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A Realization Theory for Perturbed Linear Systems*

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In this paper we present a theory which characterizes LTI state-space realizations of perturbed rational transfer function matrices. Our approach is to model system perturbations as sequences in the space of rational matrices. First, we give a definition of convergence in the space of rational matrices which is motivated by the kinds of parameter uncertainties occurring in many robust control problems. A realization theory is then established under the constraint that the realization of any convergent sequence of rational matrices should also be convergent. Next, we consider the issue of minimality of realizations and propose a method for calculating the dimension of a minimal realization of a given transfer matrix sequence. Finally, necessary and sufficient conditions are discussed under which a sequence of state-space systems is a minimal realization and under which minimal realizations of the same transfer function sequence are state-space equivalent. Relationships with standard algebraic system theoretic results are discussed.

1. INTRODUCTION

The theory of state-space realizations for strictly proper rational matrices has been thoroughly studied (e.g., see [16]). More recently, techniques for handling improper transfer matrices have been devised (see [14]). In this paper we extend those ideas to the perturbational case—i.e., where a system is described by a convergent sequence of rational matrices (possibly improper). A realization is then a sequence of (generalized) state-space systems. The problem is made nontrivial by imposing the constraint that the matrix entries of the realization sequence should also converge.

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Part of our motivation for this problem comes from the study of robust control problems—specifically, from those dealing with order uncertainty and singular perturbations. For example, the robustness problems addressed in [1–3] are based on singularly perturbed system models. Physical systems are invariably subject to some variations in parameters, often resulting in changes in model order. It is desirable, therefore, to design compensators which meet performance criteria independent of system perturbations. Many robust control theories (e.g., [4]) emphasize input–output performance characteristics. Our intention is to develop some fundamental tools for examining robustness problems associated with a system's internal structural properties.

One way to approach this problem might be through the application of algebraic system theory (see, e.g., [17]). In this setting, the transfer function sequence is viewed as a rational function over the ring c of convergent real sequences using pointwise operations. Unfortunately, we see that existing results in algebraic realization theory apply to our case only marginally. This is due to three key facts: (1) The ring c is not an integral domain. (2) Most results in algebraic realization theory deal only with the case of proper transfer functions. (3) An abstract version of the Weierstrass decomposition for matrix pencils over a ring does not yet exist. Nevertheless, our feeling is that the properties of sequences of transfer functions are sufficiently important from the point of view of robust control theory that they deserve separate treatment, not only for the sake of mimicking standard results from algebraic theory, but also in order to obtain deeper insight into the specific structure of realizations over this particular ring.

From an analytic perspective, considerable work dealing with perturbations of rational matrices has appeared (e.g., [4, 7–12]). In these papers various rational matrix topologies have been proposed, motivated by a variety of control problems. The closest of these to our work are [10–12], where a singular perturbation theory for transfer functions is developed and a specific form of realization is given. However, [10–12] do not explicitly address those problems dealing with the existence of realizations in general and, in particular, the minimal realization of perturbed systems. In [4] rational matrix convergence is characterized in terms of the "graph metric" which is used to address certain problems in local simultaneous stabilization. It is easy to show that the graph metric induces a topology which is very different from that corresponding to simple system parameter convergence. The work of [7 and 9] also treats the problem of topologizing the set of rational matrices and is closely related to ours, but again does not examine the realization problem. Our work is motivated solely by realization and robustness issues; our constructions are designed to yield the simplest definition of convergence corresponding to convergence of system parameters.

We are mainly concerned with the interplay between two types of LTI system representations. First, let $\mathbb{R}(s)$ be the set of all rational functions over $\mathbb{R}$, and let $\mathbb{R}(s)^{rm}$ be the set of $r \times m$ matrices over $\mathbb{R}(s)$. Next, consider (generalized) state-space systems

$$E\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1}$$

where E and A are $n \times n$ real matrices satisfying the standard regularity assumption $\det(sE - A) \not\equiv 0$, B is $n \times m$, C is $r \times n$, and E may be singular. For the sake of brevity, we identify the system (1) with the matrix 4-tuple $\sigma = (E, A, B, C) \in \mathbb{R}^{n(2n+m+r)}$. The transfer matrix of (1) is

$$H(s) = C(sE - A)^{-1}B = \frac{C \cdot \operatorname{adj}(sE - A) \cdot B}{\det(sE - A)} \in \mathbb{R}(s)^{rm}. \tag{2}$$

Throughout the paper we assume that the values of m and r are fixed; we consider n to be a variable.

Definition 1.1. (1) A state-space system $\sigma \in \mathbb{R}^{n(n+m+r)}$ is said to have *dimension n*. In this case, we write $\dim \sigma = n$.

(2) If a rational matrix H is of the form (2), we say that (E, A, B, C) is a *realization* of H.

With regard to parts 1 and 2, a (nonperturbational) realization theory already appears in [14]. We now summarize the main results of this theory.

Theorem 1.2 [14]. (1) *Every rational matrix has a realization.*

(2) *The minimal dimension over all realizations of H, denoted* $\mu(\cdot)$, *is* $\mu(H(s)) = \nu(H_s(s)) + \nu(1/s)H_f(1/s))$, *where* $\nu(\cdot)$ *is the MacMillan degree, and* H_s *and* H_f *are the unique strictly proper rational matrix and the polynomial matrix, respectively, satisfying* $H = H_s + H_f$.

(3) *A 4-tuple* σ *is a minimal realization of some rational matrix H if and only if* σ *is controllable and observable (as defined in* [8]).

(4) *If* $\sigma_1 = (E_1, A_1, B_1, C_1)$ *and* $\sigma_2 = (E_2, A_2, B_2, C_2)$ *are minimal realizations of the same rational matrix, there exist nonsingular matrices M and N such that* $E_2 = ME_1N$, $A_2 = MA_1N$, $B_2 = MB_1$, *and* $C_2 = C_1N$.

The results of our present work may be considered to be a generalization of Theorem 1.2 to the case of rational matrix sequences $\{H_k\}$.

In Section 2 we choose a natural definition for convergence of rational matrices. Working from this definition, we consider sequences $H_k \to H$ in $\mathbb{R}(s)^{rm}$ and attempt to characterize those sequences $\{\sigma_k\}$ in $\mathbb{R}^{n(2n+m+r)}$ such that (1) σ_k converges to some σ in the matrix sense, (2) σ_k is a realization of H_k for sufficiently large k, and (3) σ is a realization of H.

We view this approach as a way of modeling the possible perturbations in the internal structure of a system corresponding to a given perturbation in the input–output description $\{H_k\}$.

In our realization theory, we see that Theorem 1.2 part 1, remains true (Section 5). Corresponding to the expression for μ in part 2, in Section 6 we define and give an explicit expression for a degree function which equals the dimension of all "minimal realizations" of a sequence of transfer matrices. It is shown that properties 3 and 4 do not hold as stated for sequences; however, we discuss important special cases where similar statements do hold. In Sections 5 and 6 we also discuss the connections between our work and the standard algebraic realization theory (see [17]).

2. Convergence in the Space of Rational Matrices

We first consider the problem of defining a topology on $\mathbb{R}(s)$ and later on $\mathbb{R}(^s)^{rm}$. Convergence in $\mathbb{R}(s)$ is defined in the most natural way such that small perturbations in $\mathbb{R}(s)$ correspond to small perturbations in the coefficients of numerator and denominator of the rational function. To begin, we must define convergence in the set $\mathbb{R}[s]$ of all polynomials over $\mathbb{R}$. Suppose P_k, $k = 1, 2, \ldots$, and P are polynomials in $\mathbb{R}[s]$.

Definition 2.1. We say P_k *converges to* P if there exists an integer $q < \infty$ such that $\deg P_k \leq q$, $k = 1, 2, \ldots$, $\deg P \leq q$, and $a_{ik} \to a_i$, $i = 0, \ldots, q$, where $P_k(s) = a_{qk}s^q + \cdots + a_{1k}s + a_{0k}$, $k = 1, 2, \ldots$, and $P(s) = a_q s^q + \cdots + a_1 s + a_0$.

Remarks. (1) If we regard $P_k \in \mathbb{R}[s]$, $k = 1, 2, \cdots$, as functions over $\mathbb{C}$, we might be tempted to define $P_k \to P$ when $\lim_{k\to\infty} P_k(s) = P(s)$ for any $s \in \mathbb{C}$. But we note that in this definition, $\deg P_k$ may not be bounded. For example, let $P_k(s) = (1/k^k)s^k + 1$. This observation brings us to a crossroads in the theory: If we were to allow convergent polynomial sequences to have unbounded degree, the same would be true for sequences of rational functions. This would result in an undesirable situation where state-space realizations could have unbounded dimension. Hence, we insist on bounded degree based both on physical intuition and on a desire for mathematical elegance.

(2) Definition 2.1 is equivalent to the following two conditions:

(a) $\{\deg P_k | k = 1, 2, \cdots\}$ is bounded.

(b) $\lim_{k\to\infty} P_k(s) = P(s)$ for every $s \in \mathbb{C}$.

Indeed, the necessity of (a) and (b) is obvious. On the other hand, if $\{P_k\}$ satisfies (a), $P_k(s)$ and $P(s)$ can be written as in Definition 2.1. Choose $q + 1$ distinct complex numbers $\{s_1, \ldots, s_{q+1}\}$. Then, from (b), $V\xi_k \to$

$V\xi$, where

$$V = \begin{Bmatrix} s_1^q & \cdots & s_1 & 1 \\ s_2^q & \cdots & s_2 & 1 \\ \vdots & & \vdots & \vdots \\ s_{q+1}^q & \cdots & s_{q+1} & 1 \end{Bmatrix}, \qquad \xi_k = \begin{Bmatrix} a_{qk} \\ \vdots \\ a_{1k} \\ a_{0k} \end{Bmatrix}, \qquad \xi_k = \begin{Bmatrix} a_q \\ \vdots \\ a_1 \\ a_0 \end{Bmatrix}.$$

We know that the Vandermonde matrix V satisfies det $V \neq 0$ as long as $s_i \neq s_j$, $i \neq j$; therefore V^{-1} exists and $\xi_k \to \xi$ as $k \to \infty$.

(3) We can define a topology on $\mathbb{R}[s]$ which is consistent with our notion of convergence in Definition 2.1. To do so, identify every element in $\mathbb{R}[s]$ with an element in $\mathbb{R}^\infty$ according to

$$p_m s^m + \cdots + p_1 s + p_0 \leftrightarrow (p_0, p_1, \cdots, p_m, 0, 0, 0, \cdots),$$

and let

$$\mathcal{R}_{m+1} = \{(p_0, p_1, \cdots, p_m, 0, 0, \cdots) \in \mathbb{R}^\infty \mid p_i \in \mathbb{R}, i = 0, 1, 2, \cdots, m\}.$$

Then

$$\mathcal{R} = \bigcup_{k=1}^{\infty} \mathcal{R}_k$$

is the set of all polynomials. On $\mathcal{R}_k$, we take identification topology (e.g., see [19, p. 120]) with respect to the bijections f_k: $\mathbb{R}^k \to \mathcal{R}_k$ defined by $f_k(a_1, a_2, \cdots, a_k) = (a_1, a_2, \cdots, a_k, 0, \cdots)$. That is, a set $U = \{(a_1, \cdots, a_k, 0, \cdots) \mid (a_1, \cdots, a_k) \in V\}$ is open in $\mathcal{R}_k$ if and only if V is open in $\mathbb{R}^k$. On $\mathcal{R}$ we impose the inductive limit topology [19, p. 420] with respect to the $\mathcal{R}_k$—i.e., we impose on $\mathcal{R}$ the finest topology which makes the natural imbeddings $\mathcal{R}_k \subset \mathcal{R}$ continuous. It is routine to prove that $P_k \to P$ in the sense of Definition 2.1 if and only if P_k converges to P in $\mathcal{R}$.

(4) It is shown in [6, Lemma 4.3] that, if $\{P_k\}$ is convergent in $\mathbb{R}[s]$, then there exist convergent real sequences $\{\alpha_{ik}\}$, $\{\beta_{ik}\}$, and $\{\gamma_k\}$, with $\alpha_{ik} \to 0$ and lim $\gamma_k \neq 0$, such that

$$P_k(s) = \gamma_k \prod_i (\alpha_{ik} s - 1) \prod_i (s - \beta_{ik}). \tag{3}$$

In particular, if the roots of p_k are bounded, then deg p_k = deg lim P_k.

In order to define convergence in $\mathbb{R}(s)$, we adopt a standard quotient space construction over $\mathbb{R}[s] \times (\mathbb{R}[s] - \{0\})$ (e.g., see [18, p. 136]) and identify each rational function with a unique equivalence class under the relation $(a, b) \approx (c, d) \Leftrightarrow ad = bc$. We use the expression a/b to denote both a rational function and its corresponding equivalence class in $\mathbb{R}[s] \times (\mathbb{R}[s] - \{0\})$. Note that a similar construction may be employed in identifying rational matrices $N/d \in \mathbb{R}^{rm}(s)$ with equivalence classes of pairs $(N, d) \in \mathbb{R}[s]^{rm} \times (\mathbb{R}[s] - \{0\})$.

Adopting ordinary quotient set topology on $\mathbb{R}(s)$, we arrive at the following definition.

DEFINITION 2.2. Suppose h_k, $k = 1, 2, \ldots$, and h are in $\mathbb{R}(s)$. We say h_k *converges to* h in $\mathbb{R}(s)$, if there exist $n_k \to n$ and $d_k \to d$ in $\mathbb{R}[s]$, with d_k, $d \neq 0$, such that $n_k/d_k = h_k$; $k = 1, 2, \ldots$, and $n/d = h$.

Along similar lines, we now give three alternative definitions for convergence in $\mathbb{R}(s)^{rm}$.

DEFINITION 2.3. Suppose H_k, $k = 1, 2, \ldots$, and H are $r \times m$ rational matrices with components h_{ijk} and h_{ij}, respectively. We say H_k *converges to* H in $\mathbb{R}(s)^{rm}$ if $h_{ijk} \to h_{ij}$ in $\mathbb{R}(s)$ as $k \to \infty$.

DEFINITION 2.3′. Suppose H_k; $k = 1, 2, \cdots$, and H are $r \times m$ rational matrices. We say H_k *converges to* H in $\mathbb{R}(s)^{rm}$ if there exist $N_k \to N$ in $\mathbb{R}[s]^{rm}$ and $d_k \to d$ in $\mathbb{R}[s]$ such that $N_k/d_k = H_k$; $k = 1, 2, \ldots$, and $N/d = H$. (Here we assume product topology on $\mathbb{R}[s]^{rm}$ and that the quotient space constructions above are applied componentwise on $\mathbb{R}[s]^{rm} \times (\mathbb{R}[s] - \{0\})$.)

DEFINITION 2.3″. We say H_k *converges to* H in $\mathbb{R}(s)^{rm}$ if there exist $N_k \to N$ in $\mathbb{R}[s]^{rm}$ and $D_k \to D$ in $\mathbb{R}[s]^{m_2}$ with D_k and D nonsingular such that $N_k D_k^{-1} = H_k$; $k = 1, 2, \ldots$, and $ND^{-1} = H$.

Remarks. (1) It is easy to show that Definitions 2.3, 2.3′, and 2.3″ are equivalent. A fourth alternative definition is the same as 2.3″ except using left instead of right factorizations.

(2) Note that a sequence which converges in the sense of Definition 2.3 also converges in identification topology with respect to the map $\mathcal{H}$: $\mathbb{R}^{n(2n+m+r)} \to \mathbb{R}(s)^{rm}$ defined by

$$\mathcal{H}(E, A, B, C) = \frac{C \cdot \operatorname{adj}(sE - A)B}{\det(sE - A)},$$

where $(E, A, B, C) \in \mathbb{R}^{n(2n+m+r)}$. The construction of the topology on $\mathbb{R}(s)^{rm}$ shows that $\mathcal{H}$ is continuous.

(3) If $H_k \to H$ and $G_k \to G$, then $H_k + G_k \to H + G$ and $H_kG_k \to HG$; more generally, $\mathbb{R}(s)^{rm}$ is a topological ring with respect to identification topology on $\mathbb{R}(s)$ and the corresponding product topology on $\mathbb{R}(s)^{rm}$. In particular, relative topology on the subgroup of polynomial matrices $\mathbb{R}[s]^{rm}$ is the same as product topology with respect to Definition 2.1. Note that $\mathbb{R}[s]^{rm}$ is closed in $\mathbb{R}(s)^{rm}$.

We show in Section 5 that our definition of convergence is the "right" definition for the realization problem, since a sequence in $\mathbb{R}(s)^{rm}$ converges in our sense if and only if it admits a convergent sequence of state-space realizations. One view of the results of this paper is that they characterize local properties of the map $\mathcal{H}$.

3. Time-Scale Decomposition of Transfer Matrix Sequences

Clearly, any rational matrix H can be uniquely expressed as $H = H_s + H_f$, where H_s is strictly proper and H_f is a polynomial matrix. We now generalize the decomposition to the sequential case; this must be carried out in a way that preserves convergence.

Definition 3.1. (1) We say a convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$ is a *slow sequence*, if H_k is strictly proper for every k and there exists a bounded region $\Lambda \subset \mathbb{C}$ such that all poles of each H_k lie in Λ.

(2) A convergent sequence $\{H_k\}$ is called a *fast sequence* if for every $M < \infty$ there exists a $K < \infty$ such that $k > K$ implies that each pole p of H_k satisfies $|p| > M$ (all poles tend to infinity).

Remarks. (1) The set of all slow sequences in $\mathbb{R}(s)^{rm}$ forms a proper subspace of the real vector space of all convergent sequences in $\mathbb{R}(s)^{rm}$. The same statement holds for fast sequences.

(2) Any slow sequence can be expressed as $H_{sk} = N_k/d_k$ where d_k is convergent and monic for every k and $\deg N_k < \deg d_k$, where $\deg N = \max_{i,j}\{\deg n_{ij}\}$ for any polynomial matrix N. Thus $\deg \lim N_k < \deg \lim d_k$. This shows that the limit of every slow sequence is strictly proper.

(3) Since the limit of any fast sequence can have no finite poles, such a limit must be a polynomial matrix.

(4) Every convergent sequence of polynomial matrices is a fast sequence.

(5) If a sequence is both slow and fast, it must be strictly proper and have no poles whatsoever for large k; hence, the sequence must be identically zero for large k.

(6) A sequence of matrices $\{H_k\}$ is slow (fast) if and only if each component sequence $\{h_{ijk}\}$ is slow (fast).

DEFINITION 3.2. (1) We say $H_k = H_{sk} + H_{fk}$ is a *time-scale decomposition* of $\{H_k\}$ when $\{H_{sk}\}$ and $\{H_{fk}\}$ are slow and fast sequences, respectively.

(2) In a time-scale decomposition, $\{H_{sk}\}$ and $\{H_{fk}\}$ are called the *slow part* and the *fast part* of $\{H_k\}$.

Note that from Remarks 2 and 3 above, if $H_k = H_{sk} + H_{fk}$ is a time-scale decomposition of $\{H_k\}$, then $H_{sk} \to H_s$ and $H_{fk} \to H_f$, where H_s and H_f are the strictly proper part and the polynomial part of $H = \lim H_k$. Theorem 3.3 tells us that every convergent sequence $\{H_k\}$ has an essentially unique time-scale decomposition.

THEOREM 3.3. (1) *For every convergent sequence* $\{H_k\}$ *in* $\mathbb{R}(s)^{rm}$, *there exist a slow sequence* $\{H_{sk}\}$ *and a fast sequence* $\{H_{fk}\}$ *such that* $H_k = H_{sk} + H_{fk}$ *for every k.*

(2) *If* $\{\hat{H}_{sk}\}$ *and* $\{\hat{H}_{fk}\}$ *are slow and fast sequences, respectively, and* $H_k = \hat{H}_{sk} + \hat{H}_{fk}$ *for every k, then* $H_{sk} = \hat{H}_{sk}$ *and* $H_{fk} = \hat{H}_{fk}$ *for sufficiently large k.*

Proof. (1) We need only treat the case $r = m = 1$; the multivariable case can then be handled componentwise. If $h_k \to h \in \mathbb{R}(s)$, we can find $n_k \to n$ and $d_k \to d$, with $n_k/d_k = h_k$ and $n/d = h$. Since $d_k \to d$, from (3) we can write $d_k = \gamma_k d_{sk} d_{fk}$, where $d_{sk}(s) = s^\mu + b_{\mu-1,k}s^{\mu-1} + \cdots + b_{0k}$ and $d_{fk}(s) = a_{\nu k}s^\nu + a_{\nu-1,k}s^{\nu-1} + \cdots + a_{1k}s + 1$, with each $\{b_{ik}\}$ convergent, $\gamma_k \to \gamma \neq 0$, and $a_{ik} \to 0$ as $k \to \infty$. Let $n_k(s) = z_{pk}s^p + \ldots + z_{1k}s + z_{0k}$, and let $q = \max\{\nu, p - \mu\}$. We show that there exist convergent polynomial sequences $n_{sk} = x_{\mu-1,k}s^{\mu-1} + \cdots + x_{1k}s + x_{0k}$ and $n_{fk} = y_{q-1,k}s^{q-1} + \cdots + y_{1k}s + y_{0k}$ such that $n_k/d_k = n_{sk}/d_{sk} + n_{fk}/d_{fk}$. Equivalently, we need to show that

$$\gamma_k(n_{sk}d_{fk} + n_{fk}d_{sk}) = n_k. \tag{4}$$

Note that Eq. (4) may be written in matrix form

$$\begin{bmatrix} \mathcal{A}_{1k} & \mathcal{B}_{1k} \\ \mathcal{A}_{2k} & \mathcal{B}_{2k} \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = z_k, \tag{5}$$

where

$$\begin{bmatrix} \mathcal{A}_{1k} \\ \mathcal{A}_{2k} \end{bmatrix} = \gamma_k \begin{bmatrix} 1 & & \\ a_{1k} & \ddots & 1 \\ \vdots & \ddots & a_{1k} \\ a_{\nu k} & \ddots & \vdots \\ & & a_{\nu k} \\ 0 & \cdots & 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{B}_{1k} \\ \mathcal{B}_{2k} \end{bmatrix} = \gamma_k \begin{bmatrix} b_{0k} & & \\ \vdots & \ddots & b_{0k} \\ b_{\mu-1,k} & \ddots & \vdots \\ 1 & \ddots & b_{\mu-1,k} \\ & & 1 \end{bmatrix}$$

$$x_k = \begin{bmatrix} x_{0k} \\ \vdots \\ x_{\mu-1,k} \end{bmatrix}, \qquad y_k = \begin{bmatrix} y_{0k} \\ \vdots \\ y_{q-1,k} \end{bmatrix}, \qquad z_k = \begin{bmatrix} z_{0k} \\ \vdots \\ z_{pk} \\ 0 \end{bmatrix},$$

with $\mathcal{A}_{1k}$, $\mathcal{A}_{2k}$, $\mathcal{B}_{1k}$, $\mathcal{B}_{2k}$, and z_k having dimensions $\mu \times \mu$, $\mu \times q$, $q \times q$, and $(\mu + q) \times 1$, respectively. Also note that $\mathcal{A}_{1k} \to \gamma I_\mu$, $\mathcal{B}_{2k} \to \gamma I_\nu + M$, where M is nilpotent and upper triangular, and $\mathcal{A}_{2k} \to 0$ as $k \to \infty$. Hence, there exists a $K < \infty$ such that (5) has a unique solution when $k > K$. For $k \leq K$, let h_{sk} be any strictly proper rational function and let $h_{fk} = h_k - h_{sk}$.

(2) We have $H_{sk} + H_{fk} = \hat{H}_{sk} + \hat{H}_{fk}$ for sufficiently large k. Hence,

$$H_{sk} - \hat{H}_{sk} = \hat{H}_{fk} - H_{fk}. \tag{6}$$

But the left side of (6) is slow, and the right side is fast. Hence, both sides are identically zero for large k. ∎

To conclude this section, we note that a time-scale decomposition of any transfer matrix sequence of the form $H_k = C_k(sE_k - A_k)^{-1}B_k$, where $\{(E_k, A_k, B_k, C_k)\}$ is convergent in $\mathbb{R}^{n(2n+m+r)}$ and $\det(s \cdot \lim E_k - \lim A_k) \not\equiv 0$, can be achieved by invoking the perturbational form of the classical Weierstrass decomposition for matrix pencils as developed in [7]. Indeed, from [7, p. 147], there exist convergent nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$, with $\lim M_k$ and $\lim N_k$ nonsingular, such that

$$M_kE_kN_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \qquad M_kA_kN_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}, \tag{7}$$

where $\lim A_{fk}$ is nilpotent. Let

$$\begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix} = M_kB_k, \qquad [C_{sk} \quad C_{fk}] = C_kN_k. \tag{8}$$

Then

$$H_k(s) = C_{sk}(sI - A_{sk})^{-1}B_{sk} + C_{fk}(sA_{fk} - I)^{-1}B_{fk}. \tag{9}$$

From Definition 3.1 it is clear that

$$H_{sk}(s) = C_{sk}(sI - A_{sk})^{-1}B_{sk} \tag{10}$$

$$H_{fk}(s) = C_{fk}(sA_{fk} - I)^{-1}B_{fk} \tag{11}$$

are slow and fast sequences, respectively. Hence, (9) is a time-scale decomposition of $\{H_k\}$.

4. The Characteristic Polynomial Sequence

In this section we investigate several useful properties of the sequence of characteristic polynomials corresponding to a convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$. We first extend the conventional definition of the characteristic polynomial to improper transfer matrices. Recall that the characteristic polynomial Δ of a strictly proper rational matrix H_s is defined as the least common monic denominator of all minors of H_s.

Definition 4.1. If H is a rational matrix with $H = H_s + H_f$ for some strictly proper H_s and polynomial matrix H_f, the *characteristic polynomial* Δ of H is defined as the characteristic polynomial of H_s.

Consider the sequence of characteristic polynomials $\{\Delta_k\}$ corresponding to $\{H_k\}$. Since $H_k = N_k/d_k$, it follows that Δ_k divides $d_k^{\min\{r,m\}}$ for each k; thus, boundedness of $\{\deg d_k\}$ ensures boundedness of $\{\deg \Delta_k\}$. Let $\eta = \overline{\lim}\{\deg \Delta_k\}$, and note that $\deg \Delta_k \le \eta$ for sufficiently large k. For all such k, Δ_k can thus be uniquely identified with a point $\langle \Delta_k \rangle$ in the real projective space $\mathbb{P}^\eta$ (see, e.g., [20]) according to

$$s^i + \alpha_{i-1}s^{i-1} + \dots + \alpha_0 \mapsto (0, \dots, 0, 1, \alpha_{i-1}, \dots, \alpha_0) \in \mathbb{R}^{\eta+1}.$$

In fact, there is a one-to-one correspondence between $\mathbb{P}^\eta$ and the set of monic polynomials Δ with deg $\Delta \leq \eta$. These observations lead to the following definition.

DEFINITION 4.2. Let $\{H_k\}$ be any convergent sequence in $\mathbb{R}(s)^{rm}$, and let Δ_k be the characteristic polynomial of H_k. Set

$$\rho_k = \begin{cases} \langle \Delta_k \rangle, & \deg \Delta_k \leq \eta \\ \langle 1 \rangle, & \deg \Delta_k > \eta \end{cases}.$$

The sequence $\{\rho_k\}$ is called the *characteristic polynomial* (CP) of $\{H_k\}$.

It is easy to show that $\{\rho_k\}$ converges if and only if there exists a real sequence $\{\gamma_k\}$ such that $\{\gamma_k \Delta_k\}$ converges to a non-zero limit $\Delta \in \mathbb{R}[s]$. In this case, $\lim \rho_k = \langle \Delta \rangle$. We now present several pathological situations that can arise in dealing with the CP.

EXAMPLE 4.3. (1) The following example illustrates that when $\{H_k\}$ is convergent, the corresponding CP may not converge. Consider the sequence

$$H_k(s) = \begin{cases} \dfrac{(s+2)}{(s+1)(s+2+1/k)}, & k \text{ even} \\[2ex] \dfrac{(s+3)}{(s+1)(s+3+1/k)}, & k \text{ odd} \end{cases},$$

and let $H(s) = 1/(s+1)$. We may write $H_k = N_k/d_k$, where $N_k = (s+2)(s+3)$ and

$$d_k = \begin{cases} (s+1)\left(s+2+\dfrac{1}{k}\right)(s+3), & k \text{ even} \\[2ex] (s+1)(s+2)\left(s+3+\dfrac{1}{k}\right), & k \text{ odd} \end{cases}.$$

Thus $H_k \to H$, but

$$\Delta_k(s) = \begin{cases} (s+1)\left(s+2+\dfrac{1}{k}\right), & k \text{ even} \\[2ex] (s+1)\left(s+3+\dfrac{1}{k}\right), & k \text{ odd} \end{cases}.$$

Clearly, $\{\rho_k\}$ is not convergent. Note, however, that $\{H_k\}$ can be divided into two subsequences with convergent CP's according to $H_k^{(1)} = H_{2k-1}$ and $H_k^{(2)} = H_{2k}$.

(2) In some cases, $\{\rho_k\}$ may converge even though $\{\Delta_k\}$ does not. Consider the rational sequence

$$H_k(s) = \frac{1}{((1/k)s + 1)(s + 2)}.$$

In this case, $\{H_k\}$ has the CP determined by $\Delta_k(s) = (s + k)(s + 2)$, but $\{\rho_k\}$ converges, since $(1/k)\Delta_k(s) = (1/k)(s + k)(s + 2) \to s + 2$.

(3) Finally, we note that convergence of $\{\rho_k\}$ (or even $\{\Delta_k\}$) does not guarantee that $\lim \Delta_k$ is the characteristic polynomial of $\lim H_k$. For example, let

$$H_k(s) = \frac{s + 2}{(s + 1)\left(s + 2 + \frac{1}{k}\right)}.$$

Then $\Delta_k(s) = (s + 1)\left(s + 2 + \frac{1}{k}\right) \to (s + 1)(s + 2)$, but $H_k(s) \to \frac{1}{s + 1}$.

Next we examine some basic properties of the CP with respect to the time-scale decomposition. First we need a simple result for individual systems.

LEMMA 4.4. *Suppose* $H = H_1 + H_2$, *where* H_1 *and* H_2 *have no common poles, and let* Δ, Δ_1, and Δ_2 *be the characteristic polynomials of* H, H_1, *and* H_2, *respectively. Then* $\Delta = \Delta_1\Delta_2$.

Proof. From the definition of the CP we can assume without loss of generality that H_1 and H_2 are strictly proper. Suppose (A_1, B_1, C_1) and (A_2, B_2, C_2) are minimal realizations of H_1 and H_2; then $\Delta_i(s) = \det(sI - A_i)$. If we let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad C = [C_1 \quad C_2],$$

then (A, B, C) is a minimal realization of H with CP

$$\Delta(s) = \det(sI - A_1)\det(sI - A_2). \quad \blacksquare$$

In particular, for any time-scale decomposition, Lemma 4.4 implies that, when k is sufficiently large, we have

$$\Delta_k = \Delta_{sk}\Delta_{fk}, \tag{12}$$

where Δ_k, Δ_{sk}, and Δ_{fk} are the characteristic polynomials of H_k, H_{sk}, and H_{fk}, respectively.

LEMMA 4.5. *Let $H_k = H_{sk} + H_{fk}$ be a time-scale decomposition.*

(1) *If Δ_{sk} is the characteristic polynomial of H_{sk} and $\Delta \in \mathbb{R}[s]$ is monic, then $\langle\Delta_{sk}\rangle \to \langle\Delta\rangle$ if and only if $\Delta_{sk} \to \Delta$.*

(2) *If $\{\rho_{fk}\}$ is the* CP *of H_{fk}, then $\rho_{fk} \to \langle 1\rangle$.*

(3) *The* CP *of $\{H_k\}$ is convergent if and only if the* CP *of $\{H_{sk}\}$ is convergent. When the two* CP's *converge, their limits coincide.*

Proof. (1) Sufficiency is obvious. To show necessity, observe that there must exist a real sequence $\{\gamma_k\}$ such that $\gamma_k\Delta_{sk} \to \Delta$ and that $\{\Delta_{sk}\}$ has bounded roots. From (3), $\{\gamma_k\}$ converges. Since Δ_{sk} and Δ are monic, $\gamma_k \equiv 1$.

(2) We have $\rho_{fk} = \langle\Delta_{fk}\rangle$, where

$$\Delta_{fk} = \begin{cases} \prod_i (s + \lambda_{ik}), & H_{fk} \text{ is not a polynomial matrix.} \\ 1, & H_{fk} \text{ is a polynomial matrix.} \end{cases}$$

Here the λ_{ik} satisfy the property that, for every $M < \infty$, there exists a $K < \infty$ such that $|\lambda_{ik}| > M$ for each i and each $k > K$. Let

$$\gamma_k = \begin{cases} \prod_i \dfrac{1}{\lambda_{ik}}, & H_{fk} \text{ is not a polynomial matrix,} \\ 1, & H_{fk} \text{ is a polynomial matrix.} \end{cases}$$

Then $\gamma_k\Delta_{fk} \to 1$ in $\mathbb{R}[s]$.

(3) The result follows immediately from (12) and part 2. ▌

The final result of this section focuses on the observation made in Example 4.3, part 1, that the CP of a sequence $\{H_k\}$ in $\mathbb{R}^{rm}(s)$ which is not convergent can sometimes be decomposed into convergent subsequences. We can in fact demonstrate that a finite decomposition of this sort can always be achieved.

THEOREM 4.6. *If $H_k \to H$, then $\{\rho_k\}$ has finitely many limit points.*

Proof. From Lemma 4.5, part 3, we need only consider the case where $\{H_k\}$ is a slow sequence. Let $H_k = N_k/d_k$ and $H = N/d$. From the definition of the characteristic polynomial, Δ_k divides $d_k^{\min\{r,m\}}$ for each k. But $d_k^{\min\{r,m\}} \to d^{\min\{r,m\}}$, so the unique monic representation of each limit point of $\{\rho_k\}$ must divide $d^{\min\{r,m\}}$. The result then follows from the fact that any polynomial over $\mathbb{R}$ has finitely many monic divisors. ∎

COROLLARY 4.7. *If $H_k \to H$, then there exist finitely many strictly increasing sequences $\{k_j^i\}$, $i = 1, \ldots, \pi$, of positive integers such that*

(1) $\{k_j^i | i = 1, 2, \ldots, \pi; j = 1, 2, \ldots\} = \{1, 2, 3, \ldots\}$,

(2) $\{k_j^p | j = 1, 2, \ldots\} \cap \{k_j^q | j = 1, 2, \ldots\} = \phi$ *when* $p \neq q$,

(3) *each* $\{H_{k_j^i}\}$ *has convergent* CP.

Proof. From Theorem 4.6, there are only finitely many limit points $\rho^1, \ldots, \rho^\pi \in \mathbb{P}^\eta$ of $\{\rho_k\}$. Since $\mathbb{P}^\eta$ is a compact Hausdorff space, each open subset U of $\mathbb{P}^\eta$ satisfying $\{\rho^1, \ldots, \rho^\pi\} \subset U$ contains a tail of $\{\rho_k\}$. Indeed, otherwise there would exist a subsequence of $\{\rho_k\}$ with no limit point, contradicting compactness of $\mathbb{P}^\eta$. Let $U_1, \ldots, U_\pi$ be nonintersecting neighborhoods of $\rho^1, \ldots, \rho^\pi$, respectively; then there exists a $K < \infty$ such that $\{\rho_k\} \subset \cup U_i$ for $k > K$. Let $k_j^1 = j; j = 1, \ldots, K$. The remaining k_j^i may then be defined iteratively according to $k_j^i = \min(\{k | \rho_k \in U_i\} - \{k_q^i | q < j\})$. If $V_i \subset U_i$ is another neighborhood of ρ_i, then by compactness of $\mathbb{P}^\eta$ there must be a tail of the subsequence $\{\rho_{k_j^i}\}$ contained in V_i; hence, $\rho_{k_j^i} \to \rho^i$. ∎

5. EXISTENCE OF REALIZATIONS

We are now ready to formally define realizations of a given transfer matrix sequence $\{H_k\}$ and discuss their existence. We base our definition of a realization of $\{H_k\}$ on the standard definition of a realization of a single rational matrix H as in Theorem 1.2.

DEFINITION 5.1. (1) Suppose $\{H_k\}$ converges in $\mathbb{R}(s)^{rm}$. We say a sequence $\{\sigma_k\}$ in $\mathbb{R}^{n(2n+m+r)}$ is a *realization* of $\{H_k\}$, if there exists an integer K and a $\sigma \in \mathbb{R}^{n(n+m+r)}$ such that σ_k is a realization of H_k when $k > K$ and $\sigma_k \to \sigma$ in $\mathbb{R}^{n(2n+m+r)}$.

(2) A realization $\{\sigma_k\}$ in $\mathbb{R}^{n(2n+m+r)}$ is said to have *dimension n*.

Note that the dimension of a realization $\{\sigma_k\}$ is given simply by dim σ_k for any k. If $H_k \to H$, then continuity of $\mathscr{H}$ implies that σ is a realization of H in the conventional sense. We show that there exists a realization for any convergent sequence $\{H_k\}$; this generalizes part 1 of Theorem 1.2

to sequences and demonstrates that the definition of convergence in $\mathbb{R}(s)^{rm}$ outlined in Section 2 is the correct one for our purposes.

To simplify subsequent discussion, we make use of the mapping $\mathcal{G}$: $\mathbb{R}(s)^{rm} \to \mathbb{R}(s)^{rm}$ defined by $\mathcal{G}(H)(s) = -(1/s)H(1/s)$. It is easy to see that $\mathcal{G}$ is an isomorphism on $\mathbb{R}(s)^{rm}$ and that $\mathcal{G}^{-1} = \mathcal{G}$. Some elementary properties of $\mathcal{G}$ follow.

LEMMA 5.2. *Let $H \in \mathbb{R}(s)^{rm}$, and let $\{H_k\}$ be convergent in $\mathbb{R}(s)^{rm}$.*

(1) *(E, A, B, C) is a realization of H if and only if (A, E, B, C) is a realization of $\mathcal{G}(H)$.*

(2) $\mu(H) = \mu(\mathcal{G}(H))$.

(3) *If the characteristic polynomial of H is $\Delta(s) = s^n + \eta_{n-1}s^{n-1} + \cdots + \eta_0$, then the characteristic polynomial of $\mathcal{G}(H)$* is $\tau(s) = \gamma(\eta_0 s^n + \cdots + \eta_{n-1}s + 1)$ *for some $\gamma \neq 0$.*

Proof. (1) Suppose $H(s) = C(sE - A)^{-1}B$. Then $\mathcal{G}(H)(s) = -(1/s)C(1/s)E - A)^{-1}B = C(sA - E)^{-1}B$.

(2) From part 1, if H has a realization of degree n, then so does $\mathcal{G}(H)$. The converse follows from $\mathcal{G}(\mathcal{G}H)) = H$.

(3) Let (E, A, B, C) be a minimum realization of H; then (A, E, B, C) is a minimal realization of $\mathcal{G}(H)$. Hence, $\Delta(s) = \gamma_1 \det(sE - A)$ for some $\gamma_1 \neq 0$ (for details, see [14]), and

$$\begin{aligned}\tau(s) &= \gamma_2 \det(sA - E) = \gamma_2(-s)^n \det\left(\frac{1}{s}E - A\right)\\ &= \frac{\gamma_2}{\gamma_1}(-s)^n\left(\left(\frac{1}{s}\right)^n + \eta_{n-1}\left(\frac{1}{s}\right)^{n-1} + \cdots + \eta_0\right)\\ &= (-1)^n \cdot \frac{\gamma_2}{\gamma_1}(\eta_0 s^n + \cdots + \eta_{n-1}s + 1). \quad \blacksquare\end{aligned}$$

LEMMA 5.3. *If $\{H_k\}$ is a fast sequence, then $\{\mathcal{G}(H_k)\}$ is a slow sequence.*

Proof. Since all poles of H_k tend to infinity, we can write

$$H_k(s) = \frac{N_k(s)}{\Pi_{i=1}^{p}(\alpha_{ik}s - 1)},$$

where $\{N_k\}$ is convergent and each $\alpha_{ik} \to 0$ as $k \to \infty$. Let $q = \max_k\{\deg n_{ijk}, p - 1\}$. Then

$$\mathcal{G}(H_k)(s) = (-1)^{p+1}\frac{s^q N_k(1/s)}{s^{q-p+1}\Pi(s - \alpha_{ik})}.$$

Note that $s^q N_k(1/s)$ is a polynomial matrix, each of whose elements has degree at most q. Clearly, $\mathcal{G}(H_k)$ has bounded poles, and the demoninator of $\mathcal{G}(H_k)$ has degree $q + 1$, so $\mathcal{G}(H_k)$ has bounded poles, and the denominator of $\mathcal{G}(H_k)$ is strictly proper. ▌

Now consider a time-scale decomposition $H_k = H_{sk} + H_{fk}$ of an arbitrary $\{H_k\}$ in $\mathbb{R}(s)^{rm}$. Suppose $\{H_{sk}\}$ and $\{\mathcal{G}(H_{fk})\}$ have realizations of the form $\{(I, A_{sk}, B_{sk}, C_{sk})\}$ and $\{(I, A_{fk}, B_{fk}, C_{fk})\}$. Then each $H_{fk} = \mathcal{G}(\mathcal{G}(H_{fk}))$ has $(A_{fk}, I, B_{fk}, C_{fk})$ as one of its realizations. Defining

$$E_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \qquad A_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}, \qquad B_k = \begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix}, \qquad C_k = [C_{sk}\, C_{fk}],$$

it is easy to check that $\{(E_k, A_k, B_k, C_k)\}$ is a realization of $\{H_k\}$. Therefore, we only need to prove existence of realizations for slow sequences.

THEOREM 5.4. *Every slow sequence has a realization of the form* $\{(I, A_k, B_k, C_k)\}$.

Proof. First we treat the case $r = m = 1$. Let $H_k = N_k/d_k$, where $\{N_k\}$ and $\{d_k\}$ are convergent in $\mathbb{R}[s]$. Then $d_k = s^q + \alpha_{q-1,k}s^{q-1} + \cdots + \alpha_{0k}$ and $N_k = \beta_{q-1,k}s^{q-1} + \beta_{q-2,k}s^{q-2} + \cdots + \beta_{0k}$, where α_{ik} and β_{ik} converge as $k \to \infty$. To obtain a realization of $\{H_k\}$ of the desired form, set

$$A_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_{0k} & -\alpha_{1k} & \cdots & -\alpha_{q-1,k} \end{bmatrix}, \qquad B_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C_k = [\beta_{0k} \quad \beta_{1k} \cdots \beta_{q-1,k}].$$

Now we consider the general case. By the definition of convergence in $\mathbb{R}(s)^{rm}$, every component sequence $\{h_{ijk}\}$ is convergent. Suppose $\{(I, A_k^{ij}, B_k^{ij}, C_k^{ij})\}$ is a realization of $\{h_{ijk}\}$. Let

$$A_k^i = \operatorname{diag}\{A_k^{ij} | j = 1, 2, \ldots, m\}, \qquad B_k^i = \operatorname{diag}\{B_k^{ij} | j = 1, 2, \cdots, m\},$$

$$C_k^i = [C_k^{i1} \cdots C_k^{im}]$$

and

$$A_k = \operatorname{diag}\{A_k^i | i = 1, 2, \ldots, r\}, \qquad B_k = \begin{bmatrix} B_k^1 \\ \vdots \\ B_k^r \end{bmatrix},$$

$$C_k = \operatorname{diag}\{C_k^i | i = 1, 2, \cdots, r\}.$$

A simple calculation verifies that $\{(I, A_k, B_k, C_k)\}$ is a realization of $\{H_k\}$. ▌

Combining the time-scale decomposition with Lemma 5.3 and Theorem 5.4, we arrive at the following result.

COROLLARY 5.5. *Every convergent sequence in* $\mathbb{R}(s)^{rm}$ *has a realization.*

Theorem 5.4 (but not Corollary 5.5) may also be proven in an abstract algebraic framework as outlined in [17, Chap. 4]. Briefly, consider the commutative ring c of convergent sequences in $\mathbb{R}$ using pointwise operations, and let the set of $r \times m$ proper rational real matrices be denoted by $\mathbb{R}_p(s)^{rm}$. A convergent sequence $\{H_k\}$ in $\mathbb{R}_p(s)^{rm}$ may then be viewed as a formal power series over the ring of $r \times m$ matrices with elements in c. Indeed, we may expand each element of each H_k about $s = \infty$, yielding the series

$$H_k = \sum_{i=1}^{\infty} \left(\frac{1}{s}\right)^i H_{ik}, \tag{13}$$

where the sequences $\{H_{ik}\}$ are convergent. From this point there are two ways to proceed. First, one can prove realizability by constructing a certain infinite-dimensional Hankel matrix from the H_{ik}. It must then be shown that the span of the columns of the Hankel matrix is a finitely generated module over c. A second approach is to show that the formal power series (13) is "rational" in a certain algebraic sense. This immediately guarantees realizability. Both conditions can be demonstrated in our framework fairly easily; however, our proof of Theorem 5.4 is more direct and is sufficient for our purposes.

6. MINIMALITY

In section 5 we showed that every convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$ has a convergent realization $\{\sigma_k\}$. In this section, we explore the issue of minimality of a realization.

DEFINITION 6.1. (1) If n is the smallest integer such that $\{H_k\}$ has a realization of dimension n, and $\{\sigma_k\}$ is a realization of $\{H_k\}$ with dim $\sigma_k = n$, then we say $\{\sigma_k\}$ is a *minimal realization* of $\{H_k\}$.

(2) If a sequence of state-space systems $\{\sigma_k\}$ is a minimal realization of its transfer matrix sequence, we say $\{\sigma_k\}$ is *minimal.*

Obviously, all minimal realizations of $\{H_k\}$ have the same dimension. This fact enables us to define a degree function δ on the set of convergent rational matrix sequences by setting $\delta\{H_k\}$ equal to the dimension of any minimal realization of $\{H_k\}$. In this section we develop a simple expression for $\delta\{H_k\}$ for slow sequences and then extent it to the general case. Next, we examine a natural conjecture for determining whether a sequence $\{\sigma_k\}$ is minimal and relate minimal realizations of the same $\{H_k\}$ in a manner analogous to Theorem 1.2, part 4. Finally, we relate our results to the realization theory outlined in [17] for algebraic systems over the ring c.

In our development it is helpful to exploit various properties of the mapping which takes each state-space system into a particular choice of numerator and denominator of its transfer function. Specifically, define $\Gamma_n : \mathbb{R}^{n(n+m+r)} \to \mathbb{R}^{n(rm+1)+1}$ according to $\Gamma_n(A, B, C) = (C \cdot \operatorname{adj}(sI - A)B, \det(sI - A))$.

Here we have identified $\mathcal{R}_k$, as defined in Remark 3 after Definition 2.1, with $\mathbb{R}^k$. Note that Γ_n is continuous; if $\Gamma_n(A, B, C) = (N, d)$, then (I, A, B, C) is a realization of N/d. Also note the distinction between Γ_n and $\mathcal{H}$, as defined in Section 2. We denote $\operatorname{Im} \Gamma_n = \Gamma_n(\mathbb{R}^{n(n+m+r)})$.

The following series of lemmas leads us to the first main thorem of this section.

LEMMA 6.2. *Consider any pair (N, d) where d is monic,* $\deg d = n$, *and N/d is strictly proper with characteristic polynomial Δ. Then $(N, d) \in \operatorname{Im} \Gamma_n$ if and only if Δ divides d.*

Proof (Sufficient). Suppose $d(s) = \Delta(s) \prod_{i=1}^{p} (s + \beta_i)$, and let (I, A, B, C) be a minimal realization of N/d. Define

$$A^* = \begin{bmatrix} A & 0 \\ 0 & \Sigma \end{bmatrix}, \qquad B^* = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad C^* = [C \quad 0],$$

where

$$\Sigma = \begin{bmatrix} -\beta_1 & & & \\ & -\beta_2 & & \\ & & \ddots & \\ & & & -\beta_p \end{bmatrix}.$$

Then $\det(sI - A^*) = \det(sI - A)\det(sI - \Sigma) = \Delta(s) \prod_{i=1}^{p} (s + \beta_i) = d(s)$. Since $C^*(sI - A^*)^{-1}B^* = C(sI - A)^{-1}B = N(s)/d(s)$, $C^* \operatorname{adj}(sI - A^*)B^* = \det(SI - A^*)(N(s)/d(s) = N(s)$. Hence, $(N, d) = \Gamma_n(A^*, B^*, C^*)$.

(Necessary) Suppose $(N, d) = \Gamma_n(A, B, C)$; then (I, A, B, C) is a realization of N/d. From [16, Theorems 5–18], we can find a similarity transformation T such that

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \qquad CT = [0 \quad C_2 \quad C_3],$$

where (A_{22}, B_2, C_2) is a minimal realization of N/d. Note that $\det(sI - A_{22}) = \Delta(s)$. Thus $d(s) = \det(sI = A) = \det(sI - A_{11}) \det(sI - A_{33}) \Delta(s)$. ▮

From Corollary 4.7, $\{H_k\}$ can be decomposed into π sequences $\{H_k^{(i)}\}$, $i = 1, 2, \ldots, \pi$, where we define $H_j^{(i)} = H_{k_j^i}$. Each sequence has convergent CP satisfying $\langle\Delta_k^{(i)}\rangle \to \langle\Delta^{(i)}\rangle$, $i = 1, 2, \ldots, \pi$, where $\Delta^{(i)}$ is monic. If $\{H_k\}$ is slow, then, from Lemma 4.5, part 1, $\Delta_k^{(i)} \to \Delta^{(i)}$. Let $\hat{\Delta} = \text{LCM}\,\{\Delta^{(i)}, \ldots, \Delta^{(\pi)}\}$, where LCM denotes the least common multiple. Also define.

$$\hat{\Delta}_k^{(i)} = \Delta_k^{(i)} \frac{\hat{\Delta}}{\Delta^{(i)}}, \qquad i = 1, 2, \ldots, \pi. \tag{14}$$

Note that each $\hat{\Delta}_k^{(i)}$ is a polynomial and that, if $\{H_k\}$ is slow, $\hat{\Delta}_k^{(i)} \to \Delta$.

LEMMA 6.3. *Let $\{H_k\}$ be a slow sequence with $H_k \to H$, and suppose H has characteristic polynomial Δ. Then Δ divides $\hat{\Delta}$.*

Proof. According to Corollary 4.7, $\{H_k\}$ can be decomposed into π subsequences with convergent CP's $\Delta_k^{(i)} \to \Delta^{(i)}$. If Δ divides $\Delta^{(i)}$ for each i, then Δ divides $\hat{\Delta}$. Hence, it suffices to treat the case where $\{H_k\}$ has convergent CP $\Delta_k \to \hat{\Delta}$.

Let $p = \min\{r, m\}$, and consider the sequence $\{\tilde{H}_k\}$ of $1 \times \sum_{i=1}^{p} \binom{r}{i}\binom{m}{i}$ rational matrices, each $\tilde{H}_k$ consisting of the minors of H_k of all orders. Obviously, $\tilde{H}_k \to \tilde{H}$, where $\tilde{H}$ is defined similarly. It follows from elementary arguments that $\tilde{H}_k$ has characteristic polynomial Δ_k (same as H_k) and that, for any polynomial q, $q\tilde{H}_k$ is a polynomial matrix if and only if Δ_k divides q. In particular, $\Delta_k\tilde{H}_k$ is a polynomial matrix. Since $\mathbb{R}[s]^j$ is closed in $\mathbb{R}(s)^j$ for any j, $\hat{\Delta}\tilde{H}$ is a polynomial matrix. Thus, the characteristic polynomial Δ of $\tilde{H}$ (and H) divides $\hat{\Delta}$. ▮

LEMMA 6.4. *Γ_n is an open mapping.*

Proof. Note that Γ_n is multilinear; thus, it is a composition of functions on Euclidean spaces $\mathbb{R}^p$ of the form $f(x_1, \ldots, x_p) = \pm x_i x_j$ and $g(x_1, \ldots, x_p) = x_1 + \cdots + x_p$. Since f and g are open, compositions of open maps are open, and products of opens sets are open, it follows that Γ_n is open. ▮

LEMMA 6.5. *Let X and Y be topological spaces with X first countable, and let Q: $X \to Y$ be an onto, open, and continuous map. For any conver-*

gent sequence $\{y_k\}$ *in* Y *with* $y_k \to y \in Y$ *and any* $x \in Q^{-1}(y)$, *there exist* $x_k \in Q^{-1}(y_k)$, $k = 1, 2, \ldots$, *such that* $x_k \to x$.

Proof. Let $\{U_i, i = 1, 2, \cdots\}$ be a countable basis of neighborhoods of x with $U_i \supset U_{i+1}$. Since Q is open, each $V_i = Q(U_i)$ is a neighborhood of y. Hence, for any V_i, we can find an integer K_i such that $y_k \in V_i$ when $k > K_i$. Furthermore, there must exist points $x_k^{(i)} \in U_i$, $k = K_i + 1, K_i + 2, \ldots$, with $Q(x_k^{(i)}) = y_k$. For $k \leq K_i$, select any $x_n^{(i)} \in Q^{-1}(y_n)$. This process defines sequences $\{x_k^{(i)}\}$, $i = 1,2, \ldots$. Without loss of generality, we may assume $K_{i+1} > K_i$. If we let $x_k = x_k^{(i)}$, $k = K_{i-1}+1, \ldots, K_i$, where $K_0 = 0$, the construction shows that each U_i contains a tail of the sequence $\{x_k\}$. Hence, $x_k \to x$. ▮

LEMMA 6.6. *Suppose* $\{H_k\}$ *is a slow sequence with* $H_k \to H$. *If there are pairs* (N_k, d_k), $(N, d) \in \operatorname{Im} \Gamma_n$ *such that* $N_k/d_k = H_k$, $N/d = H$, *and* $(N_k, d_k) \to (N, d)$, *then* $\{H_k\}$ *has an n-dimensional realization.*

Proof. Note that $\mathbb{R}^{n(n+m+r)}$ is first countable. Thus, if we restrict the range of Γ_n to $\operatorname{Im} \Gamma_n$, we may use Lemmas 6.4 and 6.5 and the fact that Γ_n is continuous to conclude that there exists a convergent sequence $(A_k, B_k, C_k) \to (A, B, C)$ in $\mathbb{R}^{n(n+m+r)}$ such that $\Gamma_n(A_k, B_k, C_k) = (N_k, d_k)$ and $\Gamma_n(A, B, C) = (N, d)$. Note that (I, A_k, B_k, C_k) is a realization of $N_k/d_k = H_k$, $k = 1, 2, \ldots$, and (I, A, B, C) is a realization of $N/d = H$. ▮

LEMMA 6.7. *If a slow sequence* $\{H_k\}$ *has an n-dimensional realization, then it has an n-dimensional realization of the form* $\{(I, A_k, B_k, C_k)\}$.

Proof. Suppose $\{H_k\}$ has a realization having dimension n. Since $\{H_k\}$ is slow, the decomposition (7)–(11) shows that $\{H_k\}$ is of the form $H_k(s) = C_{sk}(sI - A_{sk})^{-1}B_{sk}$, where A_{sk} is $q \times q$ with $q \leq n$. Define

$$A_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_k = \begin{bmatrix} B_{sk} \\ 0 \end{bmatrix}, \qquad C_k = [C_{sk} \quad 0]. \quad ▮$$

THEOREM 6.8. *For any slow sequence* $\{H_k\}$, $\delta\{H_k\} = deg\ \hat{\Delta}$.

Proof. Let $n_0 = \deg \hat{\Delta}$. We first show that there exists an n_0-dimensional realization of $\{H_k\}$. Let $\hat{N}_k^{(i)} = \hat{\Delta}_k^{(i)} H_k^{(i)}$ and $\hat{N} = \hat{\Delta} H$; then $\hat{N}_k^{(i)}$ and $\hat{N}$ are polynomial matrices with $(\hat{N}_k^{(i)}, \hat{\Delta}_k^{(i)}) \to (\hat{N}, \hat{\Delta})$. Since all poles of H_k are bounded, Remark 4 after Definition 2.1 shows that $\deg \hat{\Delta}_k^{(i)} = \deg \hat{\Delta} = n_0$. Thus, from (14) and Lemmas 6.2 and 6.3, $(\hat{N}_k^{(i)}, \hat{\Delta}_k^{(i)})$, $(\hat{N}, \hat{\Delta}) \in \operatorname{Im} \Gamma_{n0}$. Suppose $\{(\hat{N}_k, \hat{\Delta}_k)\}$ is constructed by setting $\hat{N}_{k_j^i} = \hat{N}_j^{(i)}$ and $\hat{\Delta}_{k_j^i} = \hat{\Delta}_j^{(i)}$ whenever $H_{k_j^i} = H_j^{(i)}$, where the k_j^i are defined by the composition of the following.

COROLLARY 4.7. *Then* $(\hat{N}_k, \hat{\Delta}_k) \in \operatorname{Im} \Gamma_{n_0}$. *The desired result follows from Lemma* 6.6.

It remains to show that n_0 is the minimal dimension over all realizations of $\{H_k\}$. Suppose $\{H_k\}$ has an n-dimensional realization. Then, from Lemma 6.7, it has an n-dimensional realization of the form $\{(I, A_k, B_k, C_k)\}$. Let $(N_k, d_k) = \Gamma_n(A_k, B_k, C_k)$ and $(N, d) = \Gamma_n(\lim(A_k, B_k, C_k))$; from Lemma 6.2, Δ_k divides d_k for every k. Letting $d_j^{(i)} = d_{k_j^i}$, $\Delta_k^{(i)}$ divides $d_k^{(i)}$. Since Γ_n is continuous, $d_k^{(i)} \to d$; thus, closure of $\mathbb{R}[s] \subset \mathbb{R}(s)$ guarantees that each $\Delta^{(i)}$ divides d. Thus $\hat{\Delta}$ divides d, and

$$n = \deg d \geq \deg \hat{\Delta} = n_0. \quad \blacksquare$$

The following result offers one method of calcualting $\delta\{H_k\}$ for an arbitrary convergent sequence $\{H_k\}$ in $\mathbb{R}(s)^{rm}$.

THEOREM 6.9. *If* $H_k = H_{sk} + H_{fk}$ *is a time-scale decomposition, then*

$$\delta\{H_k\} = \delta\{H_{sk}\} + \delta\{H_{fk}\}.$$

Proof. Suppose $\{(E_k, A_k, B_k, C_k)\}$ is a minimal realization of $\{H_k\}$. Appealing to (7)–(11), it suffices to show that $\{\sigma_{sk}\} = \{(I, A_{sk}, B_{sk}, C_{sk})\}$ and $\{\sigma_{fk}\} = \{(A_{fk}, I, B_{fk}, C_{fk})\}$ are minimal realizations of $\{H_{sk}\}$ and $\{H_{fk}\}$. Suppose there exists a realization $\{(\bar{\sigma}_{sk}\} = \{(\bar{E}_{sk}, \bar{A}_{sk}, \bar{B}_{sk}, \bar{C}_{sk})\}$ of $\{H_{sk}\}$ with $\dim \bar{\sigma}_{sk} < \dim \sigma_{sk}$. By Lemma 6.7, we may assume that $\bar{E}_{sk} = I$. Let

$$\bar{E}_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \qquad \bar{A}_k = \begin{bmatrix} \bar{A}_{sk} & 0 \\ 0 & I \end{bmatrix}, \qquad \bar{B}_k = \begin{bmatrix} \bar{B}_{sk} \\ B_{fk} \end{bmatrix}, \qquad \bar{C}_k = [\bar{C}_{sk} \quad C_{fk}].$$

Then $\{\bar{\sigma}_k\} = \{(\bar{E}_k, \bar{A}_k, \bar{B}_k, \bar{C}_k)\}$ is a realization of $\{H_k\}$ with $\dim \bar{\sigma}_k < \dim \sigma_k$. This is a contradiction. A similar argument shows the minimality of $\{\sigma_{fk}\}$. $\blacksquare$

Thus, one way to find $\delta\{H_k\}$ is to first perform a time-scale decomposition $H_k = H_{sk} + H_{fk}$ and then to use Theorem 6.8 to find $\delta\{H_{sk}\}$ and $\delta\{H_{fk}\} = \delta\{\mathcal{G}(H_{fk})\}$. Fortunately, the next theorem simplifies this task and shows how to calculate $\delta\{H_k\}$ without resorting to time-scale decomposition. Recall that, for any $H \in \mathbb{R}(s)^{rm}$, H_f denotes the polynomial part of H.

THEOREM 6.10. *Suppose* $H_k \to H$. *Then* $\delta\{H_k\} = \max_i \overline{\lim}_k (\deg \hat{\Delta}_k^{(i)} + \mu((H_k^{(i)})_f))$.

Proof. Suppose $H_k = H_{sk} + H_{fk}$ is a time-scale decomposition of $\{H_k\}$, and let

$$H_{sj}^{(i)} = H_{sk_j^i}, \qquad H_{fj}^{(i)} = H_{fk_j^i}.$$

It is clear that $H_k^{(i)} = H_{sk}^{(i)} + H_{fk}^{(i)}$ is also a time-scale decomposition and that $H_{sk}^{(i)}$ and $H_{fk}^{(i)}$ have characteristic polynomials $\Delta_{sk}^{(i)}$ and $\Delta_{fk}^{(i)}$, respectively. From (12),

$$\Delta_k^{(i)} = \Delta_{sk}^{(i)} \Delta_{fk}^{(i)}.$$

From (15) and Lemma 4.5, part 2, $\Delta_{sk}^{(i)} \to \Delta^{(i)}$. Hence, from Theorem 6.8, $\delta\{H_{sk}\} = \deg \hat{\Delta}$. Also, since each $\Delta_{sk}^{(i)}$ is convergent and monic, $\deg \Delta_{sk}^{(i)} = \deg \Delta^{(i)}$ for large k.

From (3), $\langle \Delta_{fk} \rangle$ is of the form $\langle \Delta_{fk} \rangle = \langle \prod_{i=1}^{p} (\alpha_{ik} s - 1) \rangle = \langle \varepsilon_{pk} s^p + \cdots + \varepsilon_{1k} s + 1 \rangle$, where $\varepsilon_{ik} \to 0$ as $k \to \infty$. Note that some of the ε_{ik} may vanish: so from Lemma 5.2, part 3, $\mathscr{G}(H_{fk})$ has characteristic polynomial of the form $\tau_k(s) = s^q + \varepsilon_{1k} s^{q-1} + \cdots + \varepsilon_{qk}$, where q may depend on k. Therefore, the limit of any convergent subsequence of $\{\tau_k\}$ is of the form s^q. Suppose $\hat{\tau}$ is the least common multiple of these limits; then $\hat{\tau} = s^{\hat{q}}$, where $\hat{q} = \overline{\lim}_k \deg(\tau_k) = \overline{\lim}_k \mu(\mathscr{G}(H_{fk})) = \overline{\lim}_k \mu(H_{fk})$. The last equality is obtained from Lemma 5.2, part 2. Arguing as in Lemma 5.2, part 2, $\delta\{\mathscr{G}(H_k)\} \doteq \delta\{H_k\}$ for any $\{H_k\}$. Hence, from Theorem 6.8, we have $\delta\{H_{fk}\} = \delta\{\mathscr{G}(H_{fk})\} = \deg \hat{\tau} = \hat{q}$. Theorem 1.2, part 2, and Theorem 6.8 show that $\mu(H_{fk}) = \deg \Delta_{fk} + \mu(H_k)_f)$. From Theorem 6.9,

$$\begin{aligned} \delta\{H_k\} &= \deg \hat{\Delta} + \overline{\lim} (\deg \Delta_{fk} + \mu((H_k)_f)) \\ &= \overline{\lim_k} (\deg \hat{\Delta} + \deg \Delta_{fk} + \mu((H_k)_f)) \\ &= \max_i \overline{\lim_k} (\deg \hat{\Delta} + \deg \Delta_{fk}^{(i)} + \mu((H_k^{(i)})_f)). \end{aligned}$$

It remains to prove

$$\deg \hat{\Delta}_k^{(i)} = \deg \hat{\Delta} + \deg \Delta_{fk}^{(i)}. \tag{16}$$

By the definition of $\hat{\Delta}_k^{(i)}$,

$$\deg \hat{\Delta}_k^{(i)} = \deg \hat{\Delta} + \deg \hat{\Delta}_k^{(i)} - \deg \Delta^{(i)}. \tag{17}$$

Since $\deg \hat{\Delta}_{sk}^{(i)} = \deg \Delta^{(i)}$, it follows from (15) that

$$\deg \Delta_k^{(i)} = \deg \Delta^{(i)} + \deg \Delta_{fk}^{(i)}. \tag{18}$$

Combining (17) and (18), we obtain (16). ∎

COROLLARY 6.11. *Suppose $H_k \to H$. If the* CP *of $\{H_k\}$ is convergent, then*

$$\delta\{H_k\} = \overline{\lim_k} \mu(H_k).$$

Proof. In this case, $\pi = 1$ and $\hat{\Delta}_k^{(1)}$, so

$$\delta\{H_k\} = \overline{\lim_k}\,(\deg \Delta_k + \mu((H_k)_f)) = \overline{\lim_k}\,(\mu(H_k)_s + \mu((H_k)_f)) = \overline{\lim_k}\,\mu(H_k). \quad \blacksquare$$

Our next goal is to generalize part 3 of Theorem 1.2. An obvious conjecture is that a realization $\{\sigma_k\}$ of $\{H_k\}$ is minimal if and only if each σ_k is controllable and observable (as defined in [8]). While controllability and observability for every k (or, indeed, for infinitely many k) are clearly sufficient for minimality, the next examples demonstrate how necessity can fail.

EXAMPLE 6.12. (1) Even for a slow $\{H_k\}$, minimality of $\{\sigma_k\}$ does not imply controllability and observability even at a single point. Consider the sequence $\{H_k\}$ in Example 4.3, part 1. We can decompose $\{H_k\}$ into two subsequences

$$H_k^{(1)} = \frac{s+2}{(s+1)(s+2+(1/2k-1)}, \qquad H_k^{(2)} = \frac{s+3}{(s+1)(s+3+(1/2k)}.$$

For $H_k^{(1)}$ and $H_k^{(2)}$, the CP's are $\Delta_k^{(1)} = (s+1)(s+2+1/2k-1) \to (s+1)(s+2) = \Delta^{(1)}$ and $\Delta_k^{(2)} = (s+1)(s+3+1/2k) \to (s+1(s+3) = \Delta^{(2)}$. Thus $\hat{\Delta}(s) = \text{LCM}\,\{\Delta^{(1)}, \Delta^{(2)}\} = (s+1)(s+2)(s+3)$. Since $\{H_k\}$ is a slow sequence, Theorem 6.9 indicates that $\delta\{H_k\} = \deg \hat{\Delta} = 3$. Every minimal realization $\{\sigma_k\}$ of $\{H_k\}$ must have dimension 3, but $\mu(\sigma_k) = 2$ for each k; hence, no σ_k is controllable and observable.

(2) In this example, the CP converges, but controllability and observability on a subsequence is the most that can be achieved. Let

$$H_k(s) = \begin{cases} 1, & k \text{ even} \\ \dfrac{1}{((1/k)\,s+1)^2}, & k \text{ odd} \end{cases}.$$

A simple calculation yields $\delta\{H_k\} = 2$, but $\mu(H_k) = 1$ when k is even, so any realization must contain infinitely many terms which are not controllable and observable.

The next result offers a weak extension of Theorem 1.2, part 3, to the sequential case.

THEOREM 6.13. *Consider a convergent sequence* $\{\sigma_k\} = \{(E_k, A_k, B_k, C_k)\}$ *and let* $H_k = C_k(sE_k - A_k)^{-1}B_k$. *If the* CP *of* $\{H_k\}$ *converges and* $\{\sigma_k\}$ *is minimal, then there exists a subsequence* $\{\sigma_{k_j}\}$ *such that* σ_{k_j} *is controllable and observable for every* j. *If, in addition,* $\{H_k\}$ *is a slow sequence, then* $\lim \sigma_k$ *is controllable and observable, so* σ_k *is controllable and observable for every sufficiently large* k.

Proof. From Corollary 6.11, there exists a subsequence $\{H_{k_j}\}$ of $\{H_k\}$ such that $\delta\{H_k\} = \mu(H_{k_j})$ for all j. Therefore, each σ_{k_j} has dimension $\mu(H_{k_j})$ and must be controllable and observable. If $\{H_k\}$ is a slow sequence and the CP converges, Lemma 4.5, part 1, shows that $\Delta_k \to \Delta$, where Δ is the characteristic polynomial of $H = \lim H_k$. Since each Δ_k is monic, for large k we have $\mu(H_k) = \deg \Delta_k = \deg \Delta = \mu(H)$. From Corollary 6.11, $\dim \sigma = \delta\{H_k\} = \mu(H_k)$; hence, σ is controllable and observable. ∎

Restricting attention to slow sequences, Example 6.12, part 1, has special significance from an abstract algebraic perspective. It is easy to show that algebraic controllability and observability over the ring c of convergent real sequences, as defined in [17, Chap. 2], is equivalent to controllability and observability of σ_k for sufficiently large k. Thus, the linear systems over c do not satisfy the property that minimality implies algebraic controllability and observability.

We conclude this section by examining the problem of extending Theorem 1.2, part 4, to the sequential case. The following examples show that, in our case, two minimal realizations of the same sequence $\{H_k\}$ may not be related by nonsingular transformation (cf. [17, Theorem 4.19]).

EXAMPLE 6.14. (1) In fact, two minimal realizations may not be related by nonsingular transformation for any value of k. To see this, let $\{H_k\}$ again be the slow sequence given in Example 4.3, part 1. In Example 6.12, part 1, we showed that $\delta\{H_k\} = 3$. Consider the two minimal realizations $\{(I, A_{1k}, B_k, C_{1k})\}$ and $\{(I, A_{2k}, B_k, C_{2k})\}$, where

$$A_{1k} = \begin{cases} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\left(6 + \frac{3}{k}\right) & -\left(11 + \frac{4}{k}\right) & -\left(6 + \frac{1}{k}\right) \end{bmatrix}, & k \text{ even} \\[2ex] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\left(6 + \frac{2}{k}\right) & -\left(11 + \frac{3}{k}\right) & -\left(6 + \frac{1}{k}\right) \end{bmatrix}, & k \text{ odd} \end{cases}$$

$$A_{2k} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\left(6 + \frac{5}{k} + \frac{1}{k^2}\right) & -\left(11 + \frac{7}{k} + \frac{1}{k^2}\right) & -\left(6 + \frac{2}{k}\right) \end{bmatrix}$$

$$B_k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C_{1k} = [6 \quad 5 \quad 1],$$

$$C_{2k} = \begin{cases} \left[6 + \dfrac{2}{k} \quad 5 + \dfrac{1}{k} \quad 1\right], & k \text{ even} \\ \left[6 + \dfrac{3}{k} \quad 5 + \dfrac{1}{k} \quad 1\right], & k \text{ odd} \end{cases}$$

Suppose there exist convergent nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$ such that $M_k N_k = I$, $M_k A_{1k} N_{1k} = A_{2k}$, $M_k B_{1k} = B_{2k}$, $C_{1k} N_k = C_{2k}$ for each k. Then $N_k = M_k^{-1}$ and $A_{2k} = M_k A_{1k} M_k^{-1}$. But a simple calculation shows that A_{1k} and A_{2k} have different spectra, yielding a contradiction for each value of k.

(2) When the CP converges, Theorem 6.13 implies that there exists a subsequence on which every minimal realization is controllable and observable. Hence, Theorem 1.2, part 4, guarantees that for any two minimal realizations there exist nonsingular sequences $\{M_k\}$ and $\{N_k\}$ which relate the various matrices. However, it may not be possible to find $\{M_k\}$ and $\{N_k\}$ which converge. Consider the sequence given by

$$H_k(s) = \left(\frac{1}{k}\right) \bigg/ \left(\frac{1}{k} s + 1\right)$$

with realizations $(1/k, -1, 1/k, 1)$ and $(1/k, -1, 1, 1/k)$. We have immediately $M_k \cdot 1/k = 1$, so $M_k = k$.

However, there is still one interesting case where a result is possible.

THEOREM 6.15. *Suppose $\{H_k\}$ is a convergent sequence in $\mathbb{R}(s)^{rm}$ and let $\{(E_{ik}, A_{ik}, B_{ik}, C_{ik})\}$, $i = 1, 2$, be two minimal realizations of $\{H_k\}$ with $(E_{ik}, A_{ik}, B_{ik}, C_{iK})$ controllable and observable for every i, k. Further, assume that each $(E_i, A_i, B_i, C_i) = \lim (E_{ik}, A_{ik}, B_{ik}, C_{ik})$ is controllable and observable. Then there exist nonsingular matrix sequences $\{M_k\}$ and $\{N_k\}$ with $M_k \to M$ and $N_k \to N$, M and N nonsingular, such that $M_k E_{1k} N_k = E_{2k}$, $M_k A_{1k} N_k = A_{2k}$, $M_k B_{1k} = B_{2k}$, and $C_{1k} N_k = C_{2k}$ for every k.*

Proof. Applying the decomposition (7), (8) to $\{(E_{ik}, A_{ik}, B_{ik}, C_{ik})\}$ yields decomposing matrices $\tilde{M}_{ik} \to \tilde{M}_i$ and $\tilde{N}_{ik} \to \tilde{N}_i$ and decomposed system matrices A_{isk}, A_{ifk}, B_{isk}, B_{ifk}, C_{isk}, and C_{ifk}, with $\lim A_{ifk}$ nilpotent. This determines in two ways the same time-scale decomposition $H_k = H_{sk} +$

H_{fk} given by $H_{sk}(s) = C_{isk}(sI - A_{isk})^{-1}B_{isk}$, $H_{fk}(s) = C_{ifk}(sA_{ifk}-I)^{-1}B_{ifk}$. Note that, for sufficiently large k, each of the subsystems $(I, A_{isk}, B_{isk}, C_{isk})$ and $(A_{ifk}, I, B_{ifk}, C_{ifk})$ must be controllable and observable. From [16, p. 208], the similarity transformation $T_{sk} = (V_{2sk}^T V_{2sk})^{-1} V_{2sk}^T V_{1sk}$, where V_{isk} is the observability matrix of the pair (A_{isk}, C_{isk}), takes $(I, A_{1sk}, B_{1sk}, C_{1sk})$ into $(I, A_{2sk}, B_{2sk}, C_{2sk})$. Furthermore, $\{T_{sk}\}$ converges to the nonsingular matrix $T_s = (V_{2s}^T V_{2s})^{-1} V_{2s}^T V_{1s}$. A similar construction yields $T_{fk} \to T_f$. A straightforward calculation shows that the sequences

$$M_k = \tilde{M}_{2k}^{-1}\begin{bmatrix} T_{sk} & 0 \\ 0 & T_{fk} \end{bmatrix}\tilde{M}_{1k}, \qquad N_k = \tilde{N}_{1k}\begin{bmatrix} T_{sk}^{-1} & 0 \\ 0 & T_{fk}^{-1} \end{bmatrix}\tilde{N}_{2k}^{-1}$$

yield the desired result. ∎

Our final result follows with the aid of Theorem 6.13.

COROLLARY 6.16. *If* $\{H_k\}$ *is a slow sequence with convergent* CP *and* $\{(E_{ik}, A_{ik}, B_{ik}, C_{ik})\}$, $i = 1, 2$, *are any two minimal realizations, then, for sufficiently large k, there exist nonsingular matrix sequences* $\{M_k\}$ *and* $\{N_k\}$ *and nonsingular matrices M and N such that* $M_k \to M$, $N_k \to N$, $M_kE_{1k}N_k = E_{2k}$, $M_kA_{1k}N_k = A_{2k}$, $M_kB_{1k} = B_{2k}$, *and* $C_{1k}N_k = C_{2k}$ *for every k.*

7. Concluding Remarks

The problem discussed in this paper is the realization of convergent transfer matrix sequences with convergent generalized state-space sequences. Just as state-space sequences may be decomposed according to time-scale behavior, a time-scale decomposition for any rational matrix sequence may also be achieved. We have shown that convergence of the CP of a sequence of rational matrices is a crucial issue in the minimal realization problem. It was proved that, when the characteristic polynomial of a rational matrix sequence is not convergent, the rational sequence can be decomposed into finitely many subsequences in such a way that each subsequence has convergent CP. Our results demonstrate that the general problem can be reduced to finitely many subproblems, each of which can be handled using a simpler theory. It is hoped that our results will complement the robustness literature at large.

References

1. H. K. Khalil, On the robustness of output feedback control methods to modeling errors, *IEEE Trans. Automat. Control* **26** (1981).

2. H. K. KHALIL, A further note on the robustness of feedback control methods to modeling errors, *IEEE Trans. Automat. Control* **29** (1984).
3. M. VIDYASAGAR, Robust stabilization of singularly perturbed systems, *Systems Control Lett.* **5** (1985).
4. M. VIDYASAGAR, The graph metric for unstable plants and robustness estimates for feedback stability, *IEEE Trans. Automat. Control* **29** (1984).
5. J. D. COBB, Linear compensator designs based exclusively on input–output information are never robust with respect to unmodelled dynamics, *IEEE Trans. Automat. Control* **33** (June 1988), 559–563.
6. J. D. COBB, "Descriptor Variable and Generalized Singularly Perturbed Systems: A Geometric Approach," Ph.D. thesis, Department of Electrical Engineering, University of Illinois at Urbana–Champaign, 1980.
7. J. D. COBB, Global analyticity of a geometric decomposition for linear singularly perturbed systems, *Circuits, Systems, Signal Process.* **5,** No. 1, special issue on semistate systems (1986), 139–152.
8. J. D. COBB, Controllability, observability, and duality in singular systems, *IEEE Trans. Automat. Control* (Dec. 1984), 1076–1082.
9. M. HAZEWINKEL, On families of linear systems: Degeneration phenomena, *in* "Algebraic and Geometric Methods in Linear Systems Theory," Lectures in Applied Mathematics, Vol. 18, AMS, Providence, RI, 1980.
10. R. W. BROCKETT, Some geometric questions in the theory of linear systems, *IEEE Trans. Automat. Control* **21** (1976).
11. D. W. LUSE AND H. K. KHALIL, Frequency domain results for systems with slow and fast dynamics, *IEEE Trans. Automat. Control* **30** (1985).
12. D. W. LUSE, Frequency domain results for systems with multiple time scales, *IEEE Trans. Automat. Control* **31,** No. 10 (1986).
13. D. W. LUSE, State-space realization of multiple-frequency-scale transfer matrices, preprint.
14. G. VERGHESE, "Infinite-Frequency Behavior in Generalized Dynamical Systems," Ph.D. thesis, Department of Electrical Engineering, Stanford University, December 1978.
15. T. KAILATH, "Linear Systems," Prentice–Hall, Englewood Cliffs, NJ, 1980.
16. C. T. CHEN, "Linear System Theory and Design," Holt, Rinehart, and Winston, New York, 1984.
17. J. W. BREWER, J. W. BUNCE, AND F. S. VAN VLECK, "Linear Systems over Commutative Rings," Dekker, New York, 1986.
18. J. R. MUNKRES, "Topology: A First Course," Prentice–Hall, Englewood Cliffs, NJ, 1975.
19. J. DUGUNDJI, "Topology," Allyn and Bacon, Rockleigh, NJ.
20. Y. MATSUSHIMA, "Differentiable Manifolds," Dekker, New York.

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