Linear Circuits, Systems and Signal Processing:
Theory and Application
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A REALIZATION THEORY FOR PERTURBED SINGULAR SYSTEMS[1]

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1. INTRODUCTION

The theory of state-space realizations for strictly proper rational matrices has been thoroughly studied. More recently, techniques for handling improper transfer matrices have been devised (see [12]). In this paper we extend those ideas to the case where a system is described by a convergent *sequence* of rational matrices. A realization is then a sequence of state-space systems. The problem is made nontrivial by imposing the constraint that the matrix entries of the realization sequence should also converge.

We consider sequences of rational $r \times m$ matrices $H_k(s)$ which are convergent in a natural sense to be described in detail in Section 2. We desire to realize such sequences with corresponding sequences of state-space systems of the form

$$E\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where E and A are $n \times n$, B is $n \times m$, C is $r \times n$, and E may be singular. For the sake of brevity, we identify the system (1) with the matrix 4-tuple (E,A,B,C). The transfer matrix of (1) is

$$H(s) = C(sE - A)^{-1}B = \frac{C\cdot adj(sE - A)\cdot B}{det(sE - A)} \tag{2}$$

The entries of H(s) are elements of $\mathbb{R}(s)$, the set of all rational functions over $\mathbb{R}$.

2. CONVERGENCE IN THE SPACE OF RATIONAL MATRICES

<u>Definition 1</u> Suppose P_k; k=1,2,... and P are in $\mathbb{R}[s]$. We say P_k converges to

[1]This work was supported by NSF Grant No. ECS-8612948.

P ($P_k \longrightarrow P$) if there is an integer q and sequences $a_i^{(k)} \longrightarrow a_i$ in $\mathbb{R}$ such that

$$P_k(s) = a_q^{(k)}s^q + \cdots + a_1^{(k)}s + a_0^{(k)}, \quad P(s) = a_q s^q + \cdots + a_1 s + a_0$$

In order to define convergence in $\mathbb{R}(s)^{rm}$, we view each rational function in $\mathbb{R}(s)$ as an equivalence class determined by the relation $\frac{a}{b} \approx \frac{c}{d} \Leftrightarrow ad = bc$ where $a,b,c,d \in \mathbb{R}[s]$ and $b,d \neq 0$. To each element $p \in \mathbb{R}[s]$ corresponds a unique equivalence class [p]. When no confusion can arise, we simply write p instead of [p].

Definition 2 Suppose h_k; $k = 1,2,\ldots$, and h are in $\mathbb{R}(s)$. If there exist sequences $n_k \longrightarrow n$ and $d_k \longrightarrow d$ in $\mathbb{R}[s]$ such that $n_k/d_k \in [h_k]$; $k = 1,2,\ldots$, $\frac{n}{d} \in [h]$, then we say that the sequence $\{h_k\}$ converges to h ($h_k \longrightarrow h$) in $\mathbb{R}(s)$.

We note that $\mathbb{R}[s]$ is a subset of $\mathbb{R}(s)$, so convergence in $\mathbb{R}[s]$ can be viewed as a special case of convergence in $\mathbb{R}(s)$. We now address convergence in $\mathbb{R}(s)^{rm}$.

Definition 3 Suppose H_k; $k = 1,2,\ldots$ and H are $r \times m$ rational matrices. We say $H_k \longrightarrow H$ in $\mathbb{R}(s)^{rm}$ if every component sequence $h_{ij}^{(k)} \longrightarrow h_{ij}$ in R(s).

3. EXISTENCE OF REALIZATIONS OF $\{H_k\}$

Suppose we have a convergent sequence $H_k \longrightarrow H$ in $\mathbb{R}[s]^{rm}$. We would like to define $\{(E_k,A_k,B_k,C_k)\}$ to be a realization sequence of $\{H_k\}$ if (E_k,A_k,B_k,C_k) is a realization of H_k for $k = 1,2,\ldots$ and if (E_k,A_k,B_k,C_k) converges to some realization (E,A,B,C) of H. For technical reasons, we need to make a slightly less restrictive definition.

Definition 4 Suppose $H_k \longrightarrow H$ in $\mathbb{R}(s)^{rm}$. We call a sequence $\{(E_k,A_k,B_k,C_k)\}$ a realization of $\{H_k(s)\}$ if there exists an integer K such that (E_k,A_k,B_k,C_k) is a realization of H_k when $k > K$ and if $(E_k,A_k,B_k,C_k) \longrightarrow (E,A,B,C)$ in $\mathbb{R}^{n(2n+m+r)}$, where (E,A,B,C) is some realization of H. The integer n is called the dimension of the realization.

The following theorem guarantees the existence of a realization sequence for any convergent $\{H_k\}$.

Theorem 1 Every convergent rational matrix sequence in $\mathbb{R}(s)^{rm}$ has a realization.

Sketch of Proof

First, we let $\Sigma_n = \{(E,A,B,C) \in R^{n(2n+m+r)} \mid \det(sE-A) \not\equiv 0\}$ and define a map

$G_n: \Sigma_n \longrightarrow \mathbb{R}[s]^{rm} \times \mathbb{R}[s]$ according to $G_n(E,A,B,C) = (C\cdot adj(sE-A)B, det(sE-A))$. We say a pair $(N,d) \in \mathbb{R}[s]^{rm} \times \mathbb{R}[s]$ is n-dimensional realizable if $(N,d) \in G_n(\Sigma_n)$. We call the inverse image (E,A,B,C) of (N,d) with respect to G_n a realization of (N,d). It is obvious that if (E,A,B,C) is a realization of (N,d) then it is also a realization of the transfer matrix $\frac{N}{d}$ in the usual sense. We require a series of lemmas.

Lemma 1 The map G_n is open and continuous.

Lemma 2 Let X and Y be topological spaces with X first countable, and let $Q: X \longrightarrow Y$ be an onto, open, and continuous map. For any convergent sequence $\{y_k\}$ in Y with $y_k \longrightarrow y \in Y$, there exist $x_k \in Q^{-1}(y_k)$; $k = 1,2,\ldots$ and $x \in Q^{-1}(y)$ such that $x_k \longrightarrow x$ in X.

Lemma 3 If $H_k \longrightarrow H$, then there exist an integer n and pairs (N_k,d_k); $k = 1,2,\ldots$ and $(N,d) \in G_n(\Sigma_n)$ such that $\frac{N_k}{d_k} \in [H_k]$, $\frac{N}{d} \in [H]$, and $(N_k,d_k) \longrightarrow (N,d)$.

Now we suppose $H_k \longrightarrow H$; then we can find a convergent sequence $(N_k,d_k) \longrightarrow (N,d)$ in $G_n(\Sigma_n)$ with the properties in Lemma 3. It is easy to see that Σ_n is first countable. Restricting the range of G_n to $G_n(\Sigma_n)$ and using Lemmas 1 and 2, we conclude that, for the convergent sequence (N_k,d_k), we can always find a convergent sequence $(E_k,A_k,B_k,C_k) \longrightarrow (E,A,B,C)$ in Σ_n such that $G_n(E_k,A_k,B_k,C_k) = (N_k,d_k)$ and $G_n(E,A,B,C) = (N,d)$. Notice that (E_k,A_k,B_k,C_k) is a realization of $\frac{N_k}{d_k} \in [H_k]$; $k=1,2,\ldots$, and (E,A,B,C) is a realization of $\frac{N}{d} \in [H]$. □

Theorem 1 says that we can always find a realization of $\{H_k\}$ with some dimension n. In the next section we address the problem of calculating the minimal value of n.

4. MINIMAL REALIZATION OF $\{H_k\}$

First we recall a result about the minimal realization of rational matrices in $R(s)^{rm}$ by singular systems [12]: Let $H = H_s + H_f$, where H_s and H_f are respectively the strictly proper part and the polynomial part of H. Define the δ-degree of H according to $\delta(H(s)) = \nu(H_s(s)) + \nu(\frac{1}{s}H_f(\frac{1}{s}))$, where $\nu(\cdot)$ is McMillan degree [14]. It can be shown that $\delta(H)$ is the degree of any minimal realization of H (see [12]).

Definition 5 Suppose $\{(E_k, A_k, B_k, C_k)\}$ is an n-dimensional realization of a convergent sequence $\{H_k\}$. If the dimension of every realization of $\{H_k\}$ is no less than n, we call $\{(E_k, A_k, B_k, C_k)\}$ minimal and we define the δ-degree of $\{H_k\}$ according to $\delta\{H_k\} = n$.

Note that there is a distinction between the δ-degree of an individual rational matrix H and a sequence $\{H_k\}$. To continue our development, we need to extend the definition of the characteristic polynomial to improper matrices.

Definition 6 For a rational matrix $H = H_s + H_f$ we define the characteristic polynomial (or simply C.P.) of H as that of H_s when $H_s \neq 0$ and as 1 when $H_s = 0$. We denote the C.P. of H by Δ.

Now we need to consider the sequence of C.P.'s $\{\Delta_k\}$ corresponding to $\{H_k\}$. We first note that, from Definitions 1 and 3', $H_k = N_k/d_k$ for some convergent sequences $\{N_k\}$ and $\{d_k\}$, where $\deg d_k \leq \mu$ for some integer μ. Since the strictly proper part of each H_k is of the form $H_{sk} = N_{sk}/d_k$, where N_{sk} is some (not necessarily convergent) matrix sequence, it follows that $\deg \Delta_k \leq \mu \cdot \min\{r,m\}$. Let $\eta = \max\{\deg \Delta_k\}$. Then each Δ_k may be uniquely identified with a point in the real projective space $\mathbb{P}^\eta$ according to $s^i + \alpha_{i-1}s^{i-1} + \dots + \alpha_0 \mapsto [0,\dots,0,1,\alpha_{i-1},\dots,\alpha_0]$, where $i \leq \eta$. Clearly, the sequence $\{\Delta_k\}$ converges iff there exists a real sequence $\{c_k\}$ such that $\{c_k\Delta_k\}$ converges in the sense of Definition 1. It is also important to note that convergence of $\{H_k\}$ and $\{\Delta_k\}$ does not necessarily imply that $\lim \Delta_k$ is the C.P. of $\lim H_k$.

We will see that convergence of the C.P. sequence plays an important role in the minimal realization problem. We first treat the case where $\{\Delta_k\}$ converges. In this case we can give a simple formula for calculating $\delta\{H_k\}$.

Theorem 2 Suppose $H_k \longrightarrow H$. If $\{H_k\}$ has convergent C.P. sequence $\{\Delta_k\}$ and $\Delta_k \longrightarrow \phi$ for some polynomial ϕ, then the δ-degree of $\{H_k\}$ is

$$\delta\{H_k\} = \max\{\delta(H_f) + \deg \phi,\ \overline{\lim_{k\to\infty}}\, (\delta(H_{fk}) + \deg \Delta_k)\}$$

Here we assume H_{fk} is the polynomial part of H_k; $k = 0,1,2,\dots$

We note that even though $\{H_k\}$ is convergent, the corresponding C.P. sequence may not converge.

<u>Theorem 3</u> If $H_k \longrightarrow H$, then there exist strictly increasing sequences $\{k_j^{(\alpha)}\}$; $\alpha = 1,\dots,p$ of positive integers such that

1) $\{k_j^{(\alpha)} | \alpha = 1,\dots,p;\ j = 1,2,\dots\} = \{1,2,3,\dots\}$
2) $\{k_j^{(\alpha)} | j = 1,2,\dots\} \cap \{k_j^{(\beta)} | j = 1,2,\dots\} = \phi$ when $\alpha \neq \beta$.
3) $\{H_{k_j^{(\alpha)}}\}$ has convergent C.P. sequence.

Theorem 3 shows that any convergent sequence $\{H_k\}$ can be "decomposed" into finitely subsequences, each with convergent characteristic polynomial. Theorem 3 allows us to generalize Theorem 2. Suppose that $\{H_k\}$ has been decomposed into p subsequences $\{H_k^{(\alpha)}\}$; $\alpha = 1,2,\dots,p$ with convergent C.P. sequences $\Delta_k^{(\alpha)} \longrightarrow \phi^{(\alpha)}$; $\alpha = 1,2,\dots,p$. Let $\phi^* = \underset{1\le\alpha\le p}{\mathrm{LCM}} \{\phi^{(\alpha)}\}$, where LCM denotes the least common multiple. Theorem 4 requires the use of the polynomial sequences

$$d_k^{(\alpha)} = \frac{\Delta_k^{(\alpha)}}{\phi^{(\alpha)}}\cdot\phi^*;$$ $\alpha = 1,2,\dots,p$ and the sequence $\{d_k^*\}$ determined by $d^*_{k_j^{(\alpha)}} = d_j^{(\alpha)}$.

<u>Theorem 4</u> Suppose $H_k \longrightarrow H$; then the δ-degree of $\{H_k\}$ is given by

$$\begin{aligned}\delta\{H_k\} &= \max\{\delta(H_f)+\deg \phi^*, \overline{\lim_{k\to\infty}}(\delta(H_{fk})+\deg d_k^*))\} \\ &= \max\{\delta(H_f)+\deg \phi^*, \overline{\lim_{k\to\infty}}(\delta(H_{fk}^{(\alpha)})+\deg d_k^{(\alpha)});1\le\alpha\le p\}\end{aligned}$$

where $H_{fk}^{(\alpha)}$ is the polynomial part of $H_k^{(\alpha)}$.

5.MINIMALITY OF $\{(E_k,A_k,B_k,C_k)\}$

In this section we describe a partial generalization of the standard result from realization theory for fixed systems [12] which says that a system (E,A,B,C) is a minimal realization of some H(s) if and only if (E,A,B,C) is controllable and observable. We assume that the C.P. of $\{H_k\}$ converges and that the system (1) is SISO. The proof of our result is based on the following lemma.

<u>Lemma 4</u> Consider a sequence of scalar rational functions $h_k \longrightarrow h$. If $\{h_k\}$ has C.P. sequence $\Delta_k \longrightarrow \phi$, then, for sufficiently large k, $\delta(h_k) \ge \delta(h) + \deg \phi - \deg \Delta = \delta(h_f) + \deg \phi$, where Δ is the C.P. of h.

<u>Theorem 5</u> Consider a sequence $\xi_k = (E_k,A_k,B_k,C_k) \longrightarrow (E,A,B,C)$ with $m = r = 1$ and let $h_k(s) = C_k(sE_k - A_k)^{-1}B_k$. If the C.P. of $\{h_k\}$ converges, then $\{\xi_k\}$ is

minimal if and only if there exists a subsequence of $\{\xi_k\}$ consisting entirely of controllable and observable systems.

The sufficiency part of Theorem 5 is obvious. Necessity is based on Lemma 4 used in conjunction with Theorem 2. Together these imply that $\delta\{h_k\} = \overline{\lim_{k\to\infty}} \delta(h_k)$. It follows that there exists a subsequence of $\{h_{k_j}\}$ such that $\delta\{h_k\} = \delta(h_{k_j})$ for all j. Hence, the minimal realization of each h_{k_j} has dimension $\delta(h_{k_j})$, and so each corresponding state-space system is controllable and observable.

7. CONCLUDING REMARKS

The basic problem discussed in this paper is the perturbation of a singular system in the frequency domain and the relationship with its realization in the time domain. We have shown that whether the characteristic polynomial sequence corresponding to a sequence of rational matrices is convergent constitutes a crucial piece of information in the minimal realization problem. We have proved (Theorem 3) that, when the characteristic polynomial of the original sequence is not convergent, we can decompose the original sequence into finitely many subsequences in such a way that each subsequence has convergent characteristic polynomial sequence. Our results indicate that the general problem can be reduced to finitely many subproblems each of which can be handled using a simpler theory.

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