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Fundamental Properties of the Manifold of Singular and Regular Linear Systems

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We construct a real analytic manifold $\mathscr{L}$ of systems of the form $E\dot{x} = Ax + Bu$, $y = Cx$ and show that $\mathscr{L}$ is the "completion," with respect to solutions, of the set of regular (state–space) systems, i.e., those systems with nonsingular E. Other geometric and analytic properties of $\mathscr{L}$ are established, including genericity of the regular systems. © 1986 Academic Press, Inc.

1. INTRODUCTION

In this paper we are interested in properties of regular (state-space) systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

at infinity. Here, the real matrix A is $n \times n$, B is $n \times m$, and C is $p \times n$. To illustrate behavior at infinity, consider the simple example where $n = 2$, $m = p = 0$, and

$$A = \begin{bmatrix} -k & -k^2 \\ 0 & -k \end{bmatrix}.$$

Choosing $x(0) = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, we obtain the solution

$$x_k(t) = \begin{bmatrix} -k^2 t e^{-kt} \\ e^{-kt} \end{bmatrix}.$$

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As $k \to \infty$,

$$x_k \to \begin{bmatrix} -\delta \\ 0 \end{bmatrix} \tag{2}$$

where δ is the unit impulse.

Thus we have a sequence of regular systems whose solution sequence converges, yet the entries of A fail to converge. In fact, there does not even exist a regular system which achieves the right-hand side of (2). In this sense, one might say that the class of regular systems is not "complete"; apparently, the space contains "holes" which can be approximated but not achieved by regular points. It is natural to ask what sort of systems are represented by those holes or, equivalently, *what is the smallest space of systems which is complete with respect to its own solutions and which contains the regular systems*?

The appearance of an impluse as the limit of *unforced* solutions of (1) suggests that the answer might involve the so-called singular systems

$$E\dot{x} = Ax + Bu \tag{3a}$$

$$y = Cx \tag{3b}$$

where E is a singular matrix. Such systems have been studied extensively (e.g., see [4–10]). In [10] it is shown that a necessary and sufficient condition for existence and uniqueness of solutions in (3) for every $x(0)$ and u is that

$$\det(sE - A) \not\equiv 0. \tag{4}$$

We note that premultiplication of (3a) by any nonsingular $n \times n$ matrix M has no effect whatever on the equation's solutions (not even a coordinate change!). One consequence is that, since $M = E^{-1}$ transforms (3) into the form (1), (3) may be considered regular if E is nonsingular. Hence, (3) represents both regular and singular systems if the rank of E is not specified.

One is tempted to say that systems (3) form a Euclidean space of 4-tuples (E, A, B, C) minus the algebraic variety of points violating (4). However, in view of the preceding comments, we are more interested in the quotient set determined by the equivalence relation

$$(E, A, B, C) \approx (ME, MA, MB, C) \qquad \forall \text{ nonsingular } M.$$

It is the structural properties of the quotient set which will be explored in subsequent sections.

The idea of studying the geometry of spaces of linear systems is by no means new (e.g., see [1–3]) and has been exploited with great utility in such areas as system identification and disturbance decoupling. Results in [1, 2] are particularly closely related to our work in that the same sort of completeness issue is addressed. There is a major difference, however, between our efforts and those of other researchers in that other work in the area has centered around rational (transfer) function representations of systems which by nature capture only input–output properties. It is often of crucial interest as well to study internal structural properties of systems, i.e., those properties of (3) dealing with the variable x.

Perhaps the main reason for this lack of interest in studying the space of linear systems (1) is that, when regular systems alone are considered, the structure of the space is obvious: Each system of the form (1) determines a point in the Euclidean space of all triples (A, B, C). Hence, the class of regular systems inherits, among other things, the structure of a real-analytic manifold of dimension $n(n+m+p)$. We will see, however, that the situation changes drastically when singular systems are brought into the picture.

2. Preliminaries

Fix positive integers n, m, and p, denote by V the algebraic variety in $\mathbf{R}^{n(2n+m+p)}$ of points violating (4), and define the open, dense subset

$$\Sigma(n, m, p) = \mathbb{R}^{n(2n+m+p)} - V.$$

Also let

$$\Sigma^i(n, m, p) = \{(E, A, B, C) \in \Sigma(n, m, p) \mid \operatorname{rank} E = i\}; \quad i = 0, \ldots, n.$$

When the arguments (n, m, p) are intended, we may simply write Σ or Σ^i. Clearly, the Σ^i are disjoint and

$$\Sigma = \bigcup_{i=0}^{n} \Sigma^i.$$

We use the notation

$$(E_1, A_1, B_1, C_1) \approx (E_2, A_2, B_2, C_2)$$

iff $C_1 = C_2$ and there exists a nonsingular M such that $E_1 = ME_2$, $A_1 = MA_2$, and $B_1 = MB_2$. This determines an equivalence relation on $\mathbf{R}^{n(2n+m+p)}$. The corresponding quotient sets of Σ and Σ^i are denoted $\mathscr{L}(n, m, p)$ and $\mathscr{L}^i(n, m, p)$, respectively. Again, the $\mathscr{L}^i$ are disjoint and

$$\mathscr{L} = \bigcup_{i=0}^{n} \mathscr{L}^i.$$

We signify by $[E, A, B, C]$ the equivalence class determined by $(E, A, B, C) \in \Sigma$.

Appealing to the theory of distributions (e.g., see [11]), let $\mathscr{D}$ be the space of C^∞ test functions $\phi: \mathbf{R} \to \mathbf{R}$ with compact support and let $\mathscr{D}_+$ be the space of distributions (continuous linear functionals on $\mathscr{D}$) with support in $[0, \infty)$. By $\mathscr{D}_p$ we mean the *piecewise* C^∞ *distributions*: Each $f \in \mathscr{D}_p$ also belongs to $\mathscr{D}_+$ and there exist finitely many points $t_1, \ldots, t_k$ in any bounded interval and a piecewise C^∞ function f_{P} such that $f = f_{\mathrm{P}}$ on each (t_i, t_{i+1}). f_{P} is the *piecewise* C^∞ *part* of f and $f_{\mathrm{I}} = f - f_{\mathrm{P}}$ the *impulsive part*. We may impose a strong and a weak topology on $\mathscr{D}_{\mathrm{p}}$ as follows: Let $\mathscr{T}_{\mathrm{W}}$ be the relative topology induced by the standard weak* topology on $\mathscr{D}_+$. $\mathscr{T}_{\mathrm{s}}$ is determined by the basis of neighborhoods of the origin consisting of all finite intersections of sets of the form

$$V_\phi = \{ f \in \mathscr{D}_{\mathrm{p}} \mid |\langle f, \phi \rangle| < 1 \}$$

and

$$V_{ab\delta\varepsilon} = \{ f \in \mathscr{D}_{\mathrm{p}} \mid m\{ t \in [a, b] \mid |f_{\mathrm{P}}(t)| \geq \delta \} < \varepsilon \}$$

where $0 < a < b$ and m denotes Lebesgue measure. It is easily seen that convergence to the origin in $\mathscr{T}_{\mathrm{s}}$ is equivalent to convergence in $\mathscr{T}_{\mathrm{W}}$ along with almost uniform convergence of the piecewise C^∞ parts on compact subintervals of $(0, \infty)$. We also use the symbols $\mathscr{T}_{\mathrm{W}}$ and $\mathscr{T}_{\mathrm{s}}$ to refer to the corresponding product topologies on the spaces $\mathscr{D}_{\mathrm{p}}^i \times \mathbf{R}^j$.

Define the family of solution mappings

$$\Phi_{x_0 u}: \Sigma \to \mathscr{D}_{\mathrm{p}}^{n+p} \times \mathbf{R}^{n+p}$$

according to

$$\Phi_{x_0 u}(E, A, B, C) = (x, Cx, x_0, Cx_0)$$

where x is the solution of (3a) corresponding to initial condition $x_0 \in \mathbf{R}^n$ and input $u \in \mathscr{D}_{\mathrm{p}}^m$. Using linear operator terminology, we know from [4] that for each $\sigma \in \Sigma$ there exist subspaces $S \oplus F = \mathbf{R}^n$ and linear maps $A_s: S \to S$ and $A_f: F \to F$, A_f nilpotent, such that

$$\begin{aligned} x = &\left(e(A_s)\, P_{SF} - \sum_{i=1}^{q-1} \delta^{i-1} A_f^i P_{FS} \right) x_0 \\ &+ \left(e(A_s)\, P_{SF} - \sum_{i=0}^{q-1} \delta^i A_f^i P_{FS} \right) * Bu. \end{aligned} \tag{5}$$

Here, $e(A_s) \in \mathscr{D}_{\mathrm{p}}^{n^2}$ is the locally integrable function defined by

$$e(A_s)(t) = e^{tA_s}, \qquad t \geq 0,$$

P_{SF}: $\mathbf{R}^n \to S$ and P_{FS}: $\boldsymbol{R}^n \to F$ are projection operators, q in the index of A_f, δ^i denotes the ith derivative, and $*$ denotes convolution.

We are careful to include information about the initial conditions x_0 and Cx_0 of x and y as part of the solution $\Phi_{x_0 u}(\sigma)$ since singular systems exhibit a loss of initial information in x and y. For example, the sequence of systems

$$-\frac{1}{k}\dot{x} = x$$

$$y = x$$

has solutions $y_k(t) = x_k(t) = e^{-kt}x_0$ from which x_0 is easily determined. However, in the limit the solution is $y = x = 0$ for every x_0. The importance of maintaining information concerning the initial value of y is especially clear when one deals with the theory of observability [5]. Such information will also be crucial in our study of completeness.

In addition to the family $\{\Phi_{x_0 u}\}$, we define the map

$$\Phi\colon \Sigma(n, 0, 0) \to \mathscr{D}_{\mathrm{p}}^{n^2}$$

according to

$$\Phi(\sigma) = e(A_s)\, P_{SF} - \sum_{i=1}^{q-1} \delta^{i-1} A_f^i P_{FS}.$$

If bases $\{v_1,..., v_d\}$ of S and $\{v_{d+1},..., v_n\}$ of F are chosen, then using matrix terminology we may construct the similarity transformation

$$T = [v_1 \cdots v_n]$$

yielding

$$T^{-1}\langle\Phi(\sigma), \phi\rangle\, T = \begin{bmatrix} \int_0^\infty \phi(t)\, e^{tA_s}\, dt & 0 \\ 0 & -\sum_{i=1}^{q-1} (-1)^i \phi^i(0)\, A_f^i \end{bmatrix}$$

for every $\sigma \in \Sigma$, $\phi \in \mathscr{D}$ where A_s and A_f are $d \times d$ and $(n-d) \times (n-d)$ matrices.

Since equivalent points in Σ determine the same solution of (3) for every x_0 and u, we may define the induced maps

$$\Psi_{x_0 u}\colon \mathscr{L} \to \mathscr{D}_{\mathrm{p}}^{n+p} \times \mathbf{R}^{n+p}$$

$$\Psi\colon \mathscr{L}(n, 0, 0) \to \mathscr{D}_{\mathrm{p}}^{n^2}$$

on the quotient set $\mathscr{L}$. If $\mu: \Sigma \to \mathscr{L}$ and $\nu: \Sigma(n, 0, 0) \to \mathscr{L}(n, 0, 0)$ are the natural surjections, then

$$\Phi_{x_0 u} = \Psi_{x_0 u} \circ \mu$$

and

$$\Phi = \Psi \circ \nu.$$

We say a differentiable manifold [12] is analytic if it has an atlas $\{\phi_\alpha\}$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is analytic for all choices of α and β. An analytic mapping between manifolds $f: \mathscr{M}_1 \to \mathscr{M}_2$ is one where $\psi \circ f \circ \phi^{-1}$ is analytic for any charts ϕ and ψ of $\mathscr{M}_1$ and $\mathscr{M}_2$. By an analytic diffeomorphism we mean a bijection f such that both f and f^{-1} are analytic. If such an f exists, we say $\mathscr{M}_1$ and $\mathscr{M}_2$ are diffeomorphic.

Finally, in any topological space we signify the boundary of a set Ω by $\partial\Omega$. An infinite sequence with elements x_k is denoted (x_k).

3. A Manifold of Linear Systems

Before addressing the issue of completeness, we wish to consider certain fundamental properties of the quotient set $\mathscr{L}$. First, we establish its manifold structure.

THEOREM 1. *$\mathscr{L}(n, m, p)$ is a real-analytic manifold of dimension $n(n+m+p)$.*

Proof. We appeal to the standard Grassman construction as in [12]. It will be shown that $\mathscr{L}$ is an open subset of $G_n(\mathbf{R}^{2n+m}) \times \mathbf{R}^{pn}$ where $G_n(\mathbf{R}^{2n+m})$ is the Grassman manifold of n-dimensional subspaces of $\mathbf{R}^{2n+m}$. Hence, $\mathscr{L}$ is an $n(n+m+p)$-dimensional real-analytic manifold.

From [10] we know (4) guarantees that the matrix $[E\ A]$ and, hence, $[E\ A\ B]$ have full rank. Then

$$\mathscr{L}(n, m, 0) \subset G_n(\mathbf{R}^{2n+m}).$$

From [12, p. 92] we have that $G_n(\mathbf{R}^{2n+m})$ is a quotient manifold of an open subset of $\mathbf{R}^{n(2n+m)}$ under the submersion μ determined by $\approx$. Since

$$\Sigma(n, m, 0) = \mu^{-1}(\mathscr{L}(n, m, 0)) = \{(E, A, B) \mid (4) \text{ holds}\}$$

is open and $G_n(\mathbf{R}^{2n+m})$ has identification topology with respect to μ, $\mathscr{L}(n, m, 0)$ is open. Hence,

$$\mathscr{L}(n, m, p) = \mathscr{L}(n, m, 0) \times \mathbf{R}^{pn} \subset G_n(\mathbf{R}^{2n+m}) \times \mathbf{R}^{pn}$$

is open. ∎

We note that even $\mathscr{L}(n, m, 0)$ is not compact. For example, the sequence (ξ_k) in $\mathscr{L}(n, m, 0)$, where

$$\xi_k = \left[\frac{1}{k} E, \frac{1}{k} A, B\right]$$

for some given E, A, B satisfying (4), converges in the compact manifold $G_n(\mathbf{R}^{2n+m})$ but not in $\mathscr{L}(n, m, 0)$. The points missing in $\mathscr{L}(n, m, 0)$ are precisely those violating (4) and hence those which to not exhibit existence and uniqueness of solution. Such points cannot be considered well-defined systems, and so should not be included in the system space. This is in contrast to the compact manifolds described in [1, 2].

The next result characterizes in more detail the subsets $\mathscr{L}^i \subset \mathscr{L}$; $i = 0, ..., n$.

THEOREM 2. (1) *$\mathscr{L}^n(n, m, p)$ is an open, dense submanifold of $\mathscr{L}(n, m, p)$, diffeomorphic to $\mathbf{R}^{n(n+m+p)}$.*

(2) *$\mathscr{L}^k(n, m, p)$ is a regular submanifold of $\mathscr{L}$ with dimension $n(m+p+2k)-k^2$ for $k = 0, ..., n-1$.*

(3) *$\partial\mathscr{L}^k(n, m, p) \supset \bigcup_{i=0}^{k-1} \mathscr{L}^i(n, m, p)$; $k = 1, ..., n$.*

(4) *$\mathscr{L}^0(n, m, p)$ is diffeomorphic to $\mathbf{R}^{n(m+p)}$.*

Proof. (1) First, note that

$$\mu^{-1}(\mathscr{L}^n(n, m, 0)) = \{(E, A, B) \mid \det E \neq 0\}$$

is open and dense in $\Gamma(n, m, 0)$. Since $\mathscr{L}(n, m, 0)$ has identification topology with respect to μ,

$$\mathscr{L}^n = \mathscr{L}^n(n, m, 0) \times \mathbf{R}^{pn}$$

is open and dense in $\mathscr{L}$. A standard Grassman chart takes

$$\mathscr{L}^n(n, m, 0) \to \mathbf{R}^{n(n+m)}$$

according to

$$[E, A, B] \to [E^{-1}A \quad E^{-1}B].$$

The chart is onto and hence a diffeomorphism between $\mathscr{L}^n(n, m, 0)$ and $\mathbf{R}^{n(n+m)}$. Thus, $\mathscr{L}^n$ is diffeomorphic to $\mathbf{R}^{n(n+m+p)}$.

(2) Construct an atlas on $\mathscr{L}^k(n, m, 0)$ by restricting standard Grassman charts. For example, let $U_1 \subset G_n(\mathbf{R}^{2n+m})$ be the open set of all $[E, A, B]$ where the first k columns of E and the first $n-k$ columns of A

from a basis for $\mathbf{R}^n$. Each point ξ in U_1 may be represented by a matrix of the form

$$\begin{bmatrix} I_k & T_2 & 0 & \\ 0 & T_3 & I_{n-k} & T_1 \end{bmatrix}.$$

The corresponding chart

$$\phi_1 : U_1 \to \mathbf{R}^{n(k+m)} \times \mathbf{R}^{k(n-k)} \times \mathbf{R}^{(n-k)^2}$$

is defined by

$$\phi_1(\xi) = (T_1, T_2, T_3).$$

The restriction $\tilde{\phi}_1 = \phi_1 \mid U_1 \cap \mathscr{L}^k(n, m, 0)$ satisfies $\tilde{\phi}_1(\xi) = (T_1, T_2, 0)$ and, hence, $\tilde{\phi}_1$ may be considered to have range

$$\mathbf{R}^{n(k+m)} \times \mathbf{R}^{k(n-k)} = \mathbf{R}^{n(m+2k)-k^2}.$$

Other choices of independent columns—(4) guarantees that n can always be found in the first $2n$ columns—give maps $\tilde{\phi}_2, \tilde{\phi}_3, \ldots$. Clearly, the domains of the $\tilde{\phi}_i$ cover $\mathscr{L}^k(n, m, 0)$ and, since each $\phi_i \circ \phi_j^{-1}$ is a diffeomorphism, so is each $\tilde{\phi}_i \circ \tilde{\phi}_j^{-1}$. Thus, $\{\tilde{\phi}_1, \tilde{\phi}_2, \ldots\}$ is an atlas for $\mathscr{L}^k(n, m, 0)$.

To show submanifold structure, we need to prove that the injection

$$j\colon \mathscr{L}^k(n, m, 0) \to G_n(\mathbf{R}^{2n+m})$$

has constant rank $n(m+2k) - k^2$. But this is clearly true since, for example,

$$\phi_1 \circ j \circ \tilde{\phi}_1^{-1} \colon (T_1, T_2) \to (T_1, T_2, 0).$$

For any submanifold, manifold topology is at least as strong as relative topology. Hence, for regularity of $\mathscr{L}^k(n, m, 0)$ we need only show that every open set in $\mathscr{L}^k(n, m, 0)$ can be extended to an open set in $G_n(\mathbf{R}^{2n+m})$. Consider an open $V \subset \mathscr{L}^k(n, m, 0)$ and let $V_i = V \cap U_i$. Then each V_i is open and $V = \bigcup V_i$. Since $\mathbf{R}^{n(m+2k)-k^2}$ can be imbedded in $\mathbf{R}^{n(n+m)}$ as a regular submanifold, $\tilde{\phi}_i(V_i)$ can be extended to an open set W_i. Furthermore, $\phi_i^{-1}(W_i)$ is open so $\bigcup \phi_i^{-1}(W_i)$ is also. But

$$\begin{aligned}
\left(\bigcup \phi_i^{-1}(W_i)\right) \cap \mathscr{L}^k(n, m, 0) &= \bigcup \left(\phi_i^{-1}(W_i) \cap \phi_i^{-1}(\mathbf{R}^{n(m+2k)-k^2})\right) \\
&= \bigcup \phi_i^{-1}(\tilde{\phi}_i(V_i)) \\
&= \bigcup V_i \\
&= V
\end{aligned}$$

so V has an open extension. It follows that

$$\mathscr{L}^k = \mathscr{L}^k(n, m, 0) \times \mathbf{R}^{pn}$$

is regular in $\mathscr{L}$.

(3) Let $i<k$ and $\sigma \in \Sigma^i(n, m, 0)$. Choose any open $U \subset \Sigma(n, m, 0)$ with $\sigma \in U$. Then there exists $\tau = (E, A, B) \in U$ such that rank $E = k$. Thus, $\tau \in \Sigma^k(n, m, 0)$ and $\sigma \in \partial\Sigma^k(n, m, 0)$. Therefore,

$$\bigcup_{i=0}^{k-1} \Sigma^i(n, m, 0) \subset \partial\Sigma^k(n, m, 0).$$

Applying the submersion μ, we obtain

$$\begin{aligned}\bigcup_{i=0}^{k-1} \mathscr{L}^i(n, m, 0) &= \bigcup_{i=0}^{k-1} \mu(\Sigma^i(n, m, 0))\\ &= \mu\left(\bigcup_{i=0}^{k-1} \Sigma^i(n, m, 0)\right)\\ &\subset \mu(\partial\Sigma^k(n, m, 0))\\ &= \partial\mu(\Sigma^k(n, m, 0))\\ &= \partial\mathscr{L}^k(n, m, 0)\end{aligned}$$

since $\mathscr{L}(n, m, 0)$ inherits quotient set topology. The result follows from

$$\mathscr{L}^i = \mathscr{L}^i(n, m, 0) \times \mathbf{R}^{pn}.$$

(4) $\mathscr{L}^0(n, m, 0)$ has an atlas consisting of one chart:

$$[0, I, B] \rightarrow B.$$

Hence, $\mathscr{L}^0(n, m, 0)$ is diffeomorphic to $\mathbf{R}^{nm}$ and $\mathscr{L}^0$ is diffeomorphic to $\mathbf{R}^{n(m+p)}$. ∎

Since each $\mathscr{L}^i$, $i<n$, consists of equivalence classes $[E, A, B, C]$ with E singular, we call $\mathscr{L}^0, \ldots, \mathscr{L}^{n-1}$ the *singular submanifolds* of $\mathscr{L}$. Note that the set of regular systems (1) is precisely $\mathscr{L}^n$ which we refer to as the *regular submanifold*. In order to argue convincingly that $\mathscr{L}$ is not too "large," we need to show that $\mathscr{L}^n$ is in fact dense in $\mathscr{L}$. We will actually prove a stronger result in the next section.

4. Properties of the Solution Mappings

In this section various relationships between the structure of $\mathscr{L}$ and the family $\{\Psi_{x_0u}\}$ will be established including completeness of $\mathscr{L}$. To begin with, we note that the equivalence relation $\approx$ on Σ is closely connected with the family $\{\Phi_{x_0u}\}$. For example, we have already seen that equivalent points in Σ determine the same solutions of (3). We now show that the converse is also true. Let D be any dense subspace of $\mathscr{D}_{\mathrm{p}}^m$ with respect to $\mathscr{T}_{\mathrm{W}}$.

THEOREM 3. *For any* $\sigma_1, \sigma_2 \in \Sigma(n, m, p)$, $\sigma_1 \approx \sigma_2$ *iff*

$$\Phi_{x_0u}(\sigma_1) = \Phi_{x_0u}(\sigma_2)$$

for every $x_0 \in \mathbf{R}^n$, $u \in D$.

Proof. Necessity is obvious. To prove sufficiency we note that, for $u = 0$, (3) has solutions

$$\left(e(A_{s1})\, P_{S_1F_1} - \sum_{i=1}^{q_1-1} \delta^{i-1} A_{f1}^i P_{F_1S_1}\right) x_0$$
$$= \left(e(A_{s2})\, P_{S_2F_2} - \sum_{i=1}^{q_2-1} \delta^{i-1} A_{f2} P_{F_2S_2}\right) x_0$$

corresponding to σ_1 and σ_2. From linear independence,

$$e(A_{s1})\, P_{S_1F_1} x_0 = e(A_{s2})\, P_{S_2F_2} x_0$$

for every $x_0 \in \mathbf{R}^n$. Hence,

$$e^{tA_{s1}} P_{S_1F_1} = e^{tA_{s2}} P_{S_2F_2}$$

for every t, so their images S_1 and S_2 coincide. The same holds for their kernels F_1 and F_2. It follows that

$$e^{tA_{s1}} = e^{tA_{s2}}$$

so $A_{s1} = A_{s2}$. Also from linear independence, $i = 1$ gives $A_{f1} = A_{f2}$.

Now choose a sequence (u_k) in D with $u_k \to \delta v$ in $\mathscr{T}_{\mathrm{W}}$ where $V \in \mathbf{R}^m$, and let $x_0 = 0$. Taking limits yields

$$\left(e(A_{s1})\, P_{S_1F_1} B_1 - \sum_{i=0}^{q_1-1} \delta^i A_{f1}^i P_{F_1S_1} B_1\right) v$$
$$= \left(e(A_{s2})\, P_{S_2F_2} B_2 - \sum_{i=0}^{q_2-1} \delta^i A_{f2}^i P_{F_2S_2} B_2\right) v.$$

From linear independence,

$$e^{tA_{s1}} P_{S_1F_1} B_1 v = e^{tA_{s2}} P_{S_2F_2} B_2 v$$

for every v, so $P_{S_1F_1} B_1 = P_{S_2F_2} B_2$. Setting $i=0$ gives $P_{F_1S_1} B_1 = P_{F_2S_2} B_2$, so $B_1 = B_2$.

Finally, we note that

$$C_1 x_0 = C_2 x_0$$

for every x_0, so $C_1 = C_2$ and $\sigma_1 \approx \sigma_2$. ∎

Theorem 3 shows that the family of incuded maps $\{\Psi_{x_0u}\}$ distinguishes points in the quotient set $\mathscr{L}$. That is, for any $\xi_1, \xi_2 \in \mathscr{L}$ with $\xi_1 \neq \xi_2$, there exist $x_0 \in \mathbf{R}^n$, $u \in D$ such that $\Psi_{x_0u}(\xi_1) \neq \Psi_{x_0u}(\xi_2)$.

Our next result states that not only is the class of regular system $\mathscr{L}^n$ dense in $\mathscr{L}$ with respect to manifold topology, but $\mathscr{L}^n$ also satisfies a density property involving solutions.

THEOREM 4. *For any $\xi \in \mathscr{L}(n, m, p)$ there exists a sequence (ξ_k) in $\mathscr{L}^n(n, m, p)$ with $\xi_k \to \xi$ such that*

$$\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$$

in $\mathscr{T}_s$ for every $x_0 \in \mathbf{R}^n$ and $u \in \mathscr{D}_p^m$.

Proof. If $\xi \in \mathscr{L}^n(n, m, p)$, the result is obvious. Since $\mathscr{L}(n, m, p)$ inherits quotient set topology, we need only show that every system σ of the form (3) with E singular can be approximated by a sequence $\sigma_k = (E_k, A_k, B_k, C_k) \in \Sigma^n(n, m, p)$ such that $\Phi_{x_0u}(\sigma_k) \to \Phi_{x_0u}(\sigma)$. Recall that σ has a decomposition $S \oplus F = \mathbf{R}^n$. If we let $E_k | S = I$, $E_k | F = A_f - (1/k) I$, $A_k | S = A_s$, $A_k | F = I$, $B_k = B$, and $C_k = C$, then $\sigma_k \in \Sigma^n$ and $P_{SF} x_k = P_{SF} x$ for every k. To show $P_{FS} x_k \to P_{FS} x$ for every x_0 and u, we set $A_{fk} = A_f - (1/k) I$ and note that

$$A_{fk}^{-1} = -\sum_{i=0}^{q-1} k^{i+1} A_f^i$$

so

$$\begin{aligned} A_{fk}^{-j} e^{tA_{fk}^{-1}} &= \left(-\sum_{i=0}^{q-1} k^{i+1} A_f^i\right)^j \prod_{i=0}^{q-1} e^{-k^{i+1} t A_f^i} \\ &= e^{-kt} \left(-\sum_{i=0}^{q-1} k^{i+1} A_f^i\right)^j \prod_{i=1}^{q-1} \sum_{r=0}^{q-1} \frac{(-k^{i+1} t)^r}{r!} A_f^{ir}. \end{aligned} \qquad (6)$$

Each entry in (6) is of the form $\psi_j(k, t)\, e^{-kt}$ where ψ_j is a polynomial function. Also, there exists a sufficiently large integer j such that all entries of

$$A_{fk}^j e^{tA_{fk}^{-1}} = \sum_{i=0}^{j} \binom{j}{i} \left(\frac{-1}{k}\right)^{j-i} A_f^i e^{tA_{fk}^{-1}}$$

$$= \left(-\frac{1}{k}\right)^{j-(q-1)} e^{-kt} \sum_{i=0}^{q-1} \binom{j}{i} \left(-\frac{1}{k}\right)^{q-1-i} A_f^i e^{tA_{fk}^{-1}}$$

are of the form $\phi_j(1/k, t)\, e^{-kt}$ where ϕ_j is again a polynomial.

We treat the forced and natural parts of $P_{FS}x_k = x_{fk} + x_{nk}$ separately. To show (x_{nk}) converges almost uniformly, simply note that

$$\psi_0(k, t)\, e^{-kt} \to 0$$

uniformly on $[\varepsilon, \infty)$ for any $\varepsilon > 0$. Hence

$$e(A_{fk}^{-1})\, P_{FS} x_0 \to 0$$

almost uniformly for every x_0. We proved in [6] that

$$e(A_{fk}^{-1}) \to -\sum_{i=1}^{q-1} \delta^{i-1} A_f^i$$

in the weak* topology of $\mathscr{D}_{\mathrm{p}}^{n^2}$ so x_{nk} converges in $\mathscr{T}_{\mathrm{s}}$ to the desired limit.

For the forced response, decompose $u = u_{\mathrm{I}} + u_{\mathrm{P}}$. The piecewise C^∞ part of the forced response due to u_{I} converges almost uniformly on compact intervals since

$$e(A_{fk}^{-1}) * \delta^j A_{fk}^{-1} = \sum_{i=0}^{j-1} \delta^{j-i-1} A_{fk}^{-(i+1)} + A_{fk}^{-(j+1)} e(A_{fk}^{-1})$$

and $\psi_{j+1}(k, t)\, e^{-kt}$ converges uniformly on each $[\varepsilon, \infty)$. For u_{p}, note that there exist functions $u_i\colon \mathbf{R} \to \mathbf{R}^m$, where u_i has support in $[0, \infty)$ and is C^∞, such that

$$u_{\mathrm{P}}(t) = \sum_{i=0}^{\infty} u_i(t - t_i).$$

To show almost uniform convergence of $e(A_{fk}^{-1}) * A_{fk}^{-1} P_{FS} B u_P$ on compact intervals we need only show convergence of $e(A_{fk}^{-1}) * A_{fk}^{-1} P_{FS} Bu_i$ for each i. Integration by parts gives

$$\int_0^t e^{(t-\tau)A_{fk}^{-1}} A_{fk}^{-1} P_{FS} Bu_i(\tau)\, d\tau = \int_0^{t^+} A_{fk}^j e^{(t-\tau)A_{fk}^{-1}} P_{FS} Bu_i^{(j+1)}(\tau)\, d\tau$$

$$+ \sum_{r=0}^{j} A_{fk}^r e^{tA_{fk}^{-1}} P_{FS} Bu_i^{(r)}(0^+)$$

$$- \sum_{r=0}^{j} A_{fk}^r P_{FS} Bu_i^{(r)}(t) \tag{7}$$

where the derivatives of u_i are taken in the ordinary function sense. If $j \geqslant q-1$, the last term in (7) converges uniformly on compact intervals to the piecewise C^∞ part of $-\sum_{r=0}^{q-1} A_f^r P_{FS} u^r$. Each element of the second term is a linear combination of terms of the form $\psi_0(k, t)\, e^{-kt}$ with convergent coefficients. For sufficiently large j, each entry of the first term in (7) is of the form $\int_0^t \phi_j(1/k, t-\tau)\; e^{-k(t-\tau)} v(\tau)\, d\tau$ where v is continuous. Almost uniform convergence follows from

$$\sup_{t \in [0,\infty)} \left| \int_0^t \phi_j\left(\frac{1}{k}, \tau\right) e^{-k\tau} v(t-\tau)\, d\tau \right| \leqslant M \int_0^\infty \left| \phi_j\left(\frac{1}{k}, \tau\right) \right| e^{-k\tau}\, d\tau$$

$$\leqslant M \sum_i |\alpha_i| \int_0^\infty \frac{\tau^{n_i}}{k^{m_i}} e^{-k\tau}\, d\tau$$

$$= M \sum_i |\alpha_i| \frac{n_i!}{k^{m_i+n_i+1}}$$

where $M < \infty$ and the α_i are constants. Therefore, we have that the piecewise C^∞ part of x_{fk} converges almost uniformly on compact intervals to the corresponding part of x. To show weak* convergence, simply note that

$$e(A_{fk}^{-1}) * A_{fk}^{-1} P_{FS} Bu = (e(A_{fk}^{-1}) * P_{FS} Bu) - P_{FS} Bu$$

$$\to \left(-\sum_{i=1}^{q-1} \delta^i A_f * P_{FS} Bu \right) - P_{FS} Bu$$

$$= -\sum_{i=0}^{q-1} A_f^i P_{FS} Bu^i$$

since convolution is a continuous operation on $\mathscr{D}_{\mathrm{p}}$. Hence, (x_{fk}) converges as desired in $\mathscr{T}_{\mathrm{s}}$. ∎

Now we are finally in a position to prove completeness of $\mathscr{L}$. First, we need a technical lemma.

LEMMA. *If* (ξ_k) *is a sequence in* $\mathscr{L}(n, 0, 0)$ *with* $(\Psi(\xi_k))$ *coverging in* $\mathscr{D}_+^{n^2}$ *in the weak* sense, then there exist convergent sequences* (E_k) *and* (A_k) *in* $\mathbf{R}^{n^2}$ *such that the pencil* $s \cdot \lim E_k - \lim A_k$ *is regular and* $\xi_k = [E_k, A_k]$ *for every* k. *In other words,* (ξ_k) *converges.*

Proof. Suppose $\Psi(\xi_k) \to Z$ and choose $\phi \in \mathscr{D}$ such that $\int_0^\infty \phi(t)\, dt \neq 0$. The eigenvalues $\eta_1, ..., \eta_n$ of $\langle Z, \phi \rangle$ can be indexed so that $\eta_1, ..., \eta_r \neq 0$ and $\eta_{r+1}, ..., \eta_n = 0$. We have

$$\langle \Psi(\xi_k), \phi \rangle \to \langle Z, \phi \rangle$$

so the eigenvalues η_{ik} of $\langle \Psi(\xi_k), \phi \rangle$ can be indexed to guarantee that $\eta_{ik} \to \eta_i$ for every i. Choose a rectifiable Jordan curve Γ encircling $\eta_1, ..., \eta_r$ but not the origin. Then we may define the projection matrix

$$P = \frac{1}{2\pi i} \oint_\Gamma (sI - \langle Z, \phi \rangle)^{-1}\, ds.$$

For sufficiently large k, Γ encloses $\eta_{1k}, ..., \eta_{rk}$ as well, but does not enclose $\eta_{r+1,k}, ..., \eta_{nk}$. Hence,

$$P_k = \frac{1}{2\pi i} \oint_\Gamma (sI - \langle \Psi(\xi_k), \phi \rangle)^{-1}\, ds$$

defines a sequence of projections satisfying $P_k \to P$. This construction yields the eigenspace decompositions $Q_k \oplus R_k = Q \oplus R = \mathbf{R}^n$ where

$$Q_k = \operatorname{Im} P_k, \qquad R_k = \operatorname{Im}(I - P_k)$$
$$Q = \operatorname{Im} P, \qquad R = \operatorname{Im}(I - P).$$

As in [4], ξ_k determines subspaces $S_k \oplus F_k = \mathbf{R}^n$ and linear operators A_{sk} and A_{fk}. Let $d_k = \dim S_k$ and suppose A_{sk} has eigenvalues $\lambda_{1k}, ..., \lambda_{d_k k}$, indexed so that

$$\eta_{ik} = \int_0^\infty \phi(t)\, e^{\lambda_{ik} t}\, dt.$$

For large k and $j - i > r$, we know that $\eta_{ik} \neq \eta_{jk}$ so $\lambda_{ik} \neq \lambda_{jk}$. Hence, $\{\lambda_{1k}, ..., \lambda_{rk}\}$ and $\{\lambda_{r+1,k}, ..., \lambda_{d_k k}\}$ induce an eigenspace decomposition $S'_k \oplus S''_k = S_k$. Since S'_k and S''_k are A_{sk}-invariant, they are also $\langle \Psi(\xi_k), \phi \rangle$-invariant. Note that S'_k, S''_k, and F_k have characteristic polynomials $\prod_{i=1}^r (s - \eta_{ik})$, $\prod_{i=r+1}^{d_k} (s - \eta_{ik})$, and $\prod_{i=d_k+1}^n (s - \eta_{ik})$, respectively. Thus, $S'_k = Q$ and $S''_k \oplus F_k = R_k$.

Choose linearly independent columns $v_1, ..., v_r$ of P and $v_{r+1}, ..., v_n$ of

$I-P$ and define $V=[v_1 \cdots v_n]$. The same choice of columns in P_k and $I-P_k$ gives $V_k=[v_{1k},..., v_{nk}]$ with $V_k \to V$ and V_k nonsingular for large k. Next, choose bases $\{w_{r+1,k},..., w_{d_k k}\}$ of $V_k^{-1}S_k''$ and $\{w_{d_k+1,k},..., w_{nk}\}$ of $V_k^{-1}F_k$. Since $S_k'' \subset R_k$ and $V_k^{-1}R_k = \operatorname{Im}[{}^0_I]$, it follows that

$$[w_{r+1,k} \cdots w_{nk}] = \begin{bmatrix} 0 \\ W_k \end{bmatrix}$$

for some $W_k \in \mathbf{R}^{(n-r)^2}$. Let

$$T_k = V_k \begin{bmatrix} I & 0 \\ 0 & W_k \end{bmatrix}.$$

Since the columns of T_k form bases of S_k', S_k'', and F_k, for any $\psi \in \mathscr{D}$

$$T_k^{-1} \langle \Psi(\xi_k), \psi \rangle \, T_k$$
$$= \begin{bmatrix} \int_0^\infty \psi(t)\, e^{tA'_{sk}}\, dt & 0 & 0 \\ 0 & \int_0^\infty \psi(t)\, e^{tA''_{sk}}\, dt & 0 \\ 0 & 0 & -\sum_{i=1}^{q_k-1} (-1)^{i-1} \psi^{i-1}(0)\, A_{fk}^i \end{bmatrix}$$

where $A'_{sk} \in \mathbf{R}^{r^2}$, $A''_{sk} \in \mathbf{R}^{(d_k-r)^2}$, and $A_{fk} \in \mathbf{R}^{(n-d_k)^2}$ is nilpotent. If, for some $i>r$, $\lambda_{ik}=0$ for infinitely many k, there exists a subsequence

$$\eta_{ik_j} = \int_0^\infty \phi(t)\, dt \neq 0.$$

But $\eta_{ik} \to 0$ for $i>r$, so $\lambda_{r+1,k},..., \lambda_{d_k k} \neq 0$ and A''_{sk} is nonsingular for large k. Define

$$E_k = T_k \begin{bmatrix} I & 0 & 0 \\ 0 & A''^{-1}_{sk} & 0 \\ 0 & 0 & A_{fk} \end{bmatrix} T_k^{-1}$$
$$A_k = T_k \begin{bmatrix} A'_{sk} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} T_k^{-1}.$$

Observe that

$$V_k^{-1} \langle \Psi(\xi_k), \psi \rangle \, V_k =$$
$$\begin{bmatrix} \int_0^\infty \psi(t)\, e^{tA'_{sk}}\, dt & 0 \\ 0 & W_k \begin{bmatrix} \int_0^\infty \psi(t)\, e^{tA''_{sk}} & 0 \\ 0 & -\sum_{i=1}^{q_k-1} (-1)^{i-1} \psi^{i-1}(0)\, A_{fk}^i \end{bmatrix} W_k^{-1} \end{bmatrix}$$

converges for every $\psi \in \mathscr{D}$. Hence, there exist $Z_s \in \mathscr{D}_+^{r^2}$ and $Z_f \in \mathscr{D}_+^{(n-r)^2}$ such that

$$e(A'_{sk}) \to Z_s \tag{8}$$

and

$$W_k \begin{bmatrix} e(A''_{sk}) & 0 \\ 0 & -\sum_{i=1}^{q_k-1} \delta^{i-1} A^i_{fk} \end{bmatrix} W_k^{-1} \to Z_f.$$

Differentiation of (8) gives

$$A'_{sk} e(A'_{sk}) + \delta I \to \dot{Z}_s$$

so

$$A'_{sk} \int_0^\infty \phi(t)\, e^{tA'_{sk}}\, dt \to \langle \dot{Z}_s, \phi \rangle - \phi(0)\, I.$$

Since $\langle Z_s, \phi \rangle$ has eigenvalues $\eta_1, \ldots, \eta_r \neq 0$, we may write

$$\left(\int_0^\infty \phi(t)\, e^{tA'_{sk}}\, dt \right)^{-1} \to \langle Z_s, \phi \rangle^{-1}$$

and hence

$$A'_{sk} \to (\langle \dot{Z}_s, \phi \rangle - \phi(0)\, I) \langle Z_s, \phi \rangle^{-1} \triangleq A_s.$$

From [11, p. 43] and the fact that we consider only distributions with support in $[0, \infty)$, there exist unique Y, $Y_k \in \mathscr{D}_+^{n^2}$ such that

$$\dot{Y}_k = W_k \begin{bmatrix} e(A''_{sk}) & 0 \\ 0 & -\sum_{i=1}^{q_k-1} \delta^{i-1} A^i_{fk} \end{bmatrix} W_k^{-1}$$

and

$$\dot{Y} = Z_f.$$

Since $\dot{Y}_k \to \dot{Y}$, $Y_k \to Y$. But

$$\begin{aligned} Y_k &= W_k \begin{bmatrix} A''^{-1}_{sk}(e(A''_{sk}) - \theta I) & 0 \\ 0 & A_{fk}(-\sum_{i=1}^{q_k-1} \delta^{i-1} A^i_{fk} - \theta I) \end{bmatrix} W_k^{-1} \\ &= \left(W_k \begin{bmatrix} A''^{-1}_{sk} & 0 \\ 0 & A_{fk} \end{bmatrix} W_k^{-1} \right) (\dot{Y}_k - \theta I) \end{aligned}$$

where θ is the unit step. Furthermore,

$$\dot{Y}_k - \theta I \to Z_f - \theta I$$

and $\langle Z_f, \phi\rangle - \int_0^\infty \phi(t)\, dt\, I$ is nonsingular, since $\langle Z_f, \phi\rangle$ has eigenvalues $\eta_{r+1},\ldots, \eta_n = 0$ and $\int_0^\infty \phi(t)\, dt \neq 0$. Thus

$$W_k \begin{bmatrix} A_{sk}''^{-1} & 0 \\ 0 & A_{fk} \end{bmatrix} W_k^{-1} \to \langle Y, \phi\rangle \left(\langle Z_f, \phi\rangle - \int_0^\infty \phi(t)\, dt\, I \right)^{-1} \triangleq A_f.$$

To conclude, we note that

$$E_k = V_k \begin{bmatrix} I & 0 \\ 0 & W_k \begin{bmatrix} A_{sk}''^{-1} & 0 \\ 0 & A_{fk} \end{bmatrix} W_k^{-1} \end{bmatrix} V_k^{-1} \to V \begin{bmatrix} I & 0 \\ 0 & A_f \end{bmatrix} V^{-1}$$

and

$$A_k = V_k \begin{bmatrix} A_{sk}' & 0 \\ 0 & I \end{bmatrix} V_k^{-1} \to V \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix} V^{-1}.$$

The limiting pencil is regular since

$$\det(s \cdot \lim E_k - \lim A_k) = \det(sI - A_s) \cdot \det(sA_f - I) \not\equiv 0.$$

Finally, observe that

$$\begin{aligned} \Psi([E_k, A_k]) &= T_k \Phi \left(\begin{bmatrix} I & 0 & 0 \\ 0 & A_{sk}''^{-1} & 0 \\ 0 & 0 & A_{fk} \end{bmatrix}, \begin{bmatrix} A_{sk}' & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right) T_k^{-1} \\ &= T_k \begin{bmatrix} e(A_{sk}') & 0 & 0 \\ 0 & e(A_{sk}'') & 0 \\ 0 & 0 & -\sum_{i=1}^{q_k-1} \delta^{i-1} A_{fk}^i \end{bmatrix} T_k^{-1} \\ &= \Psi(\xi_k). \end{aligned}$$

Let π: $\mathscr{D}_{\mathrm{p}}^{n+p} \times \mathbf{R}^{n+p} \to \mathscr{D}_{\mathrm{p}}^{n}$ be the projection map onto the first n coordinates. Then, for any $\xi \in \mathscr{L}(n, 0, 0)$.

$$\Psi(\xi)\, x_0 = \pi(\Psi_{x_0 0}(\xi))$$

so Theorem 3 implies $\xi_k = [E_k, A_k]$.

It should be noted that the constructions in the foregoing proof always assume that k is sufficiently large. For small values of k, simply choose any $(E_k, A_k) \in \xi_k$. ∎

We comment that in the proof of the Lemma, the matrix A_f may not be nilpotent, depending on the choice of ϕ. The notation we use here is therefore slightly different from that used in (5) or [7].

Of general interest, but not directly related to the rest of our results, is the following

COROLLARY. *As a subset of* $\mathscr{D}_+^{n^2}$, $\mathscr{E} = \{e(M) \mid M \in \mathbf{R}^{n^2}\}$ *has weak* closure equal to* Im Ψ.

Proof. Although $\mathscr{D}_+^{n^2}$ is not first countable, it is routine to verify that the proof of the Lemma works for nets $(\Psi(\xi_k))$ as well as for sequences. In fact, the same is true for Theorem 2 in [6] and the main theorem in [7]. The construction of the Lemma yields sets (E_k) and (A_k) satisfying the desired properties. From [7] it follows that there exist convergent matrix nets (M_k) and (N_k) such that

$$M_k E_k N_k = \begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}$$

$$M_k A_k N_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}$$

where (A_{fk}) and (A_{sk}) converge and $\lim A_{fk}$ is nilpotent. From [6] we have

$$\Phi\left(\begin{bmatrix} I & 0 \\ 0 & A_{fk} \end{bmatrix}, \begin{bmatrix} A_{sk} & 0 \\ 0 & I \end{bmatrix}\right) \to \begin{bmatrix} e(\lim A_{sk}) & 0 \\ 0 & -\sum_{i=1}^{q-1} \delta^{i-1}(\lim A_{fk})^i \end{bmatrix}.$$

Hence $\lim \Psi(\xi_k) \in \operatorname{Im} \Psi$ and Im Ψ is closed.

To show density of $\mathscr{E}$ in Im Ψ, simply note that

$$\Psi(\mathscr{L}^n(n, 0, 0)) = \mathscr{E}.$$

Density then follows from Theorem 4. ∎

We are now in a position to prove that $\mathscr{L}$ is complete with respect to the family $\{\Psi_{x_0 u}\}$.

THEOREM 5. *If* (ξ_k) *is a sequence in* $\mathscr{L}(n, m, p)$ with $(\Psi_{x_0 u}(\xi_k))$ converging in $\mathscr{T}_{\mathrm{W}}$ for every $x_0 \in \mathbf{R}^n$, $u \in \mathscr{D}_{\mathrm{p}}^m$, then (ξ_k) converges in $\mathscr{L}(n, m, p)$.

Proof. Let ω: $\mathscr{L} \to \mathscr{L}(n, 0, 0)$ be defined by

$$\omega([E, A, B, C]) = [E, A].$$

Then

$$\pi(\Psi_{x_0 u}(\xi_k)) = \Psi(\omega(\xi_k))\, x_0$$

for every x_0, where π: $\mathscr{D}^{n+p} \times \mathbf{R}^{n+p} \to \mathscr{D}^n_{\mathrm{p}}$ projects onto the first n coordinates. Hence, the Lemma guarantees that the sequence $(\omega(\xi_k))$ converges or, in other words, there exist sequences (E_k) and (A_k) converging to a regular pencil with

$$\omega(\xi_k) = [E_k, A_k].$$

It remains to show that there are convergent sequences (B_k) and (C_k) such that

$$\xi_k = [E_k, A_k, B_k, C_k]. \tag{9}$$

Choose any (B_k) and (C_k) satisfying (9). (Indeed, there is only one choice for each.) Appealing to the construction in the proof of the Lemma, we have from (5) that for $x_0 = 0$ and $u = \delta v$

$$x_k = V_k \begin{bmatrix} e(A'_{sk})\, B'_{sk} \\ W_k \begin{bmatrix} e(A''_{sk})\, A''_{sk} B''_{sk} \\ -\sum_{i=0}^{q_k - 1} \delta^i A^i_{fk} B_{fk} \end{bmatrix} \end{bmatrix} v$$

where

$$T_k^{-1} B_k = \begin{bmatrix} B'_{sk} \\ B''_{sk} \\ B_{fk} \end{bmatrix}.$$

Since (V_k) converges and v is arbitrary,

$$e(A'_{sk})\, B'_{sk} \to \tilde{Z}_s$$

for some $Y_s \in \mathscr{D}^{rm}_{\mathrm{p}}$. Thus,

$$B'_{sk} \to \langle Z_s, \phi \rangle^{-1} \langle \tilde{Z}_s, \phi \rangle.$$

Next, we note that for some $\tilde{Z}_f$

$$W_k \begin{bmatrix} e(A''_{sk})\, A''_{sk} & 0 \\ 0 & -\sum_{i=0}^{q_k-1} \delta^i A^i_{fk} \end{bmatrix} W_k^{-1} \left(W_k \begin{bmatrix} B''_{sk} \\ B_{fk} \end{bmatrix} \right) \to \tilde{Z}_f.$$

We know that A''_{sk} has eigenvalues $\lambda_{r+1,k}, \ldots, \lambda_{d_k k}$ with $1/\lambda_{ik} \to 0$. Hence, $\int_0^\infty \phi(t)\, e^{tA''_{sk}} A''_{sk}\, dt$ has eigenvalues

$$\begin{aligned} \lambda_{ik} \eta_{ik} &= \lambda_{ik} \int_0^\infty \phi(t)\, e^{\lambda_{ik} t}\, dt \\ &= -\phi(0) - \int_0^\infty \dot{\phi}(t)\, e^{\lambda_{ik} t}\, dt \\ &\to -\phi(0). \end{aligned}$$

$-\sum_{i=0}^{q_k-1}(-1)^i\,\phi^i(0)\,A_{fk}^i$ has all its eigenvalues at $-\phi(0)$ so, if we choose $\phi(0)\neq 0$,

$$W_k\begin{bmatrix}\int_0^\infty \phi(t)\,e^{tA''_{sk}}A''_{sk}\,dt & 0\\ 0 & -\sum_{i=0}^{q_k-1}(-1)^i\,\phi^i(0)\,A_{fk}^i\end{bmatrix}W_k^{-1}\to U$$

where U is a nonsingular matrix. Therefore,

$$W_k\begin{bmatrix}B''_{sk}\\ B_{fk}\end{bmatrix}\to U^{-1}\langle\tilde{Z}_f,\phi\rangle$$

and

$$B_k=V_k\begin{bmatrix}B'_{sk}\\ W_k\begin{bmatrix}B''_{sk}\\ B_{fk}\end{bmatrix}\end{bmatrix}$$

also converges.

Finally, (C_k) converges since $C_k x_0$ converges for every $x_0\in\mathbf{R}^n$. ∎

In summary, $\mathscr{L}$ is complete since convergence of solutions in (3) guarantees convergence of system parameters. Also, $\mathscr{L}$ is the smallest completion of $\mathscr{L}^n$ possible in the sense that $\mathscr{L}^n$ is dense in $\mathscr{L}$ (Theorem 4) and that there are only enough points in the singular submanifolds $\mathscr{L}^0,\dots,\mathscr{L}^{n-1}$ to distinguish the solutions of (3) (Theorem 3). Note that the parts of the proofs of Theorems 3 and 5 dealing with the output matrix C depend heavily upon initial information in Ψ_{x_0u}. If such information is dropped, it becomes easy to construct examples where Theorems 3 and 5 fail.

5. Miscellaneous Properties of $\mathscr{L}(n, m, p)$

In this section we list some of the more obvious properties of the manifold $\mathscr{L}$ which have not yet been mentioned, and draw some connections with prior work done by ourselves and others. To begin with, besides being an open dense submanifold of a certain Grassman manifold, there is at least one situation where $\mathscr{L}$ has an even more familiar structure.

THEOREM 6. *$\mathscr{L}(1, 1, 0)$ is a Möbius band (without boundary).*

Proof. The Möbius band $\mathscr{M}$ is the quotient manifold obtained from $S^1\times\mathbf{R}$, where S^1 is the unit circle in $\mathbf{R}^2$, by identifying $(x_1, x_2)\sim(-x_1,-x_2)$. $\mathscr{L}(1, 1, 0)$ is obtained from $\mathbf{R}^3$ by identifying

$(e, a, b) \sim (me, ma, mb)$ for any $m \neq 0$. In view of (4), this yields real projective 2-space minus the b-axis. The map

$$[e, a, b] \to \left\{ \pm \frac{1}{\sqrt{e^2 + a^2}} (e, a, b) \right\}$$

is clearly an analytic diffeomorphism from $\mathscr{L}(1, 1, 0)$ onto $\mathscr{M}$. ∎

Let Γ be a rectifiable Jordan curve in the complex plane, encircling the origin, and $\Sigma_\Gamma \subset \Sigma$ the open subset of all systems with exactly r eigenvalues encircled by Γ. In [7] we constructed a real-analytic manifold $\mathscr{C}$ of canonical forms for (3) and proved that the decomposition map $d: \Sigma_\Gamma \to \mathscr{C}$, which associates with each system its canonical form, is analytic. The following result establishes a deep connection between $\mathscr{L}$ and the constructions of [7]. Let $\mathscr{L}_\Gamma = \mu(\Sigma_r)$ and $\mathscr{C}_\Gamma = d(\Sigma_\Gamma)$.

THEOREM 7. *There exists an analytic diffeomorphism $f: \mathscr{L}_\Gamma \to \mathscr{C}_\Gamma$ such that*

$$d(\sigma) = f(\mu(\sigma)), \qquad \sigma \in \Sigma_r.$$

Furthermore, $\mathscr{L}_\Gamma$ and $\mathscr{C}_\Gamma$ are open submanifolds of $\mathscr{L}$ and $\mathscr{C}$.

Proof. From [7] we have that

$$d(E, A, B, C) = (S, A_s, B_s, C_s, F, A_f, B_f, C_f)$$

for some $S \oplus F = \mathbf{R}^n$ and linear $A_s: S \to S$, $A_f: F \to F$, etc. Also,

$$\det(sE - A) = \det(sI - A_s) \cdot \det(sA_f - I)$$

so $d(\sigma) \in \mathscr{C}_\Gamma$ for any $\sigma \in \Sigma_\Gamma$. Furthermore,

$$d(E, A, B, C) = d(ME, MA, MB, C)$$

for any nonsingular M, so d induces a unique mapping

$$f: \mathscr{L}_\Gamma \to \mathscr{C}_\Gamma$$

defined by

$$f([\sigma]) = d(\sigma).$$

To see that f is one-to-one, suppose that $f([\sigma_1]) = f([\sigma_2]) = (S, \ldots, C_f)$. Then from [4] there exists a nonsingular M_1 such that $M_1 E_1 \mid S = I$, $M_1 E_1 \mid F = A_f$, $M_1 A_1 \mid S = A_s$, $M_1 A_1 \mid F = I$, $P_{SF} M_1 B_1 = B_s$, $P_{FS} M_1 B_1 = B_f$,

$C_1 | S = C_s$, and $C_1 | F = C_f$ where $\sigma_1 = (E_1, A_1, B_1, C_1)$. Similarly, there exists an M_2 satisfying analogous conditions with respect to σ_2. Hence,

$$(M_1 E_1, M_1 A_1, M_1 B_1, C_1) = (M_2 E_2, M_2 A_2, M_2 B_2, C_2).$$

Letting $M = M_1^{-1} M_2$, it follows that $\sigma_1 \approx \sigma_2$.

In order to prove f is onto, let $\omega = (S,..., C_f) \in \mathscr{C}_\Gamma$ be given and define

$$Ew = \begin{cases} w & w \in S \\ A_f w, & w \in F \end{cases} \qquad Aw = \begin{cases} A_s w, & w \in S \\ w, & w \in F \end{cases}$$

$$Bv = B_s v + B_f v, \qquad Cw = \begin{cases} C_s w, & w \in S \\ C_f w, & w \in F. \end{cases}$$

Then $f([E, A, B, C]) = \omega$ and f is bijective.

Since $\mathscr{L}_\Gamma = \mu(\Sigma_\Gamma)$ and Σ_Γ is open, $\mathscr{L}_\Gamma$ is an open submanifold of $\mathscr{L}$. We need only show that f and f^{-1} are analytic to complete the proof. Consider typical charts

$$\phi: \mathscr{L} \to \mathbf{R}^{n(n+m+p)}$$

and

$$\psi: \mathscr{C} \to \mathbf{R}^{n(n+m+p)}.$$

For example, we might choose charts ϕ_1 and ψ_1 where

$$\phi_1([I, X, Y, Z]) = (X, Y, Z)$$

and

$$\psi_1\left(\begin{bmatrix} I \\ W_1 \end{bmatrix}, X_1, Y_1, Z_1; \begin{bmatrix} I \\ W_2 \end{bmatrix}, X_2, Y_2, Z_2\right)$$
$$= (W_1, X_1, Y_1, Z_1; W_2, X_2, Y_2, Z_2).$$

ϕ_1 is defined on the open subset of $\mathscr{L}$ where E is nonsingular. ψ_1 is defined on the open set where $S \cap F = 0$ and S and F are spanned by the columns of matrices of the form $[{}^{I}_{W_1}]$ and $[{}^{I}_{W_2}]$, respectively (see [7]). Let

$$P = \frac{1}{2\pi i} \oint_\Gamma (sI - X)^{-1}\, ds.$$

Also let

$$\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}$$

be a fixed set of r linearly independent columns of P and

$$\begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}$$

a fixed set of $n-r$ linearly independent columns of $I-P$ where T_{11} and T_{12} are square. Then T_{11} and T_{12} are also nonsingular and $P=P_{SF}$. Define

$$T=\begin{bmatrix} I & I \\ T_{21}T_{11}^{-1} & T_{22}T_{12}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}=T^{-1}XT$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}=T^{-1}Y, \qquad [Z_1 Z_2]=ZT.$$

Then

$$\psi_1(f(\phi_1^{-1}(X,Y,Z)))=(T_{21}T_{11}^{-1}, X_1, Y_1, Z_1; T_{22}T_{12}^{-1}, X_2^{-1}, X_2^{-1}Y_{21}, Z_2).$$

Since the T_{ij} are analytic functions of X and all other functions are rational, $\psi_1 \circ f \circ \phi_1^{-1}$ is analytic.

We can prove that $\phi_1 \circ f^{-1} \circ \psi_1^{-1}$ is in fact rational by simply noting that

$$\phi_1(f^{-1}(\psi_1^{-1}(W_1, X_1, Y_1, Z_1; W_2, X_2, Y_2, Z_2)))$$
$$=\left(U\begin{bmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{bmatrix}U^{-1}, U\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}, [Z_1 Z_2]\,U^{-1}\right)$$

where

$$U=\begin{bmatrix} I & I \\ W_1 & W_2 \end{bmatrix}.$$

Other choices of ϕ and ψ yield analyticity as well by similar arguments. The only difference is in the choice of linearly independent columns in the matrix $[E\ A]$ and linearly independent rows in

$$\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}. \qquad \blacksquare$$

It is interesting to note that the diffeomorphism f does not extend over all of $\mathscr{L}$. For example, it is easy to prove that, when $n=m=1$ and $p=0$, taking either $r=0$ or $r=1$ yields $\mathscr{C}=\mathbf{R}^2$. Theorem 6 shows that $\mathscr{L}$ is not diffeomorphic to any piece of $\mathscr{C}$.

A final observation concerns the connection between the manifold $\mathscr{L}$ and classical singular perturbation theorey [13]. Suppose a parametrization ψ: $[0, 1] \to \mathscr{L}$ is given with $\psi(0)$ singular and $\psi(\varepsilon)$ regular for $\varepsilon > 0$. Then ψ represents an equivalence class of families

$$\begin{aligned} E(\varepsilon)\,\dot{x} &= A(\varepsilon)\,x + B(\varepsilon)\,u \\ y &= C(\varepsilon)\,x \end{aligned} \tag{10}$$

where $E(\varepsilon)$ is singular only when $\varepsilon = 0$. If, in addition, the functions E, A, B, and C satisfy some smoothness property (e.g., analyticity), (10) is singulary perturbed. It is well known that analyticity of E, A, B, and C is not sufficient to guarantee convergence of solutions in (10) as $\varepsilon \to 0^+$.

Our theory gives information regarding the converse. Suppose the coefficient matrices in (10) are not smooth, but it is known that solutions converge. For example, the family of systems

$$\begin{bmatrix} \dfrac{1}{\varepsilon} & \varepsilon \\ 0 & \sin\dfrac{1}{\varepsilon} \end{bmatrix} \dot{x} = \begin{bmatrix} -\dfrac{1}{\varepsilon} & \dfrac{1-\varepsilon}{\varepsilon} \\ 0 & -\dfrac{1}{\varepsilon}\sin\dfrac{1}{\varepsilon} \end{bmatrix} x \tag{11}$$

has solutions

$$x_\varepsilon(t) = \begin{cases} e^{-t}x_{01} + \dfrac{\varepsilon}{1-\varepsilon}\,(e^{-t} - e^{-t/\varepsilon})\,x_{02} \\ e^{-t/\varepsilon}x_{02} \end{cases}$$

where

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

As $\varepsilon \to 0^+$, $x_\varepsilon(t) \to e^{-t}x_{01}$. Theorem 5 says that, for any sequence $\varepsilon_k \to 0^+$, $\psi(\varepsilon_k) \to \psi(0)$ as $k \to \infty$. In other words, there exists a sequence (M_k) such that premultiplication by M_k produces convergent coefficients. In the example, premultiplication of (11) by

$$M_k = \begin{bmatrix} \varepsilon_k & -\dfrac{\varepsilon_k^2}{\sin 1/\varepsilon_k} \\ 0 & \dfrac{\varepsilon_k}{\sin 1/\varepsilon_k} \end{bmatrix}$$

yields

$$\begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_k \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x.$$

Our theory enlarges the class of systems which we conventionally think of as singularly perturbed to include systems such as (11).

6. Conclusions

Our constructions characterize the close relationships between regular and singular systems in a coordinate-free, non parametrized context. We have shown that the topological structure of the class of all linear systems (with bounded order), although not compact, can be naturally related to familiar objects from differential topology (viz., Grassman manifolds). We have also shown that such an abstract framework is a natural setting in which to study singular perturbation problems.

It is our intention to further explore the topological structure of the manifold $\mathscr{L}$, extending many of the existing results known for the space of regular systems over the complete manifold. For example, such properties as controllability and observability are known to be generic on the regular submanifold. Many interesting questions arise regarding the properties of the controllable and observable sets on $\mathscr{L}$. In particular, an understanding of these sets is required before deeper connections between our work and that of [1, 2] can be established.

It is our hope that the framework developed in this paper will be useful as a basis for later work in a variety of areas.

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