j\left(\varepsilon^{\frac{1}{q_j}}\right) \ln c. \tag{23}$$

But the left side of (23) tends to 0 as $\varepsilon \to 0^+$, while the right side tends to $g_j(0) \ln c > 0$, yielding a contradiction. Hence, $g_j(0), l_j > 0$ cannot occur.

(ii) Necessity is obvious. For sufficiency, we note that $\operatorname{Re} \lambda_j(\varepsilon) \to -\infty$ implies $l_j > 0$ and $g_j(0) < 0$ in (21) for every j. Then for every $c > 1$ and small ε,

$$\varepsilon^{\frac{l_j}{q_j}} \left(\frac{m_j}{q_j} \ln \left(\frac{1}{\varepsilon} \right) + \ln \left| h_j \left(\varepsilon^{\frac{1}{q_j}} \right) \right| \right) < -g_j \left(\varepsilon^{\frac{1}{q_j}} \right) \ln c.$$

Multiplying by $\varepsilon^{-\frac{l_j}{q_j}}$ and taking the exponential yields $|\operatorname{Im} \lambda_j(\varepsilon)| < c^{-\operatorname{Re} \lambda_j(\varepsilon)}$, so the λ_j are left exponentially bounded. $\square$

THEOREM 26. *$\exp(A^{-1}(\varepsilon))$ converges weak* as $\varepsilon \to 0^+$ iff there exist $M < \infty$ and $\delta > 0$ such that $\operatorname{Re} \lambda_j(\varepsilon) < M$ for all j and $\varepsilon \in (0, \delta)$.*

Proof (sufficient). Suppose $\exp(A^{-1}(\varepsilon))$ does not converge weak*. Then there exist $\varepsilon_k \to 0^+$ and $\phi \in \mathcal{D}$ such that $\langle \exp(A^{-1}(\varepsilon_k)), \phi \rangle$ does not converge, so $\exp(A^{-1}(\varepsilon_k))$ does not converge. From Theorem 11, the $\lambda_j(\varepsilon_k)$ are not right exponentially bounded, so there exist j_k such that $\operatorname{Re} \lambda_{j_k}(\varepsilon_k) \to \infty$, contradicting the hypothesis.

Proof (necessary). Suppose that for every M and δ there exist j and $\varepsilon \in (0, \delta)$ such that $\operatorname{Re} \lambda_j(\varepsilon) \geq M$. From Lemma 25, part (i), the λ_j are not right exponentially bounded. Hence, there exist $\varepsilon_k \to 0^+$ and j_k such that the $\lambda_{j_k}(\varepsilon_k)$ are not right exponentially bounded. From Theorem 11, $\exp(A^{-1}(\varepsilon_k))$ does not converge weak*, so $\exp(A^{-1}(\varepsilon))$ does not converge weak*. $\square$

THEOREM 27. *The following are equivalent:*

(i) $\exp(A^{-1}(\varepsilon))$ *converges a.e. on some* $(0, t_0)$.

(ii) *For every $M < \infty$, $p \geq 0$, and $t_0 > 0$ there exists $\delta > 0$ such that $\|A^{-p}(\varepsilon) e^{tA^{-1}(\varepsilon)}\| < e^{-Mt}$ for every $t > t_0$ and $\varepsilon \in (0, \delta)$.*

(iii) $\operatorname{Re} \lambda_j(\varepsilon) \to -\infty$ *as $\varepsilon \to 0^+$ for every j.*

Proof. (i)$\Rightarrow$(ii): Suppose there exist j_k, $\varepsilon_k \to 0^+$, and $M < \infty$ such that $\operatorname{Re} \lambda_{j_k}(\varepsilon_k) > -M$ for all k. Then $\lambda_j(\varepsilon_k) \not\to -\infty$ for some j. Choose any $t_0 > 0$. From Theorem 15, $\exp(A^{-1}(\varepsilon_k))$ does not converge a.e. on $(0, t_0)$. Hence, $\exp(A^{-1}(\varepsilon))$ does not converge a.e. on $(0, t_0)$, contradicting (i).

(iii)$\Rightarrow$(ii): From Lemma 25, part (ii), the λ_j are left exponentially bounded. Suppose there exist M, p, and t_0 such that for every $k < \infty$ there exists $t > t_0$ and $\varepsilon_k \in (0, \frac{1}{k})$ with

$$\left\| A^{-p}(\varepsilon_k) e^{tA^{-1}(\varepsilon)} \right\| \geq e^{-Mt}.$$

Then Theorem 22 indicates that the $\lambda_j(\varepsilon_k)$ are not left exponentially bounded, which is a contradiction.

(ii)$\Rightarrow$(i): Choose any t_0 and $t \in (0, t_0)$, and let $p = 0$. For any $\eta > 0$, set

$$M > \frac{\ln \frac{1}{\eta}}{t}.$$

Then $\|e^{tA^{-1}(\varepsilon)}\| < e^{-Mt} < \eta$ for small ε, so $e^{tA^{-1}(\varepsilon)} \to 0$. $\square$

5. Conclusions. We have investigated convergence of sequences of matrix exponential functions in the pointwise, almost uniform, and weak* sense. We have shown that simple results can be obtained in terms of the eigenvalues of A_k^{-1} alone. The appropriate conditions are left and right exponential boundedness in $\mathbb{C}$. Perhaps the most striking result is a necessary and sufficient condition for sequential weak* convergence. In the case of real analytic $A(\varepsilon)$, we have shown that the eigenvalue conditions reduce to simple bounds on $\operatorname{Re}\lambda(\varepsilon)$, and that pointwise and almost uniform convergence are equivalent.

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