ON REALIZATION THEORY FOR GENERALIZED STATE-SPACE SYSTEMS OVER A COMMUTATIVE RING

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Abstract. The problem of finding a state-space realization for a given rational matrix over a commutative ring is considered. To simplify the problem, we assume a certain factored structure for the denominator polynomials in the matrix. Our main results state that this class of matrices is a module which can be decomposed into two independent and isomorphic submodules, each realizable via existing results for strictly proper matrices. Any rational matrix with factored denominators can be realized through this decomposition.

Key words. algebraic systems, realization theory, singular systems, singular perturbation

1. Introduction. The theory of generalized state-space systems has been developed extensively over the past decade [1]. Most of this work has centered around dynamic system equations of the form

$$E\dot{x} = Ax + Bu \qquad (1)$$
$$y = Cx$$

and their transfer function matrices

$$H(s) = C(sE - A)^{-1}B, \qquad (2)$$

where E, A, B, and C are real matrices, E and A square. In this paper we initiate the study of such systems over a commutative ring R. In particular, we are interested here in the algebraic aspects of the problem of finding a dynamic system (1) which realizes a given rational matrix (2) over R.

One motivation for examining generalized state-space systems in an algebraic setting comes from singular perturbation theory. If the class of rings R to be considered is chosen to reflect a "perturbational" structure in (1), then those classical results in singular perturbation theory which rest on purely algebraic arguments can be exposed. Several examples of "perturbational" rings follow:

1) Convergent sequences $\{x_k\}$ in $\mathbb{R}$ under the equivalence

$$\{x_k\} \approx \{y_k\} \text{ iff } x_k = y_k \text{ for large } k.$$

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Addition and multiplication are defined by

$$[x_k] + [y_k] = [x_k + y_k]$$

$$[x_k][y_k] = [x_k y_k].$$

2) C^r functions $x : U_x \to \mathbb{R}$, where $U_x \subset \mathbb{R}^k$ is a neighborhood of 0, under the equivalence

$$x \approx y \text{ iff } \exists \text{ a neighborhood } U \subset U_x \cap U_y \text{ of } 0 \text{ s.t. } x \equiv y \text{ on } U.$$

$$[x] + [y] = [x + y]$$

$$[x][y] = [xy]$$

3) Analytic functions $x : U_x \to \mathbb{C}$ with $U_x \subset \mathbb{C}$ connected, using the same equivalence as in 2). In this case, any two equivalent functions are restrictions of a single function.

An important feature shared by these examples is that each ring contains an ideal P satisfying the property that $1 + x$ is a unit of R whenever $x \in P$. In example 1), one choice of P is the family of sequences converging to 0. In 2) and 3), the set of all x with $x(0) = 0$ plays the same role. We need not restrict ourselves to these examples, however, to find such ideals. Indeed, for any ring R, the Jacobson radical J is characterized by

$$J = \{x \in R | 1 + xy \text{ is a unit for every } y \in R\}.$$

The ideals mentioned relative to 1)-3) are just the Jacobson radicals for those rings.

In order to talk about rational matrices over R, we need to first consider the ring $R[s]$ of polynomials over R and the subset $D \subset R[s]$ given by

$$\begin{aligned} D = \{\Delta \in R[s] | \Delta(s) = u(s^r + a_{r-1}s^{r-1} + \cdots + a_0)(b_q s^q + \cdots + b_1 s \\ +1), b_i \in J, u \text{ is a unit of } R\} \end{aligned} \tag{3}$$

D may be viewed as the product of the two sets

$$D_s = \{\Delta \in R[s] | \Delta(s) = u(s^r + a_{r-1}s^{r-1} + \cdots + a_0), u \text{ a unit}\}$$

and

$$D_f = \{\Delta \in R[s] | \Delta(s) = u(b_q s^q + \cdots + b_1 s + 1), b_i \in J, u \text{ a unit}\}.$$

Note that D is multiplicatively closed. We now established another useful property.

THEOREM 1.1. *D contains no zero divisor of $R[s]$.*

Proof. It is easy to see that each $\Delta \in D$ is of the form

$$\Delta(s) = v(c_{q+r}s^{q+r} + \cdots + c_{r+1}s^{r+1} + s^r + c_{r-1}s^{r-1} + \cdots + c_0), \tag{4}$$

where $c_{r+1}, \ldots, c_{q+r} \in J$ and v is a unit. Suppose $\lambda(s) = d_k s^k + \cdots + d_0$ is any polynomial over R such that $\Delta\lambda = 0$. Then

$$\Gamma \begin{bmatrix} d_0 \\ \vdots \\ d_k \end{bmatrix} = 0,$$

where

$$\Gamma = \begin{bmatrix} 1 & c_{r-1} & \cdots & c_{r-k} \\ c_{r+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{r-1} \\ c_{r+k} & \cdots & c_{r+1} & 1 \end{bmatrix}.$$

(Whenever $i < 0$ or $i > q + k$, c_i is replaced by 0.) A simple calculation shows that $\det \Gamma$ is a unit; hence, $\lambda = 0$. $\square$

Thus the R-module of fractions

$$D^{-1}R[s] = \left\{ \frac{x}{\Delta} | x \in R[s], \Delta \in D \right\}$$

is formed by imposing the equivalence relation

$$(x_1, \Delta_1) \approx (x_2, \Delta_2) \text{ iff } x_1\Delta_2 = x_2\Delta_1$$

on $R[s] \times D$ and passing to the quotient set, using the standard operations for rational functions. (See [2, p. 36].) Note that $H(s)$ in (2) belongs to $D^{-1}R[s]$ whenever $\det(sE - A) \in D$.

2. Decomposition of $D^{-1}R[s]$. The main result of this section shows that the R-module $D^{-1}R[s]$ has a natural decomposition commensurate with the factored polynomial form assumed in the definition (4) of D. Let

$$\mathcal{H}_s = \{h \in D^{-1}R[s] | h \text{ is strictly proper and has a representative with denominator in } D_s\}$$

and

$$\mathcal{H}_f = \{h \in D^{-1}R[s] | h \text{ has a representative with denominator in } D_f\}.$$

Clearly, $\mathcal{H}_s$ and $\mathcal{H}_f$ are R-submodules of $D^{-1}R[s]$, and $R[s]$ is an R-submodule of $\mathcal{H}_f$

THEOREM 2.1. $D^{-1}R[s] = \mathcal{H}_s \oplus \mathcal{H}_f$

Proof. Suppose

$$h(s) = \frac{c_k s^k + \cdots + c_0}{u(s^r + a_{r-1}s^{r-1} + \cdots + a_0)(b_q s^q + \cdots + b_1 s + 1)},$$

where $k \geq \max\{q + r - 1, r\}$, $b_i \in J$, and u is a unit. (There is no loss of generality in constraining k, since some of the leading c_i's may be 0.) Let

$$\begin{bmatrix} W \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ b_1 & & & b_1 \\ 0 & & \ddots & \vdots \\ . & & \ddots & b_q \\ . & & & 0 \\ . & & & \vdots \\ 0 & & \cdots & 0 \end{bmatrix}, \begin{bmatrix} X \\ Z \end{bmatrix} = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{r-1} & & . & 0 \\ 1 & . & & a_0 \\ 0 & . & . & \vdots \\ \vdots & \ddots & . & a_{r-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

where $W \in R^{r \times r}, X \in R^{r \times (k-r+1)}, Y \in R^{(k-r+1) \times r}, Z \in R^{(k-r+1)\times(k-r+1)}$. Note that $\det W = \det Z = 1$, so

$$\det \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \det(I - YW^{-1}XZ^{-1}). \tag{5}$$

Since $Y \in J^{(k-r+1)\times r}$, the determinant (5) is a unit. Let

$$\begin{bmatrix} d_0 \\ \vdots \\ d_{r-1} \\ e_0 \\ \vdots \\ e_{k-r} \end{bmatrix} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}^{-1} \begin{bmatrix} c_0 \\ \vdots \\ c_k \end{bmatrix}. \tag{6}$$

Equation (6) is equivalent to writing $h = h_s + h_f$, where

$$h_s(s) = \frac{d_{r-1}s^{r-1} + \cdots + d_0}{s^r + a_{r-1}s^{r-1} + \cdots + a_0} \tag{7}$$

and

$$h_f(s) = \frac{e_{k-r}s^{k-r} + \cdots + e_0}{b_q s^q + \cdots + b_1 s + 1}. \tag{8}$$

Clearly, $h_s \in \mathcal{H}_s$ and $h_f \in \mathcal{H}_f$.

To verify independence of $\mathcal{H}_s$ and $\mathcal{H}_f$, choose $h \in \mathcal{H}_s \cap \mathcal{H}_f$. Then h has both forms (7) and (8) simultaneously. Thus $q = r = 0$ and, since h is strictly proper, $h = 0$. □

3. State-Space Realization in $D^{-1}R[s]$. Given $H \in (D^{-1}R[s])^{p\times m}$, a 4-tuple (E, A, B, C) of matrices over R is said to be a realization of H if $H(s) = C(sE - A)^{-1}B$. For $E \in \mathcal{H}_s^{p\times m}$, it is shown in [3, Ch. 4] that there exists a realization of the form (I, A, B, C). If $H \in \mathcal{H}_f^{p\times m}$, we may exploit this result with the aid of the following construction. Let $d : (D^{-1}R[s])^{p\times m} \to (D^{-1}R[s])^{p\times m}$ be defined by applying the map $h(s) \to -\frac{1}{s}h(\frac{1}{s})$ to each entry of the matrix. (The manipulation of the indeterminate s is formal.)

THEOREM 3.1.

1) d is a module automorphism on $(D^{-1}R[s])^{p\times m}$.

2) d maps $\mathcal{H}_f^{p\times m}$ isomorphically onto $\mathcal{H}_s^{p\times m}$.

Proof. 1) Direct calculation shows that d is linear and that $d \circ d$ is the identity map.

2) It suffices to show that $d(\mathcal{H}_f^{p\times m}) \subset \mathcal{H}_s^{p\times m}$. Let $h \in \mathcal{H}_f$. Using the notation in (8), and assuming $k \geq q + r$, we obtain

$$-\frac{1}{s}h\left(\frac{1}{s}\right) = \frac{-e_0 s^{k-r} - \cdots - e_{k-r}}{s^{k+1} + b_1 s^k + \cdots + b_q s^{k-r-q+1}},$$

which belongs to $\mathcal{H}_s$. □

Suppose $H \in \mathcal{H}_f^{p\times m}$. Then $d(H)$ has a realization (I, A, B, C). In fact, this realization satisfies

$$C(sA - I)^{-1}B = -\frac{1}{s}C\left(\frac{1}{s}I - A\right)^{-1} B = -\frac{1}{s}d(H)\left(\frac{1}{s}\right) = H(s),$$

so (A, I, B, C) is a realization of H. Combining Theorems 2.1 and 3.1 enables us to realize any $H \in (D^{-1}R[s])^{p\times m}$. Indeed, we may decompose $H = H_s + H_f$ and choose realizations (I, A_s, B_s, C_s) and (I, A_f, B_f, C_f) of H_s and $d(H_f)$, respectively. Defining

$$E = \begin{bmatrix} I & 0 \\ 0 & A_f \end{bmatrix}, \; A = \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix}, \; B = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \; C = [C_s \;\; C_f],$$

we have

$$C(sE - A)^{-1}B = C_s(sI - A_s)^{-1}B_s + C_f(sA_f - I)^{-1}B_f = H_s + H_f = H,$$

so (E, A, B, C) is a realization of H.

4. Conclusions. We have presented a general algebraic treatment of the state-space realization problem for rational matrices H over a commutative ring, provided that the denominators of the entries of H are of the form in (4). For arbitrary rational matrices a factorization theory would have to be developed for polynomials whose leading coefficients are members of the Jacobson radical, followed by a unit.

Another important problem that shows promise is that of developing an algebraic version of the Weierstrass decomposition for regular matrix pencils over a ring. Here the definition of regularity would undoubtedly be critical. One possibility would be to call a pencil regular if its determinant polynomial is of the form (4).

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