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Topological Aspects of Controllability and Observability on the Manifold of Singular and Regular Systems

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In a previous article (J. D. Cobb, *J. Math. Anal Appl.*, Nov. 1986) we considered the class of all singular and regular linear time-invariant systems and proved some basic topological properties of that set. In this paper we examine specific implications of those results to control theory and demonstrate, among other things, that controllability and observability are generic properties even when singular systems are included in the construction. We also derive related results for other important subclasses of systems, proving that only some of the remaining fundamental system properties are generic. Finally, we extend existing results on connectedness and show that the number of connected components of the controllable and observable sets is diminished whenever singular systems are brought into the picture. © 1989 Academic Press, Inc.

1. INTRODUCTION

We study linear, time-invariant systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{n^2}$, $B \in \mathbb{R}^{nm}$, and $C \in \mathbb{R}^{pn}$. If E is singular, (1) is called a singular system (see [1–7]); otherwise (1) is a regular. We are interested in the topological properties of the set of all singular and regular systems. In particular, we wish to explore three central issues: (1) It is well-known [11] that the sets of all controllable, observable, and controllable and observable state-space systems are open and dense in the space $\mathbb{R}^{n(n+m+p)}$ of triples (A, B, C), using the natural Euclidean topology. We wish to prove analogous results for the class of systems (1). (2) Questions involving the connectedness of various subsets of $\mathbb{R}^{n(n+m+p)}$ and the space of rational matrices have been addressed in [12, 10]. We will extend those results.

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(3) Two distinct definitions of controllability have been proposed for singular systems [1, 4]; their relationships have been discussed in [6]. We claim that the definition in [4] is more acceptable from a topological viewpoint than that of [1]. A similar result will be proven for observability.

The study of topological issues is important in the areas of system identification, disturbance decoupling, and singular perturbations, as well as for achieving a fundamental understanding of dynamic processes. (For a discussion of these topics, see [13].) Another area of application, and our field of primary interest, is that of robust control. Robustness issues are characterized by system uncertainties which may, in many cases, be modelled as small perturbations in an appropriate topology. It is important to know how system properties behave under such perturbations. For example, when performing pole assignment for a state-space system, it is desirable that the system be controllable. Thus, an important question concerns whether a nominally controllable system retains controllability under small perturbations. In general, we would like to have each relevant system property hold on an open subset of the system space.

Many studies of topological aspects of systems have centered around transfer function descriptions [12–15]. Some work has been done with state-space representations [20, 21], but virtually none of it includes singular system representations. As shown in [9], this is due in part to the fact that the various geometric structures become much more complicated when singular systems are brought into the picture. Even in the case where $E = I$, some interesting questions involving controllability and observability of state-space representations can still be addressed [10].

Our work is closest to that of [13, 14] in that [13, 14] describe a method of "completing" the set of strictly proper rational matrices with respect to the time-domain behavior of the corresponding input–output operators. In [9] we propose a similar "completion" of the regular systems and obtain the space of all representations (1). More precise connections between our work and that [13, 14] can be found in Section 5.

We now briefly summarize the construction and relevant properties from [9] concerning the class of systems (1). To begin, we recall from [16] that a necessary and sufficient condition for existence and uniqueness of solutions in (1) for every initial condition and input is that the pencil (E, A) be regular, i.e.,

$$\det(sE - A) \not\equiv 0. \tag{3}$$

Non-square systems never exhibit existence and uniqueness of solutions and hence will not be considered. Let $\Sigma(n, m, p)$ be the open, dense subset of $\mathbb{R}^{n(2n+m+p)}$ consisting of all systems (1) satisfying (3). (The arguments n, m, and p will be dropped when they are clear from context.) We do not wish to work with Σ directly since it contains unnecessary redundancy. For example, a simple row interchange in (1) leads to a different point in Σ. The same holds for premultiplication by any nonsingular matrix; yet, in an intuitive sense, such transformations leave the system unaltered—not even a change of variables occurs. With these comments in mind, we define an equivalence relation on Σ according to

$$(E_1, A_1, B_1, C_1) \approx (E_2, A_2, B_2, C_2) \tag{4}$$

whenever $C_2 = C_1$, $E_2 = ME_1$, $A_2 = MA_1$, and $B_2 = MB_1$ for some nonsingular M. We denote by $\mathscr{L}(n, m, p)$ the resulting quotient set. One way to topologize $\mathscr{L}$ is to simply impose quotient set topology inherited from Σ. Equivalently, $\mathscr{L}$ receives identification topology inherited from the natural projection $\mu: \Sigma \to \mathscr{L}$.

Among the properties of $\mathscr{L}$ established in [9] are that (1) $\mathscr{L}$ is an analytic manifold of dimension $n(n+m+p)$, (2) the state-space systems can be naturally imbedded in $\mathscr{L}$ as an open, dense submanifold. (3) although $\mathscr{L}$ eliminates the redundancy of Σ, it distinguishes systems with distinct solutions and hence preserves internal structural information, (4) $\mathscr{L}$ is "complete" in the sense that distributional convergence of solutions implies convergence of system parameters, and (5) the projection map μ is a submersion and is therefore open.

Important subsets of Σ include the set Σ^i of all systems (1) with rank $E = i$ and the *singular systems*

$$\Sigma^s = \bigcup_{i=0}^{n-1} \Sigma^i.$$

Also of interest are the *slow controllable*, *fast controllable*, *controllable*, *slow observable*, *fast observable*, and *observable systems*: A 4-tuple (E, A, B, C) is slow controllable iff

$$\operatorname{rank}[\lambda E - A \quad B] = n$$

for every $\lambda \in \mathbf{C}$ and fast controllable iff

$$\operatorname{rank}[E \quad B] = n.$$

Denoting the corresponding sets of systems by Σ_{sc} and Σ_{fc}, the controllable systems are given by

$$\Sigma_c = \Sigma_{sc} \cap \Sigma_{fc}.$$

Dual expressions determine the slow observable, fast observable, and observable systems Σ_{so}, Σ_{fo}, and Σ_o. (For deeper system-theoretic interpretations of these definitions, see [6].)

The Weierstrass decomposition (see [16]) of (1) yields nonsingular matrices M and N such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & A_f \end{bmatrix}, \qquad MAN = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{2}$$

where A_f is nilpotent and

$$r = \deg \det(sE - A).$$

Let

$$\begin{bmatrix} B_s \\ B_f \end{bmatrix} = MB, \qquad [C_s \quad C_f] = CN.$$

This decomposition allows us to define the *impulse controllable* systems as those satisfying

$$\operatorname{Im} A_f + \operatorname{Ker} A_f + \operatorname{Im} B_f = \mathbb{R}^{n-r}.$$

Denote the corresponding set Σ_{ic}; Σ_{io} is defined similarly. (For further discussion of impulse controllability, see [6].)

It is easy to show that each of the classes Σ_{sc}, Σ_{fc}, etc. is invariant under the equivalence relation (4). Thus the projection map μ induces subsets $\mathscr{L}^i$, $\mathscr{L}_c$, $\mathscr{L}_o$, $\mathscr{L}_{fc}$, $\mathscr{L}_{ic}$, etc. in $\mathscr{L}$. In particular,

$$\mathscr{L}^s = \bigcup_{i=0}^{n-1} \mathscr{L}^i.$$

It is shown in [9] that the regular systems can be naturally identified with $\mathscr{L}^n = \mathscr{L} - \mathscr{L}^s$. We call $\mathscr{L}^n$ the *singular subspace* and $\mathscr{L}^s$ the *singular subspace*. It is the topological properties of the subsets of $\mathscr{L}$ that are of primary concern to us.

2. Openness

The determination of whether a class of systems constitutes an open subset of $\mathscr{L}$ is especially important from the viewpoint of robust control issues. Specifically, if the end product of a control system design satisfies some desirable property P, it is beneficial to have P invariant under small perturbations of the system, since any model inherently contains some parameter uncertainty. Topologically, this is equivalent to the property P systems' forming an open set.

In this section we consider each of the important subsets $\mathscr{L}_c$, $\mathscr{L}_o$, $\mathscr{L}_{sc}$, etc. individually and determine whether each is open in $\mathscr{L}$. If a subset is not open, we wish to characterize its interior. We begin with an easy result.

Let $[E, A, B, C] \in \mathscr{L}$ denote the equivalence class determined by the point $(E, A, B, C) \in \Sigma$; in other words,

$$[E, A, B, C] = \mu(E, A, B, C),$$

where $\mu: \Sigma \to \mathscr{L}$ is the natural projection. Further, let Σ^i denote the set of all $(E, A, B, C) \in \Sigma$ with rank $E = i$. Then $\bigcup \{\Sigma^i \mid i = k, ..., n\}$ is open and, since $\mathscr{L}$ inherits quotient set topology from Σ (see [17]),

$$\bigcup_{i=k}^{n} \mathscr{L}^i = \mu\left(\bigcup_{i=k}^{n} \Sigma^i\right)$$

is open in $\mathscr{L}$. In particular, the regular subspace $\mathscr{L}^n$ is open. It is well-known (see, e.g., [11]) that $\mathscr{L}_c \cap \mathscr{L}^n$, $\mathscr{L}_o \cap \mathscr{L}^n$, and $\mathscr{L}_{co} \cap \mathscr{L}^n$ are all open as well. Extending this idea to all of $\mathscr{L}$ requires a preliminary result.

LEMMA 2.1. *If $(E_k, A_k, B_k, C_k) \to (E, A, B, C) \in \Sigma$ and*

$$\operatorname{rank}[\lambda E - A \quad B] = n$$

for every $\lambda \in \mathbb{C}$, then for any $R < \infty$ there exists a $K < \infty$ such that $k > K$ implies

$$\operatorname{rank}[\lambda E_k - A_k \quad B_k] = n$$

for every λ satisfying $|\lambda| \leqslant R$.

Proof. Since E and A satisfy (3), we can write uniquely

$$\det(\lambda E - A) = \varphi \prod_{i=1}^{r} (\lambda - \lambda_i).$$

from Lemma 4.3 of [7],

$$\det(\lambda E_k - A_k) = \varphi_k \prod_{i=1}^{r} (\lambda - \lambda_k) \prod_{i=1}^{n-r} (\sigma_{ik}\lambda - 1),$$

where $\lambda_{ik} \to \lambda_i$ and $\sigma_{ik} \to 0$ and $k \to \infty$. Then, for some $K < \infty$, $k > K$ implies that for every i either $\sigma_{ik} = 0$ or $1/|\sigma_{ik}| > R$. Hence, we need only verify that

$$\operatorname{rank}[\lambda_{ik} E_k - A_k \quad B_k] = n$$

for sufficiently large k. But this must be true, since all entries of the matrix converge and since the rank n matrices form an open subset of $\mathbb{R}^{n(n+m)}$. ∎

THEOREM 2.2. *$\mathscr{L}_{fc}$, $\mathscr{L}_{fo}$, $\mathscr{L}_c$, and $\mathscr{L}_o$ are open.*

Proof. Since $\mathscr{L}$ inherits quotient set topology from $\mathbb{R}^{n(n+m+p)}$, a subset $W \subset \mathscr{L}$ is open iff $\mu^{-1}(W)$ is open. Hence, we need only consider the topology of 4-tuples $(E, A, B, C) \in \Sigma$. Consider a fast controllable point—i.e., one where $\operatorname{rank}[E \ \ B] = n$—and any sequence $(E_k, A_k, B_k, C_k) \to (E, A, B, C)$. Since the $n \times (n+m)$ matrices with rank n form an open subset of $\mathbb{R}^{n(n+m)}$, $\operatorname{rank}[E_k \ \ B_k] = n$ for sufficiently large k. Therefore, Σ_{fc} is open and so is $\mathscr{L}_{fc} = \mu(\Sigma_{fc})$. Openness of $\mathscr{L}_{fo}$ follows from similar arguments applied to the matrix $\left[\begin{smallmatrix} E \\ C \end{smallmatrix}\right]$.

To show that Σ_c and $\mathscr{L}_c$ are open, we consider a controllable point—i.e., one where

$$\operatorname{rank}[E \quad B] = \operatorname{rank}[\lambda E - A \quad B] = n$$

for every $\lambda \in \mathbf{C}$. Then for some integer $K_1 < \infty$, $k > K_1$ implies $\operatorname{rank}[E_k \ \ B_k] = n$. We note that for any $\lambda \neq 0$

$$[\lambda E_k - A_k \quad B_k] \begin{bmatrix} \frac{1}{\lambda} I & 0 \\ 0 & I \end{bmatrix} = [E_k \quad B_k] + \frac{1}{\lambda}[-A_k \quad 0]$$

and that the sequence A_k is bounded. Hence, there exists an $R < \infty$ such that $|\lambda| > R$ implies $\operatorname{rank}[\lambda E_k - A_k \ \ B_k] = n$ whenever $k > K_1$. From Lemma 2.1, there exists a $K_2 < \infty$ such that $\operatorname{rank}[\lambda E_k - A_k \ \ B_k] = n$ whenever $|\lambda| \leqslant R$ and $k > K_2$. Thus $k > \max\{K_1, K_2\}$ implies that $(E_k, A_k, B_k, C_k) \in \Sigma_c$ and $\mathscr{L}_c = \mu(\Sigma_c)$ is open. That $\mathscr{L}_o$ is open follows from dual arguments. ∎

We therefore have that all intersections and unions of $\bigcup \{\mathscr{L}^i \mid i = k, ..., n\}$ with the sets listed in Theorem 2.2 are also open. Unfortunately, not all subsets of interest have such a simple structure. Before proceeding with the details, we need to prove an important preliminary result which also lends further support to the definition of controllability for singular systems given in [4–6].

LEMMA 2.3. $\mathscr{L}_c = \operatorname{int}(\mathscr{L}_c \cup \mathscr{L}^s)$, $\mathscr{L}_o = \operatorname{int}(\mathscr{L}_o \cup \mathscr{L}^s)$.

Proof. We will prove analogous statements concerning Σ_c and Σ_s. To do so, consider any

$$\sigma = (E, A, B, C) = (\Sigma_c \cup \Sigma^s) - \Sigma_c = \Sigma^s - \Sigma_c.$$

We will show that there exists a sequence $\sigma_k \to \sigma$ with $\sigma_k \notin \Sigma_c \cup \Sigma^s$ for large k. Hence, σ is a boundary point of $\Sigma_c \cup \Sigma^s$; since Σ_c is open, the result follows.

Let $\sigma_k = (E-(1/k)A, B, C)$ and note that

$$\det\left(E-\frac{1}{k}A\right)=\frac{1}{k^n}\det(kE-A).$$

Thus, for sufficiently large k, $E-(1/k)A$ is nonsingular and $\sigma_k \notin \Sigma^s$. Since $\sigma \in \Sigma^s - \Sigma_c$, either $\operatorname{rank}[E\ \ B] < n$ or $\operatorname{rank}[\lambda E - A\ \ B] < n$ for some $\lambda \in \mathbf{C}$. For the first case, note that

$$\left[-k\left(E-\frac{1}{k}A\right)-A \quad B\right]\begin{bmatrix}-\frac{1}{k}I & 0\\ 0 & I\end{bmatrix}=[E\ \ B].$$

Hence, for $\lambda = -k$,

$$\operatorname{rank}\left[\lambda\left(E-\frac{1}{k}A\right)-A \quad B\right]<n$$

and $\sigma_k \notin \Sigma_c$. With regard to the second case, suppose rank $[\lambda_1 E - A\ \ B] < n$. For sufficiently large k we may choose

$$\lambda=\frac{\lambda_1 k}{k-\lambda_1}.$$

Then

$$\left[\lambda\left(E-\frac{1}{k}A\right)-A \quad B\right]\begin{bmatrix}\frac{k}{\lambda+k}I & 0\\ 0 & I\end{bmatrix}=[\lambda_1 E-A \quad B]$$

so

$$\operatorname{rank}\left[\lambda\left(E-\frac{1}{k}A\right)-A \quad B\right]<n$$

and $\sigma_k \notin \Sigma_c$.

The observability result is proven by similar reasoning. ∎

Lemma 2.3 shows that the definition of controllability given in [4–6] is "maximal" in a topological sense. Indeed, the result shows that $\mathscr{L}_c$ is the largest open subset of $\mathscr{L}$ containing the controllable regular systems, but no other regular systems. Equivalently, the set of uncontrollable systems $\mathscr{L} - \mathscr{L}_c$ is simply the closure in $\mathscr{L}$ of the uncontrollable regular systems. Similar remarks apply to observability. It is interesting that these maximal definitions were developed without such topological considerations being taken explicitly into account.

Now we use Lemma 2.3 to characterize slow controllability.

THEOREM 2.4. int $\mathscr{L}_{sc} = \mathscr{L}_c$, int $\mathscr{L}_{so} = \mathscr{L}_o$.

Proof. Clearly, $\mathscr{L}_c \subset \mathscr{L}_{sc} \subset \mathscr{L}_c \cup \mathscr{L}^s$. The result follows immediately from Lemma 2.3 and Theorem 2.2. ∎

To state the next result, we need to define the *order* of points in Σ and $\mathscr{L}$. If $\sigma = (E, A, B, C)$, let

$$\text{ord}\, \sigma = \deg \det(sE - A). \tag{5}$$

Since the right side of (5) is invariant under multiplication by M, we may unambiguously set ord $\xi =$ ord σ whenever $\xi = [\sigma]$. Further, define $\mathcal{O}^i = \{\xi \in \mathscr{L} \mid \text{ord}\, \xi = i\}$. From (5) it is easily seen that $\bigcup \{\mathcal{O}^i \mid i = k, ..., n\}$ is open for any k.

THEOREM 2.5. int $\mathscr{L}_{ic} = \mathscr{L}_{fc} \cup \mathcal{O}^{n-1}$, int $\mathscr{L}_{io} = \mathscr{L}_{fo} \cup \mathcal{O}^{n-1}$.

Proof. Since $\mathscr{L}_{fc} \supset \mathcal{O}^n$, $\mathscr{L}_{fc} \cup \mathcal{O}^{n-1} = \mathscr{L}_{fc} \cup (\mathcal{O}^n \cup \mathcal{O}^{n-1})$. But $\mathscr{L}_{fc}$ and $\mathcal{O}^n \cup \mathcal{O}^{n-1}$ are open; hence, we need only show that there exists $\sigma_k \to \sigma$ with $\sigma_k \notin \Sigma_{ic}$ for any $\sigma = (E, A, B, C) \in \Sigma_{ic} - \Sigma_{fc}$ with ord $\sigma < n - 1$. Invoking the Weierstrass decomposition (2), we may assume that A_f is in Jordan form

$$A_f = \begin{bmatrix} J_1 & & & \\ & \ddots & & \\ & & J_p & \\ & & & 0_{\alpha \times \alpha} \end{bmatrix},$$

where each J_i is nonzero and cyclic. Let

$$\begin{bmatrix} B_1 \\ \vdots \\ B_{p+1} \end{bmatrix} = B_{f'} \quad \begin{bmatrix} b_{i1} \\ \vdots \\ b_{ij_i} \end{bmatrix} = B_i, \qquad i = 1, ..., p.$$

Also, let

$$\tilde{B} = \begin{bmatrix} b_{1j_1} \\ b_{2j_2} \\ \vdots \\ b_{pj_p} \end{bmatrix}.$$

Since $\sigma \in \Sigma_{ic}$, $\tilde{B}$ has rank p (see [6]). However, $\sigma \notin \Sigma_{fc}$ so

$$\text{rank} \begin{bmatrix} \tilde{B} \\ B_{p+1} \end{bmatrix} < p + \alpha.$$

We consider two cases. First, if $p=0$, $n=\alpha$; then ord $\sigma<n-1$ guarantees $\alpha\geqslant 2$. Furthermore, rank $B_1<\alpha$ we may assume M was chosen such that

$$B_1=\begin{bmatrix} b_{11}\\ 0\\ b_{13}\\ \vdots\\ b_{1\alpha}\end{bmatrix}.$$

Setting

$$A_{fk}=\left[\begin{array}{c|c} 0 & \frac{1}{k}\quad 0\cdots 0\\ \hline 0 & \\ \vdots & 0_{(\alpha-1)\times(\alpha-1)}\\ 0 & \end{array}\right]$$

and

$$\sigma_k=\left(M^{-1}\begin{bmatrix} I & 0\\ 0 & A_{fk}\end{bmatrix}N^{-1}, A, B, C\right)$$

gives $\sigma_k\notin\Sigma_{ic}$.

If $p>0$, we must have $\alpha>0$, since otherwise $\operatorname{Im} A_f=\operatorname{Im} A_f+\operatorname{Ker} A_f$, which implies that impulse controllability and fast controllability are equivalent (see [6]). Here we can choose M such that

$$B_{p+1}=\begin{bmatrix} 0\\ b_{p+1,2}\\ \vdots\\ b_{p+1,\alpha}\end{bmatrix}.$$

Let

$$J_{pk}=\left[\begin{array}{c|c} & 0\\ & \vdots\\ J_p & 0\\ & 1/k\\ \hline 0\cdots 0 & 0\end{array}\right],\qquad A_{fk}=\begin{bmatrix} J_1 & & & & \\ & \ddots & & & \\ & & J_{p-1} & & \\ & & & J_{pk} & \\ & & & & 0_{(\alpha-1)\times(\alpha-1)}\end{bmatrix}$$

and

$$\sigma_k = \left(M^{-1} \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix} N^{-1}, A, B, C \right).$$

Again, it is easy to see that $\sigma_k \notin \Sigma_{ic}$. The analogous result for observability follows from dual arguments. ∎

It is important to note that Theorems 2.4 and 2.5 imply that $\mathscr{L}_{sc}$, $\mathscr{L}_{so}$, $\mathscr{L}_{ic}$, and $\mathscr{L}_{io}$ are not open; otherwise, we would have $\mathscr{L}_{sc} = \mathscr{L}_c$, $\mathscr{L}_{ic} = \mathscr{L}_{fc} \cup \mathcal{O}^{n-1}$, etc., which are clearly false (see [6]).

Since $\text{int}(\bigcap X_i) = \bigcap(\text{int}\, X_i)$ holds for any finite collection $\{X_i\}$ of sets, Theorems 2.4 and 2.5 may be used to calculate the interiors of a variety of subsets of $\mathscr{L}$ not explicitly mentioned.

3. Density

To prove that controllability and observability are generic properties, we still need to prove that the corresponding subsets are dense in $\mathscr{L}$. Actually, this is obvious since we know from [11] that $\mathscr{L}_{co} \cap \mathscr{L}^n$ is dense in $\mathscr{L}^n$ and, from our discussion in Section 1, that $\mathscr{L}^n$ is dense in $\mathscr{L}$. All subsets of interest are therefore dense in $\mathscr{L}$, since each contains $\mathscr{L}_{co} \cap \mathscr{L}^n$.

A more interesting question involves the topology of the singular subspace $\mathscr{L}^s$. This set is endowed with relative or subset topology inherited from $\mathscr{L}$. (In [9] it was shown that $\mathscr{L}^s$ is the union of n regular submanifolds of $\mathscr{L}$.) Questions of openness in $\mathscr{L}^s$ are trivial, given the results of Section 2 and the fact that a set of the form $X \cap \mathscr{L}^s$ is open in $\mathscr{L}^s$ whenever X is open in $\mathscr{L}$. Proving density in $\mathscr{L}^s$ is somewhat more difficult. We need one preliminary result.

LEMMA 3.1. *Suppose $A \in \mathbb{R}^{n^2}$ is singular and $b \in \mathbb{R}^n$. There exist sequences $A_k \to A$ and $b_k \to b$ such that, for every k, A_k is cyclic and singular and b_k is a cyclic generator for A_k.*

Proof. We need only consider A in Jordan form. Let $A_k = A + (1/k)\, N$, where

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}.$$

Then A_k is cyclic and singular. Since the cyclic generators of a given matrix are dense in $\mathbb{R}^n$, for each k there exists $b_k \in \mathbb{R}^n$ such that b_k is a cyclic generator for A_k and $\|b - b_k\| < 1/k$. Thus $b_k \to b$. ∎

THEOREM 3.2. *$\mathscr{L}_{co} \cap \mathscr{L}^{n-1}$ is dense in $\mathscr{L}^s$.*

Proof. Suppose $\xi = [E, A, B, C] \in \mathscr{L}^s$. The Weierstrass decomposition (2) gives matrices M, N, A_s, A_f, B_s, B_f, C_s and C_f. From [11], there exist sequences $A_{sk} \to A_s$, $B_{sk} \to B_s$, and $C_{sk} \to C$ such that (A_{sk}, B_{sk}, C_{sk}) is controllable and observable for every k. Let

$$B_f = [b_{f1} \cdots b_{fm}].$$

Lemma 3.1 guarantees the existence $A_{fk} \to A_f$, $b_k \to b_{f1}$ such that each A_{fk} is singular and cyclic and b_k is a syclic generator. Hence, (A_{fk}, B_{fk}) is controllable. By dual arguments, a sequence $C_{fk} \to C_f$ may be constructed so that (A_{fk}, C_{fk}) is observable. Since each subsystem is controllable and observable and their spectra are disjoint, the total system is controllable and observable for each k. Thus, the corresponding sequence ξ_k where

$$E_k = M^{-1} \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix} N^{-1}, \qquad A_k = M^{-1} \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix} N^{-1}$$

$$B_k = M^{-1} \begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix}, \qquad C_k = [C_{sk} \quad C_{fk}]\, N^{-1}$$

satisfies $\xi_k \to \xi$ and $\xi_k \in \mathscr{L}_{co} \cap \mathscr{L}^s$. That $\xi_k \in \mathscr{L}^{n-1}$ follows from cyclicity of A_{fk}. ∎

Hence, controllability and observability are generic properties not only of $\mathscr{L}$ but of the singular subspace $\mathscr{L}^s$ as well. The fact that any $\xi \in \mathscr{L}^s$ may further be approximated by points in $\mathscr{L}^{n-1}$ is a side benefit of the proof.

4. CONNECTEDNESS

It was shown in [12] that the space rat(n) of all strictly proper rational functions with degree n has $n+1$ connected components, indexed by the Cauchy index $\mathscr{I}(\cdot)$ (see [16]). We have addressed a similar problem in [10] for subsets of $\mathscr{L}^n$. We wish to extend the results of [10, 12] to include singular systems. It is our conjecture that bringing such systems into the picture reduces the number of connected components of the various sets of interest. This idea has important implications for identification theory (see [12, 13]).

Unfortunately, connectedness or disconnectedness of sets does not imply anything about their unions and intersections. Hence, in order to characterize the various unions and intersections of $\mathscr{L}_c$, $\mathscr{L}_{sc}$, $\mathscr{L}_{io}$, $\mathscr{L}_{fo}$, etc., we would have to treat each case individually. In order to spare ourselves this painful exercise, we will merely treat $\mathscr{L}_c$, $\mathscr{L}_o$, and $\mathscr{L}_{co}$.

Before stating our results, we need to discuss a few technical points. First, since $\mathscr{L}$ is a manifold, connectedness and path-connectedness coincide; hence, connected components may be characterized by examining which points can be joined by continuous paths. Next, observe that each point in $\mathscr{L}^n$ uniquely determines a controllability matrix. Indeed, if $\xi = [E, A, B, C]$, we may assign to ξ the matrix

$$U = [E^{-1}B \quad E^{-1}AE^{-1}B \quad \cdots \quad (E^{-1}A)^{n-1}E^{-1}B]. \tag{6}$$

U is unique since premultiplication of E, A, and B be a nonsingular M does not alter (6). Note that U reduces to the familiar definition when $E = I$. The same viewpoint may be taken to define the observability matrix

$$V = \begin{bmatrix} C \\ CE^{-1}A \\ \vdots \\ C(E^{-1}A)^{n-1} \end{bmatrix}.$$

Although U and V cannot be defined in a consistent manner on $\mathscr{L}^s$, in some cases a partial extension can be obtained.

LEMMA 4.1. (1) *Let* $m = 1$ *and* $\xi \in \mathscr{L}_c \cap \mathscr{L}^s$. *If n is even, there exists a neighborhood W of* ξ *such that* sgn det U *is a constant on* $W \cap \mathscr{L}^n$. *If n is odd and W is any neighborhood of* ξ, *there exist* $\xi_1, \xi_2 \in W \cap \mathscr{L}^n$ *with* det $U_1 > 0$ *and* det $U_2 < 0$.

(2) *Let* $p = 1$ *and* $\xi \in \mathscr{L}_o \cap \mathscr{L}^s$. *If n is odd, there exists a neighborhood W of* ξ *such that* sgn det V *is constant on* $W \cap \mathscr{L}^n$. *If n is even and W is any neighborhood of* ξ, *there exist* $\xi_1, \xi_2 \in W \cap \mathscr{L}^n$ *with* det $V_1 > 0$ *and* det $V_2 < 0$.

Proof. (1). From [8, 9] we know that, on a sufficiently small neighborhood W_1 of ξ, points $\eta = [E, A, B, C]$ can be represented by matrix 8-tuples $(N_s, A_s, B_s, C_s, N_f, A_f, B_f, C_f)$ depending continuously on η; furthermore, there exist nonsingular matrices in η, such that

$$ME[N_s \quad N_f] = \begin{bmatrix} I & 0 \\ 0 & A_f \end{bmatrix}, \qquad MA[N_s \quad N_f] = \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix}$$

$$MB = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \qquad C[N_s \quad N_f] = [C_s \quad C_f].$$

For points in $W_1 \cap \mathscr{L}^n$ this yields

$$U = \begin{bmatrix} B_s & A_s B_s & \cdots & A_s^{n-1} B_s \\ A_f^{-1} B_f & A_f^{-2} B_f & \cdots & A_f^{-n} B_f \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_f^{-n} \end{bmatrix} \bar{U},$$

where

$$\bar{U} = \begin{bmatrix} B_s & \cdots & A_s^{n-1} B_s \\ A_f^{n-1} B_f & \cdots & B_f \end{bmatrix}.$$

Since n is even, sgn det $U =$ sgn det $\bar{U}$. But, for $\eta = \xi$, A_f is nilpotent with index less than or equal to $n - r$, where $r = \operatorname{ord} \xi$. In this case,

$$\bar{U} = \begin{bmatrix} B_s & \cdots & A_s^{r-1} B_s & A_s^r B_s & \cdots & A_s^{n-1} B_s \\ 0 & \cdots & 0 & A_f^{n-r-1} B_f & \cdots & B_f \end{bmatrix}$$

which is nonsingular, since $[B_s \cdots A_s^{r-1} B_s]$ and $[A_f^{n-r-1} B_f \cdots B_f]$ must be nonsingular when $\xi \in \mathscr{L}^c$ (see [6]). Hence, there exists a neighborhood $W \subset W_1$ of ξ such that det $\bar{U}$ has constant sign on W.

For n odd, we simply note that choosing ξ_1 with sgn det $A_f =$ sgn det $\bar{U}$ and ξ_2 with sgn det $A_f = -$sgn det $\bar{U}$ gives the desired result.

(2). Here we may use arguments dual to those in (1), involving the matrix

$$V = \begin{bmatrix} C_s & \cdots & C_f A_f^{n-1} \\ \vdots & & \vdots \\ C_s A_s^{n-1} & \cdots & C_f \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & A_f^{1-n} \end{bmatrix}. \quad \blacksquare$$

In certain cases we may therefore extend sgn det U or sgn det V to some points of $\mathscr{L}^s$. More precisely, when $m = 1$ and n is even, sgn det U at a point $\xi \in \mathscr{L}_c \cap \mathscr{L}^s$ is defined to be consistent with the sign of det U at points in a sufficiently small neighborhood of ξ. Similarly, when $p = 1$ and n is odd, sgn det V may be defined for points $\xi \in \mathscr{L}_o \cap \mathscr{L}^s$. These definitions allow us to state and prove our main result on connectedness in $\mathscr{L}$.

THEOREM 4.2. (1) *If n is even and $m = 1$, $\mathscr{L}_c$ and $\mathscr{L}_{co}$ have two components each, indexed by* sgn det U; *$\mathscr{L}_o$ is connected.*

(2) *If n is even and $m > 1$, $\mathscr{L}_{co}$, $\mathscr{L}_c$, and $\mathscr{L}_o$ are each connected.*

(3) *If n is odd and $p = 1$, $\mathscr{L}_{co}$ and $\mathscr{L}_o$ have two components each, indexed by* sgn det V; *$\mathscr{L}_c$ is connected.*

(4) *If n is odd and $p > 1$, $\mathscr{L}_{co}$, $\mathscr{L}_o$, and $\mathscr{L}_c$ are each connected.*

Proof. (1). If two points ξ, $\bar{\xi} \in \mathscr{L}_c$, with $\det U > 0$ and $\det \bar{U} < 0$ exist which can be joined by a continuous path lying entirely in $\mathscr{L}_c$, [10] guarantees that the path must pass through $\mathscr{L}^s$. But Lemma 4.1, part (1), shows that this is impossible. Hence, $\mathscr{L}_c$ has at least two components. Similar statements apply also to $\mathscr{L}_{co}$.

To show connectedness of each subset of $\mathscr{L}_{co}$ on which sgn det U is constant, we first let

$$E_1(\alpha) = \begin{cases} \begin{bmatrix} 1-4\alpha & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, & \alpha \in [0, \tfrac{1}{2}] \\ \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, & \alpha \in (\tfrac{1}{2}, 1] \end{cases}$$

$$A_1(\alpha) = \begin{cases} \begin{bmatrix} n & & & \\ & n-1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, & \alpha \in [0, \tfrac{1}{2}] \\ \begin{bmatrix} -2((n+1)(\alpha-1)+\alpha) & & & & \\ & n-1 & & & \\ & & n-2 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} + (2\alpha - 1)\, I, & \alpha \in (\tfrac{1}{2}, 1] \end{cases}$$

$$B_1(\alpha) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad C_1(\alpha) = [1 \cdots 1].$$

This defines a continuous path connecting the point

$$\xi_1 = [E_1(0), A_1(0), B_1(0), C_1(0)]$$

which has Cauchy index $\mathscr{I}(H_1) = n$, with

$$\xi_2 = [E_1(1), A_1(1), B_1(1), C_1(1)]$$

which has $I(H_2)=n-2$. It is easy to show that the path lies entirely in $\mathscr{L}_{co}$ and everywhere satisfies $\det U<0$. To pass from ξ_2 to a point ξ_3 with $\mathscr{I}(H_3)=n-4$, let

$$E_2(\alpha)=\begin{cases}\begin{bmatrix}1 & & & & \\ & 1-4\alpha & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix}, & \alpha\in[0,\tfrac{1}{2}]\\[2ex] \begin{bmatrix}1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\end{bmatrix}, & \alpha\in(\tfrac{1}{2},1]\end{cases}$$

$$A_2(\alpha)=\begin{cases}\begin{bmatrix}1 & & & \\ & n & & \\ & & \ddots & \\ & & & 2\end{bmatrix}, & \alpha\in[0,\tfrac{1}{2}]\\[2ex] \begin{bmatrix}1 & & & & \\ & -2((n+1)(\alpha-1)+\alpha) & & & \\ & & n-1 & & \\ & & & \ddots & \\ & & & & 2\end{bmatrix} & \\ \qquad +(2\alpha-1)I, & \alpha\in(\tfrac{1}{2},1]\end{cases}$$

$$B_2(\alpha)=\begin{bmatrix}-1\\ 1\\ \vdots\\ 1\end{bmatrix}, \qquad C_2(\alpha)=[1\cdots 1].$$

This process can be continued to connect all subsets of $\mathscr{L}_{co}$ corresponding to different $\mathscr{I}(H)$ and with $\det U<0$. From [10] it follows that each point in $\mathscr{L}^n\cap\mathscr{L}_{co}$ can be joined to one of the ξ_1. Since $\mathscr{L}^n\cap\mathscr{L}_{co}$ is dense in $\mathscr{L}_{co}$ and $\mathscr{L}^s$ is locally connected, $\det U<0$ determines a single component.

The case where $\det U>0$ can be dealt with by using the same parametrizations as before for E and A and by letting $C_i(\alpha)=[1\cdots 1\quad -1]$, $i=1,\dots,n$, and

$$B_1(\alpha)=\begin{bmatrix}1\\ \vdots\\ 1\\ -1\end{bmatrix},\qquad B_2(\alpha)=\begin{bmatrix}-1\\ 1\\ \vdots\\ 1\\ -1\end{bmatrix},\qquad B_3(\alpha)=\begin{bmatrix}-1\\ -1\\ 1\\ \vdots\\ 1\\ -1\end{bmatrix},\ \ldots.$$

To show that $\mathscr{L}_c$ has two components, we note from [9] that $\mathscr{L}^n$ is homeomorphic to the set $\mathbb{R}^{n(n+1+p)}$ of all triples (A, B, C) and, from [10], that $\mathscr{L}_c \cap \mathscr{L}^n$ has two components indexed by sgn det U. Since $\mathscr{L}_c \cap \mathscr{L}^n$ is dense in $\mathscr{L}_c$, $\mathscr{L}_c$ is open, and $\mathscr{L}$ is locally connected, it follows that each point in $\mathscr{L}_c$ has a connected neighborhood $W \subset \mathscr{L}_c$ with $W \cap \mathscr{L}^n \neq \varphi$. Thus each point in $\mathscr{L}_c$ can be continuously joined to a point in $\mathscr{L}_c \cap \mathscr{L}^n$; thus $\mathscr{L}_c$ has at most two components. Lemma 4.1, part (1), shows that there are at least two.

If $p=1$, we know from [10] that $\mathscr{L}_o \cap \mathscr{L}^n$ has two components indexed by sgn det V. Since $\mathscr{L}_o \cap \mathscr{L}^n$ is dense in $\mathscr{L}_o$, Lemma 4.1, part (2), implies $\mathscr{L}_o$ is connected. If $p>1$, [10] shows that $\mathscr{L}_o \cap \mathscr{L}^n$ is connected, so $\mathscr{L}_o$ must also be.

(2) If $p>1$, [10] implies that $\mathscr{L}_c \cap \mathscr{L}^n$, $\mathscr{L}_o \cap \mathscr{L}^n$, and $\mathscr{L}_{co} \cap \mathscr{L}^n$ are each connected; the result follows immediately. If $p=1$, the same holds true for $\mathscr{L}_c$. We have only to show that $\mathscr{L}_{co}$ is connected, since $\mathscr{L}_{co}$ is dense in $\mathscr{L}_o$. From [10] we know that $\mathscr{L}_{co} \cap \mathscr{L}^n$ has two components indexed by sgn det V. But Lemma 4.1 implies that any $\xi \in \mathscr{L}_{co}$ has a connected neighborhood containing points in $\mathscr{L}_{co} \cap \mathscr{L}^n$ with different sgn det V. Hence, $\mathscr{L}_{co}$ is connected.

Parts (3) and (4) are dual to (1) and (2). ∎

Theorem 4.2 is slightly disappointing in that $\mathscr{L}_{co}$ is not always connected, even though singular systems have been included in the construction. However, $\mathscr{L}_{co}$ now has fewer components than in state-space theory (see [10]); furthermore, we are also in a position to prove a striking generalization of the results of [12] on connectedness of the space of degree n rational functions.

We may associate with each $\xi = [E, A, B, C] \in \mathscr{L}$ a rational function matrix

$$H(s) = C(sE-A)^{-1}B.$$

Note that H is invariant under the equivalence relation (4). We wish to extend the results of [12] on the number of connected components of rat(n) to include (not necessarily proper) transfer matrices of points in $\mathscr{L}$.

A realization theory was presented in [2], demonstrating that any rational matrix has a realization of the form (1) and that such a realization achieves the minimum value of n if and only if it is controllable and observable. The space rat(n) is precisely the set of all rational functions with nth order controllable and observable state-space realizations; hence, a natural generalization of rat(n) is the class of all rational matrices realizable by controllable and observable (i.e., minimal) systems (1).

To construct the appropriate transfer matrix space, let H be an arbitrary rational matrix. H may be uniquely decomposed

$$H = H_s + H_f,$$

where H_s is strictly proper and H_f is a polynomial matrix. We define a degree function δ on the set of all nontrivial H in the following way: When $H_f = 0$, let $\delta H = \nu H_s$, where ν is MacMillan degree. Otherwise, let

$$\delta H = \nu H_s + \max\{k + \deg T \mid T \text{ is a nonzero } k\text{th order minor of } H_f;\ k = 1, ..., \min\{m, p\}\}.$$

Let $\mathbb{R}_n^{pm}(s)$ be the set of all rational $p \times m$ matrices H with $\delta H = n$.

PROPOSITION 4.3. *$H \in \mathbb{R}^{pm}(s)$ has a realization in $\mathscr{L}_{co}(n, m, p)$ iff $H \in \mathbb{R}_n^{pm}(s)$.*

Proof. From [2], H has a realization in $\Sigma_{co}(n, m, p)$ iff $\nu H_s(s) + \nu((1/s)\, H_f(1/s)) = n$. Therefore, it suffices to show that $\delta H_f(s) = \nu((1/s)\, H_f(1/s))$. Let $H_f(s) = [h_{ij}(s)]$ and $q = \max\{\deg h_{ij}\}$. Also let $T(s)$ and $\bar{T}(s)$ be corresponding kth order minors of $H_f(s)$ and $1/s\, H_f(1/s)$, respectively. Then $\bar{T}(s)$ is of the form

$$\bar{T}(s) = \frac{d_0 s^{kq} + \cdots + d_{kq}}{s^{k(q+1)}},$$

where

$$T(s) = d_{kq} s^{kq} + \cdots + d_0.$$

The degree of the denominator of $\bar{T}$ (after cancellations) is given by

$$k(q+1) - (kq - \deg T) = k + \deg T.$$

Since Macmillan degree is obtained simply by maximizing the denominator degree over all minors, our result follows immedaitely. ▮

Any $H \in \mathbf{R}_n^{pm}(s)$ can be written uniquely in the form

$$H(s) = \frac{1}{\Delta(s)} [b_{ij}^{r-1} s^{r-1} + \cdots + b_{ij}^0] + [c_{ij}^{n-r-1} s^{n-r-1} + \cdots + c_{ij}^0] \quad (7)$$

provided $\Delta(s) = s^r + a_{r-1} s^{r-1} + \cdots + a_0$ is the least common denominator of all the minors of H_s. Some of the leading b_{ij}'s and c_{ij}'s may vanish. Placing both terms in (7) over the common denominator $\Delta(s)$ yields the unique representation

$$H(s) = \frac{1}{\Delta(s)} [d_{ij}^{n-1} s^{n-1} + \cdots + d_{ij}^0]. \quad (8)$$

Hence, a given $H \in \mathbf{R}_n^{pm}(s)$ uniquely determines a line in $\mathbf{R}^{n(pm+1)+1}$ spanned by $(0, ..., 0, 1, a_{r-1}, ..., a_0; d_{11}^{n-1}, ..., d_{11}^0; ...; d_{pm}^{n-1}, ..., d_{pm}^0)$. $\mathbf{R}_n^{pm}(s)$ is thus naturally imbedded in the real projective space $\mathbf{P}^{n(pm+1)}$ (see [18]); $\mathbf{R}_n^{pm}(s) \subset \mathbf{P}^{n(pm+1)}$ inherits subset topology. Clearly, $\mathrm{rat}(n) \subset \mathbf{R}_n^1(s)$ consists of all strictly proper points. Since convergence in $\mathrm{rat}(n)$ corresponds simply to convergence of coefficients, $\mathrm{rat}(n)$ also inherits subset topology from $\mathbf{R}_n^1(s)$; $\mathbf{R}_n^1(s)$ is thus an appropriate generalization.

Now we can demonstrate the extent to which the inclusion of singular systems affects connectedness in the transfer matrix space.

THEOREM 4.4. *$\mathbf{R}_n^{pm}(s)$ is connected.*

Proof. Consider the map $\mathscr{H}$ defined by

$$[E, A, B, C] \mapsto C(sE - A)^{-1} B$$

from $\mathscr{L}_{co}$ onto $\mathbf{R}_n^{pm}(s)$, and note that the image under $\mathscr{H}$ of $[EN, AN, B, CN]$ does not depend on the nonsingular matrix N. Also note that

$$C(sE - A)^{-1} B = \frac{1}{\det(sE - A)} C(\mathrm{adj}(sE - A)) B.$$

Since $[E, A, B, C] \in \mathscr{L}_{co}$, $\Delta(s) = \det(sE - A)$ and $\mathscr{H}$ is continuous.

Suppose $H \in \mathbf{R}_n^{pm}(s)$. Then there exists $\xi = [E, A, B, C] \in \mathscr{L}_{co}$ with transfer function H. If n is even, $m = 1$, and ξ has $\det U < 0$, let $\tilde{\xi} = [EN, AN, B, CN]$, where

$$N = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Then $\bar{\xi}$ has det $\bar{U} > 0$ and maps into the same H as does ξ. Thus, $\mathbf{R}_n^{pm}(s)$ is the image of the component of $\mathscr{L}_{co}$ with det $U > 0$ under the continuous function $\mathscr{H}$, and $\mathbf{R}_n^{pm}(s)$ is connected.

A similar proof works when n is odd and $p = 1$. In all other cases, $\mathscr{L}_{co}$ is itself connected, so $\mathbf{R}_n^{pm}(s)$ is also. ∎

We therefore have that the degree n rational matrices form a connected set regardless of the values of p and m. This observation lends further weight to our argument in [9] that $\mathscr{L}$ should be considered the natural "completion" of the class of state-space systems.

5. Related Work

In this section we wish to explore the relationships between our constructions and those of other researchers. To begin with, we note that our results constitute a sort of "deparametrized" version of singular perturbation theory (e.g., see [19]). To see this, consider a parametrized family of singular and regular systems

$$E(\omega)\dot{x} = A(\omega)x + B(\omega)u$$
$$y = C(\omega)x,$$

where ω belongs to some topological space and the functions $E(\cdot)$, $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are continuous at a point ω_0. For example, as in [19], ω might be a real positive parameter with

$$E(\omega) = \begin{bmatrix} I & 0 \\ 0 & \omega I \end{bmatrix}$$

and $\omega_0 = 0$. Our results imply that, if the system is controllable at ω_0, it is controllable for all ω sufficiently close to ω_0. Similar statements apply to observability, fast controllability, and fast observability (Theorem 2.2), but not to slow and impulse controllability and observability (Theorems 2.3 and 2.4).

We will now show that the proposed definitions of controllability and observability appearing in [1] do not stand up to small perturbations of ω. We can in fact prove a much stronger result concerning the relationship between the definitions of [1] and those of [6]. As shown in [6], a system (1) is controllable (observable) in the sense of [1] if and only if it belongs to $\mathscr{L}_{sc} \cap \mathscr{L}_{ic}$ ($\mathscr{L}_{so} \cap \mathscr{L}_{io}$). Theorem 5.1 may be viewed as a result in the same vein as those of Section 2.

THEOREM 5.1. $\mathscr{L}_c = \operatorname{int}(\mathscr{L}_{sc} \cap \mathscr{L}_{ic})$, $\mathscr{L}_o = \operatorname{int}(\mathscr{L}_{so} \cap \mathscr{L}_{io})$.

Proof. From Theorems 2.4 and 2.5,

$$\begin{aligned}\operatorname{int}(\mathscr{L}_{sc} \cap \mathscr{L}_{ic}) &= \operatorname{int} \mathscr{L}_{sc} \cap \operatorname{int} \mathscr{L}_{ic} \\ &= \mathscr{L}_c \cap (\mathscr{L}_{fc} \cup \mathcal{O}^{n-1}) \\ &= \mathscr{L}_c\end{aligned}$$

since $\mathscr{L}_c \subset \mathscr{L}_{fc}$. A similar calculation holds for $\mathscr{L}_o$. ∎

Thus we may interpret $\mathscr{L}_c$ and $\mathscr{L}_o$ as the largest generic sets consistent with the definitions of [1]. $\mathscr{L}_{sc} \cap \mathscr{L}_{ic}$ ($\mathscr{L}_{so} \cap \mathscr{L}_{io}$), although dense in $\mathscr{L}$, is not open and hence not characteristic of a robust definition of controllability (observability).

Another body of work related to ours is contained in [13, 14], where the problem of compactifying the space rat(n) is considered. Our construction in Section 4 generalizes the method of [14] by which rat(n) is imbedded in $\mathbf{P}^{2n}$ as an open, dense submanifold. It is also shown in [14] that rat(n) ($M_{1,n,1}^{cr,co}$ in the notation of [14]) has a partial compactification $\bar{M}_{1,n,1}$ which is obtained by taking all H of the form (8) with $\Delta(s) \not\equiv 0$ and projecting into $\mathbf{P}^{2n}$. Our construction is easily seen to satisfy

$$\operatorname{rat}(n) \subset \mathbf{R}_n^1(s) \subset \bar{M}_{1,n,1}.$$

We are further able to prove the following result.

PROPOSITION 5.2. *$\mathbb{R}_n^1(s)$ is an open, dense submanifold of $\mathbf{P}^{2n}$.*

Proof. Consider $H \in \operatorname{rat}(n)$ given by

$$H(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{a_n s^n + \cdots + a_0}.$$

Clearly, $H \in \mathbf{R}_n^1(s)$ iff both numerator and denominator are coprime and either $a_n \neq 0$ or $b_{n-1} \neq 0$. This determines the complement of a homogeneous variety in $\mathbf{R}^{2n+1}$ (see [18]); thus, $\mathbf{R}_n^1(s)$ is the complement of a projective variety in $\mathbf{P}^{2n}$. All results follow immediately. ∎

Proposition 5.2 implies that $\mathbf{R}_n^1(s)$ has the same compactification as does rat(n)—viz. $\mathbf{P}^{2n}$. Extension of these results to the multivariable case has not yet been achieved, even for regular systems.

6. CONCLUSIONS

Based on our previous work [9] which describes the manifold of all n-dimensional singular and regular systems, we have in this paper explored the topological properties of controllability and observability. We have shown that controllability and observability are generic properties, even within the class of singular systems. Further, we have proven that, between the two competing definitions of controllability and observability, only the more restrictive ones determine a generic property. We have also demonstrated that the manifold of singular and regular systems is the natural "completion" of the state-space systems, in the sense that the corresponding space of transfer functions $\mathbf{R}_n^{pm}(s)$ is connected. This result does not have an analogue in state-space theory and lends further support to the completeness arguments of [9]. It is our intention to use these results while examining several important problems in robust control theory.

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