Feedback and pole placement in descriptor variable systems†

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The effects and uses of applying linear feedback to continuous time descriptor systems are studied. Structural changes resulting from feedback the slow and fast parts of the trajectory separately are characterized. It is shown that under certain conditions related to controllability the poles of the slow subsystem may be shifted arbitrarily and the impulsive behaviour of the fast subsystem may be eliminated.

1. Introduction

In this paper we consider the continuous time descriptor variable system

\[ E\dot{x} = Ax + Bu \]  

where \( E \) and \( A \) are real \( n \times n \) matrices and \( B \) is real \( n \times m \). Such systems were first considered in the frequency domain by Rosenbrock (1974) and then in both discrete and continuous time by Luenberger (1977), Campbell et al. (1970), Campbell (1977), Yip and Manke (1978), Verghese (1978) and Verghese et al. (1979). Most results have centred around existence and uniqueness of solutions and modal decomposition. A theory of controllability was proposed by Yip and Manke (1978). Many examples of systems where descriptor modelling can be used to advantage were proposed by Luenberger (1977).

Our goal is to extend the theory into unexplored areas. In particular, we are interested in the effects of applying the linear feedback law

\[ u = Kx + v \]

(2)

to (1) where \( K \) is an \( m \times n \) matrix. It will be seen that the standard pole placement results concerning the application of (2) to state variable systems can be generalized. In fact, not only can finite pole shifting be accomplished, but (2) can also be used to influence the infinite poles of (1), as defined by Rosenbrock (1974).

Most of the results that we will obtain will be easier to conceptualize from a coordinate-free or geometric point of view. Thus (1) and (2) may be viewed as relations on real euclidean spaces \( X \) and \( U \) with dimensions \( n \) and \( m \) respectively taking \( E \) and \( A \) to be linear transformations on \( X \), and \( B \) a linear transformation from \( U \) into \( X \).

The canonical analytic decomposition of the pencil \( E\alpha - A \) (Gantmacher 1964) was first applied to (1) by Rosenbrock (1974). In keeping with our

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coordinate-free philosophy we introduce here the canonical geometric decomposition of (1). As done by Rosenbrock (1974) we henceforth make the standard assumption that there exists \( \lambda \in \mathbb{R} \) such that

\[
\det(\lambda E - A) \neq 0
\]

(3)

Let

\[
\det(E\delta - A) = \phi \prod_{i=1}^{k} (s - \lambda_i)^{n_i}
\]

(4)

where \( \phi \neq 0 \) and \( i \neq j \) implies \( \lambda_i \neq \lambda_j \), let \( \sigma(E, A) = \{\lambda_1, \ldots, \lambda_k\} \), and let

\[
\tau = \sum_{i=1}^{k} n_i
\]

(5)

Then \( n_i \) is simply the multiplicity of the root \( \lambda_i \) of the polynomial (4). If \( \lambda \) satisfies (3) then from (4) \( \lambda \notin \sigma(E, A) \) and we may define

\[
S = \bigoplus_{i=1}^{k} \ker \left( (\lambda E - A)^{-1} \left( \frac{1}{\lambda - \lambda_i} I \right)^{n_i} \right)
\]

(6)

and

\[
F = \ker ((\lambda E - A)^{-1} E)^{n - \tau}.
\]

(7)

Clearly \( S \) and \( F \) are \( (\lambda E - A)^{-1} E \)-invariant subspaces. The proof of the following decomposition theorem is presented in the Appendix.

**Theorem 1**

(i) \( S \oplus F = X \) with \( \dim S = \tau \).

(ii) Let \( J_1 = (\lambda E - A)^{-1} E \rvert_S, J_2 = (\lambda E - A)^{-1} E \rvert_F \).

Let \( \bar{M} \) be the linear transformation on \( X \) defined by

\[
\bar{M} \cdot x = \begin{cases} 
J_1^{-1} x & \text{if } x \in S \\
(\lambda J_2 - I)^{-1} x & \text{if } x \in F
\end{cases}
\]

and \( M = \bar{M}(\lambda E - A)^{-1} \). Then

(a) \( S \) and \( F \) are both \( ME \)- and \( MA \)-invariant,

(b) \( ME \rvert_S = I, MA \rvert_F = I \),

(c) \( L_y = ME \rvert_F \) is nilpotent,

(d) \( \det (I - L_y) = \prod_{i=1}^{k} (s - \lambda_i)^{n_i} \) where \( L_y = MA \rvert_S \).

Let \( P \) and \( Q \) be the projections on \( S \) along \( F \) and on \( F \) along \( S \) respectively. Letting (1) have initial condition \( x_0 \), Theorem 1 allows us to decompose (1) by applying \( M \) to both sides yielding

\[
\dot{x}_o = L_\delta x_o + B_\delta u, \quad \text{i.e. } \dot{x}_{so}
\]

(8)

and

\[
\bar{L}_\delta \dot{x}_t = x_t + B_\delta u, \quad \text{i.e. } \dot{x}_{of}
\]

(9)

where \( x_{so} = Ix_0, x_{of} = Qx_0, B_\delta = PMB \) and \( B_\delta = QMB \).
The solution of (8) is well known. Controversy exists, however, regarding the solutions of (9). Some authors (Campbell et al. 1976, Campbell 1977, Yip and Manke 1978) allow only one initial condition \( x_{0t} \) in (9) and hence only one solution for each choice of \( u \). The theory has been generalized by Vergheese (1978) with formal justification by Cobb (1981) to allow arbitrary initial conditions. Following along these lines we adopt the generalized solution of (9)

\[
x_t = - \sum_{i=1}^{q-1} \delta^{i-1} L_i^t x_{0t} - \sum_{i=0}^{i-1} L_i^t B_i u_i
\]

where \( q \) is the index of nilpotency of \( L_i \), \( \delta \) is the Dirac delta, and \( \delta^i, u^i \) denote the \( i \)th derivatives. As employed by Campbell et al. (1976), Campbell (1977), and Yip and Manke (1978) the restricted solution is the same as (10) but without the first term.

The form of (10) suggests that in any conventional sense the dynamics of the overall system are concentrated in (8). Hence the \( \lambda_t \) can be thought of as the (finite) eigenvalues of (1). The form of the unforced parts of the solutions of (8) and (9) suggest the labels 'slow' and 'fast' subsystems for (8) and (9). The decomposition will be valuable in developing the theory needed for choosing appropriate pole shifting feedback gains \( K \).

A geometric theory of controllability was introduced by Yip and Manke (1978) using the restricted solutions of (9). The theory is based on the standard concept of reachable states with respect to a class of smooth controls. We may take this class to be the \( C^\infty \) mappings from \([0, \infty)\) into \( U \). Since reachability is defined for \( t > 0 \) and since the restricted solution of Campbell (1977) and generalized solution (10) differ only about the origin, the theory of Yip and Manke (1978) may be applied to our situation with only minor changes. For details see the work of Cobb (1980).

The main results concerning controllability that we will need are (i) that the sets of reachable states \( \mathcal{R}_s \subset \mathcal{S} \) and \( \mathcal{R}_f \subset \mathcal{F} \) of (8) and (9) respectively are subspaces and (ii) that \( \mathcal{R} = \mathcal{R}_s \oplus \mathcal{R}_f \) is the set of reachable states of the overall system (1). Although \( \mathcal{R}_s \) and \( \mathcal{R}_f \) are defined with respect to the \( C^\infty \) controls, they are basically objects related to the structure of the system. In fact, it will be seen that these subspaces play an important role in questions related to linear feedback where, due to the nature of (10), controls which are not smooth may appear.

Before attacking the details of the closed loop structure of (2) combined with (1) we will state a useful result. It can be viewed as the geometric form of the tests for finite and infinite decoupling zeros as introduced by Rosenbrock (1970, 1974). The proof is given in the Appendix. Recall that an eigenvalue \( \lambda \) of subsystem (8) is said to be controllable if its eigenspace is reachable, i.e. \( \ker (\lambda I - L)^{n_s} \subset \mathcal{R}_s \).

**Theorem 2**

(i) An eigenvalue \( \lambda \in \sigma(E, A) \) is controllable with respect to (8) if and only if

\[
\text{Im} (\lambda I - E - A) \mid \text{Im} B - X
\]

(ii) Subsystem (9) is controllable \( (\mathcal{R}_s = \mathcal{F}) \) if and only if

\[
\text{Im} E + \text{Im} B = X
\]
2. Linear feedback

We are interested in properties of the closed loop system

\[ E\dot{x} = (A + BK)x + Bu \tag{11} \]

which results when (2) is applied to (1). To ensure that (11) is a well defined system we must make the analogous assumption to (3) that there exists \( \lambda \in \mathbb{R} \) such that

\[ \det(\lambda E - A - BK) \neq 0 \tag{12} \]

Henceforth we will denote by \( S_k, V_k, \mathcal{R}_k, L_k, \) etc. the corresponding subspaces and transformations of the closed loop system.

An important result in its own right and one that we will use later is the following.

**Lemma 1**

The controllable subspace \( \mathcal{R} \) is invariant to linear feedback. That is, \( \mathcal{R} = \mathcal{R}_k \) for all \( K \) satisfying (12).

**Proof**

We know from Campbell et al. (1976) that for each \( x_{\text{start}} \in S \) and each \( u \in C^\infty \) there exists a unique \( C^\infty \) map \( \Phi(x_{\text{start}}, u, \cdot) \) from \([0, \infty)\) into \( X \) satisfying

\[ E \frac{\partial \Phi}{\partial t} = A\Phi + Bu, \quad P\Phi(x_{\text{start}}, u, 0) = x_{\text{start}} \]

Let \( w \in \mathcal{R} \). Then there exist \( u \in C^\infty \) and \( \tau > 0 \) such that \( \Phi(x_{\text{start}}, u, \tau) = w \). Let

\[ v(t) = -K\Phi(x_{\text{start}}, u, t) + u(t) \]

Then, applying \( v \) to the closed loop system, we have

\[ E\dot{x} = (A + RK)x + Bu = (A + BK)x - BK\Phi + Bu \]

Clearly, for initial condition \( x_{\text{start}} \), \( x = \Phi \) so \( x(\tau) = \Phi(x_{\text{start}}, u, \tau) = w \), \( w \in \mathcal{R}_k \) and \( \mathcal{R} \subset \mathcal{R}_k \). Reversing the argument gives \( \mathcal{R}_k \subset \mathcal{R} \).

Little else can be said in general about the effects of arbitrary linear feedback. However, for our purposes it will be sufficient to consider only feedback of the slow and fast trajectories \( x_s \) and \( x_f \) separately.

3. Slow feedback

Let \( K \) satisfy

\[ \text{Ker } K \subset F \tag{13} \]

and let

\[ K_s = K|S \tag{14} \]

Then (2) becomes

\[ u = K_s x_s + v \tag{15} \]
Choosing bases of $S$, $F$ and $U$, and letting $\tilde{L}_s$, $\tilde{L}_f$, $\tilde{B}_s$, $\tilde{B}_f$ and $\tilde{K}_s$ be the matrix representations of the corresponding transformations gives

$$\text{Mat} \left( ME_\omega - MA - MBK \right) = \begin{bmatrix} I_n - \tilde{L}_s - \tilde{B}_s\tilde{K}_s & 0 \\ -\tilde{B}_f\tilde{K}_s & \tilde{L}_f - I \end{bmatrix}$$

(16)

Since $\det (L_f - I) = (-1)^{n-r}$, (13) implies that (12) holds. The closed loop system

$$ME\dot{x} = (MA + MBK)x, \text{ i.e. } x_0$$

(17)

has eigenvalues identical to those of $L_\omega + B_fK_s$. In (17) premultiplication by $M$ or $M^{-1}$ has no effect on the solutions. Since subsystem (8) is a state variable system with controllable subspace $\mathscr{B}_s$, we have the following result.

**Theorem 3**

An eigenvalue $\lambda_i$ of the descriptor system (1) can be assigned arbitrarily by applying slow feedback without influencing the remaining eigenvalues if and only if $\lambda_i$ is controllable.

For arbitrary linear feedback it is difficult to make general statements concerning the closed-loop structure of the overall system. However if only the slow part of the trajectory $x_0$ is involved the resulting fast subsystem is essentially unchanged. This fact acts to simplify the calculations required to decompose the closed-loop system. The following theorem makes this precise.

**Theorem 4**

If $\text{Ker} K \Rightarrow F$ then $F_\alpha = F$, $L_{1k} - L_f$ and $\mathscr{B}_{1k} \subseteq \mathscr{B}_f$.

**Proof**

Let $\lambda \in \sigma (E, A) \cup \sigma (E, A + BK)$. Then

$$\text{Mat} (\lambda E - A - BK)^{-1}E$$

$$= \begin{bmatrix} (\lambda I - \tilde{L}_s - \tilde{B}_s\tilde{K}_s)^{-1} & 0 \\ (\lambda \tilde{I}_f - I)^{-1}\tilde{B}_f\tilde{K}_s(\lambda I - \tilde{I}_s - \tilde{B}_s\tilde{K}_s)^{-1} & (\lambda \tilde{I}_f - I)^{-1}\tilde{I}_f \end{bmatrix}$$

Clearly,

$$F_\alpha = \text{Ker} ((\lambda E - A - BK)^{-1}E)^{n-r} = F$$

Also, $\text{Ker} K \Rightarrow F$ implies

$$(\lambda E - A - BK)^{-1}E F = (\lambda E - A)^{-1}E F$$

so from the construction in (ii) of Theorem 1,

$${\bar{M}}_k F = (\lambda J_{2k} - I)^{-1} = (\lambda J_2 - I)^{-1} = {\bar{M}} F$$

and

$$L_{1k} = {\bar{M}}_k (\lambda E - A - BK)^{-1}E F = {\bar{M}} (\lambda E - A)^{-1}E F = L_f$$

Finally, $\mathscr{B}_{1k} = \mathscr{B}_k \cap F_k$, $F_k = F$ and Lemma 1 together imply $\mathscr{B}_{1k} = \mathscr{B}_f$. □
4. Fast feedback

Let $K$ satisfy

$$\text{Ker } K \supset S$$

and let

$$K_f = K | F$$

(19)

If $\mathcal{K}_f$ is the matrix representation of $K_f$, then

$$\text{Mat } (MK_\delta - MA - MBK) = \begin{bmatrix} I & -\mathcal{L}_f \mathcal{K}_f \\ 0 & \mathcal{L}_f \delta - I - \mathcal{B}_f \mathcal{K}_f \end{bmatrix}$$

(20)

Clearly, the eigenvalues of the open-loop system are also eigenvalues of the closed-loop system. But $\det (\mathcal{L}_f \delta - I - \mathcal{B}_f \mathcal{K}_f)$ may not be a constant polynomial so fast feedback may induce additional eigenvalues in the system. In this case assumption (12) is equivalent to

$$\det (\lambda \mathcal{L}_f - I - \mathcal{B}_f \mathcal{K}_f) \neq 0$$

(21)

which we will adopt.

Let

$$\det (\mathcal{L}_f \delta - I - \mathcal{B}_f \mathcal{K}_f) = \psi \prod_{i=1}^{p} (\alpha_i - \beta_i)^{m_i}$$

(22)

with $\psi \neq 0$. An additional assumption that we will need is that none of the induced eigenvalues is equal to any of the other eigenvalues of the closed loop system. That is,

$$\beta_i \neq \lambda_j \quad \forall i, j$$

(23)

This assumption allows the following decomposition result.

**Theorem 5**

Let

$$D_k = \bigoplus_{i=1}^{r} \text{Ker } \left( (\lambda \mathcal{E} - A - BK)^{-1} \mathcal{E} - \frac{1}{\lambda - \mathcal{B}_i} I \right)^{m_i}$$

Then

(i) $S_k = S \oplus D_k$,

(ii) $S$ and $D_k$ and both $M_k \mathcal{E}$- and $M_k (A + BK)$-invariant with

$$M_k (A + BK) | S + \mathcal{L}_f$$

(iii) $\mathcal{P}_k = \mathcal{P}_s \oplus D_k$.

**Proof**

We have

$$\text{Mat } ((\lambda \mathcal{E} - A - BK)^{-1} \mathcal{E})$$

$$= \begin{bmatrix} (\lambda \mathcal{I} - \mathcal{L}_f)^{-1} & (\lambda \mathcal{I} - \mathcal{L}_f)^{-1} \mathcal{B}_f \mathcal{K}_f (\lambda \mathcal{L}_f - I - \mathcal{B}_f \mathcal{K}_f)^{-1} \mathcal{L}_f \\ 0 & (\lambda \mathcal{L}_f - I - \mathcal{B}_f \mathcal{K}_f)^{-1} \mathcal{L}_f \end{bmatrix}$$
From (22) and (23) \((\lambda L_j - I - B_j K_j)^{-1}L_j - (\lambda - \lambda_j)^{-1}I\) is invertible. (That the eigenvalues of \((\lambda L_j - I - B_j K_j)^{-1}L_j\) are \(1/(\lambda - \beta_j)\) can be seen from the proof of Theorem 1 in the Appendix.) Hence:

\[
S = \bigoplus_{i=1}^{\delta} \text{Ker} \left( (\lambda E - A - BK)^{-1}E - \frac{1}{\lambda - \lambda_i} I \right)^{n_i}
\]

and (i) follows from the definition of \(D_k\).

Appealing to the algorithm in (ii) of Theorem 1 we have that \(S\) and \(D_k\) are \((\lambda E - A - BK)^{-1}E\)-invariant, \(J_{\delta_k}\) and \(M_k\)-invariant, and hence \(M_k E\)-invariant. From

\[
(\lambda E - A - BK)^{-1}(A + BK) = \lambda(\lambda E - A - BK)^{-1}E - I
\]

\(M_k(A + BK)\)-invariance of \(S\) and \(D_k\) follows. From (18) if \(x \in S\) then

\[
Ex = (\lambda E - A - BK)(\lambda E - A - BK)^{-1}Ex = (\lambda E - A)(\lambda E - A - BK)^{-1}Ex
\]

so

\[
(\lambda E - A)^{-1}E|S = (\lambda E - A - BK)^{-1}E|S\]

and

\[
M_k|S = J_{\delta_k}^{-1}|S = J_1^{-1}|S = \bar{M}|S
\]

Hence

\[
M_k(A + BK)|S = \bar{M}(\lambda E - A - BK)^{-1}(A + BK)|S = \bar{M}(\lambda E - A)^{-1}E - I)|S
\]

\[-M(\lambda E - A)^{-1}A|S - L_u
\]

Finally, to prove (iii) observe that

\[
\text{Im} \left( \beta_j M E - MA + MKR \right) + \text{Im} \left( M R = S \oplus \text{Im} \left( \beta_j L_j - I - R_j K_j \right) + \text{Im} \left( R_j \right) \right)
\]

For \(x \in \mathcal{E}\) let \(x_1 = (\beta_j L_j - I)^{-1}x\) and \(x_2 = K_j x_1\). Then

\[
(\beta_j L_j - I - B_j K_j)x_1 + B_j x_2 = x
\]

so

\[
\text{Im} \left( \beta_j L_j - I - B_j K_j \right) + \text{Im} \left( B_j = F \right)
\]

Hence, from Theorem 2, part (i), \(D_k \subseteq \mathcal{R}\). Also, \(D_k \subseteq S_k\) so

\[
D_k \subset \mathcal{R} \cap S_k = \mathcal{R}_k
\]

Furthermore, by Lemma 1

\[
\mathcal{R}_k = \mathcal{R} \cap S = \mathcal{R}_k \cap S \subseteq \mathcal{R}_k \cap S_k = \mathcal{R}_k
\]

so

\[
\mathcal{R}_k \oplus D_k \subseteq \mathcal{R}_k
\]

To prove the converse let

\[
x \in \mathcal{R}_k \oplus D_k \subseteq \mathcal{R}_k \cap S_k = \mathcal{R} \cap (S \oplus D_k)
\]

Then \(x \in \mathcal{R}\) and there exist \(y \in S\), \(z \in D_k\) such that \(x = y + z\). But

\[
y = x - z \in \mathcal{R} + D_k = \mathcal{R}
\]
so
\[ x \in (\mathcal{H} \cap S) \otimes D_k = \mathcal{H} \otimes D_k \]
and
\[ \mathcal{K}_{ab} \subseteq \mathcal{H} \otimes D_k \]

We thus have a three-fold decomposition of the closed-loop system. One subsystem is essentially the open-loop slow subsystem (8) with possibly a different input transformation but with the same controllable subspace \( \mathcal{H}_s \). The second subsystem acts on \( D_k \) with the induced eigenvalues \( \beta_i \). Part (iii) states that this subsystem is controllable. Together the first two subsystems comprise the closed-loop slow subsystem. The third subsystem determines the fast trajectory. Its structure depends heavily on the feedback gain \( K_f \).

5. Elimination of impulses by fast feedback

In this section we consider the problem of eliminating the impulsive portion of (10) by applying linear feedback. We would like to eliminate the impulses in (10) for arbitrary initial conditions. Clearly this is achieved if and only if \( I_{1/f} = 0 \). First we need a lemma.

Lemma 2

Let \( Y \) and \( Z \) be euclidean spaces with \( \dim Y = \dim Z \) and let \( N : Y \to Z \) and \( G : U \to Z \) be linear transformations. There exists a linear transformation \( H : Y \to U \) such that \( N + GH \) is invertible if and only if \( \text{Im } N + \text{Im } G = Z \).

Proof

If \( N \) is invertible let \( H = 0 \). If \( N \) is singular the existence of an appropriate \( H \) is equivalent to controllability of the zero eigenvalue of \( N \) with respect to the pair \( (N, G) \). This is equivalent to
\[ \text{Im } N + \text{Im } G = Z \]

Theorem 6

The following statements are equivalent:

(i) there exists \( K \) satisfying (12) such that \( I_{1/f} = 0 \),
(ii) there exists \( K \) satisfying (12) and (18) such that \( I_{1/f} = 0 \),
(iii) \( \text{Im } L_f + \text{Im } B_f + \text{Ker } L_f = F \).

Proof

That (ii) implies (i) is obvious. Applying the decomposition of Theorem 1 to the closed-loop system corresponding to some \( K \) gives that \( I_{1/f} = 0 \) if and only if \( r_k = \text{rank } E \) where \( r_k \) is the degree of \( \det (E s - A - BK) \). From elementary matrix arguments it follows that the (rank \( E \)th coefficient of

\[
\det (M E s - M A - MBK) = \det \begin{bmatrix}
I_n - \tilde{L}_s - \tilde{B}_sK_s & -B_fK_f \\
-B_fK_s & \tilde{L}_f s - I - B_fK_f
\end{bmatrix}
\]  

(24)
is equal to the (rank \(L_j\))th coefficient of \(\det (L_j \bar{s} - I - B_j K_j)\). If a transformation \(K\) exists as in (i) then \(r_k = \text{rank } E\) and the (rank \(E\))th coefficient of (24) is non-zero. Let the linear transformation \(\bar{K}\) be defined by

\[
\bar{K}x = \begin{cases} 
0 & \text{if } x \in S \\
Kx & \text{if } x \in F 
\end{cases}
\]

Then \(\bar{K}\) satisfies (12) and (18) and the degree \(r_k\) of

\[
\det (M_{Es} - A - B\bar{K}) = \det \begin{bmatrix} L_{s} - \tilde{L}_{s} & -\tilde{B}_{s}\bar{K}_{f} \\ 0 & \tilde{L}_{s} - I - \tilde{B}_{f}\bar{K}_{f} \end{bmatrix}
\]

is \(r + \text{rank } L_j = \text{rank } E\) so \(L_{jE} = 0\).

To show the equivalence of (ii) and (iii) choose a basis \(\{e_1, \ldots, e_{p_1}; e_{p_1+1}, \ldots, e_{p_2}; \ldots; e_{p_{d-1}+1}, \ldots, e_{p_d}\}\) of \(F\) so that \(\bar{L}_j\) is in Jordan form with \(d\) blocks of sizes \(p_{i-1}, \ldots, p_i\). Let \(I + B_j K_j = [h_{ij}], p_0 = 0, h_{ij} = h_{p_j}, p_{j-1}+1\) for \(j = 1, \ldots, d,\) and \(\Theta = [t_{ij}]\). A straightforward calculation yields that the (rank \(L_j\))th coefficient of \(\det (L_j \bar{s} - I - B_j K_j)\) is just \(\det \Theta\). Hence (ii) is equivalent to

\[
\det \Theta \neq 0
\]

Note that

\[
\text{Im } L_j = \text{span } \{e_j | j = p_{i-1} + 1, \ldots, p_i - 1; i = 1, \ldots, d\}
\]

and

\[
\text{Ker } L_j = \text{span } \{e_{p_1+1}, \ldots, e_{p_{d-1}+1}\}
\]

Let

\[
T = \text{span } \{e_{p_1}, \ldots, e_{p_d}\}
\]

and let \(V\) be the projection of \(T\) along \(\text{Im } L_j\). Then

\[
\text{Mat } (V(I + B_j K_j)|\text{Ker } L_j) = \Theta
\]

but

\[
V(I + B_j K_j)|\text{Ker } L_j = V|\text{Ker } L_j + (V B_j)(K_j|\text{Ker } L_j)
\]

so from Lemma 2 an appropriate \(K_j\) may be found if and only if

\[
V(|\text{Ker } L_j + \text{Im } B_j) = \text{Im } (V|\text{Ker } L_j) + \text{Im } (V B_j) = T
\]

This is equivalent to

\[
\text{Im } L_j + \text{Ker } L_j + \text{Im } B_j + F
\]

since \(T \oplus \text{Im } L_j = F\).

Theorem 6 may be interpreted as a pole-placement theorem concerned with shifting poles at infinity into the finite portion of the complex plane. Theorem 5, part (iii), says that the shifted poles correspond to controllable eigenvalues and can thus be placed arbitrarily.
Comparing the subspace condition (iii) to Theorem 2, part (ii), we see that impulses can be eliminated under assumptions somewhat weaker than controllability. In fact, condition (iii) is equivalent to controllability of the quotient system of (9) modulo Ker \( L \). It is not surprising that controllability is related to our ability to eliminate impulsive transients.

6. Conclusions

Pole placement of the over-all descriptor system can be accomplished in two stages. First, the given system must be decomposed as in Theorem 1. If any impulsive behaviour is present it can be eliminated under the conditions and according to the procedure of Theorem 6. It is not clear from the construction in Theorem 6 whether or not assumption (23) holds for the closed-loop system. If it does not then Theorem 6 cannot be applied in the decomposition of the system. However, it seems reasonable to expect that this is a pathological case. A topic for further research might be to see if condition (23) is in fact generic.

The second stage involves the decomposition of the closed-loop system after fast feedback and the calculation of the appropriate feedback matrix to place the finite eigenvalues. As shown in Theorem 3 this can be accomplished using standard pole-placement procedures from state-variable theory.

Appendix

Proof of Theorem 1

Let

\[
\det (I_s - (\lambda E - A)^{-1}E) = s^{n-d} \prod_{i=1}^{n} (s - \eta_i)^{p_i}
\]

where

\[
d = \sum_{i=1}^{n} p_i, \quad \eta_i \neq 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad i \neq j \implies \eta_i \neq \eta_j
\]

\(n - d\) is the multiplicity of the zero eigenvalue of \((\lambda E - A)^{-1}E\). Define

\[
R_1 = \bigoplus_{i=1}^{n} \text{Ker} \left( (\lambda E - A)^{-1}E - \eta_i I \right)^{p_i}
\]

and

\[
R_2 = \text{Ker} \left( (\lambda E - A)^{-1}E \right)^{n-d}
\]

Then \(R_1 \oplus R_2 = X\), \(\dim R_1 = d\), and \(R_1\) and \(R_2\) are \((\lambda E - A)^{-1}E\)-invariant.

Let \(H_1 = (\lambda E - A)^{-1}E|_{R_1}\) and \(H_2 = (\lambda E - A)^{-1}E|_{R_2}\). Then

\[
\det (I_s - H_1) = \prod_{i=1}^{n} (s - \eta_i)^{p_i}
\]

and \(H_2\) is nilpotent. Since

\[ (\lambda E - A)^{-1}A = \lambda(\lambda E - A)^{-1}E - I, \quad (\lambda E - A)^{-1}A|_{R_1} = \lambda H_2 - I \]
and 

$$(\lambda E - A)^{-1} A | R_2 = \lambda H_2 - I$$

Define the linear transformation $\bar{N}$ on $X$ according to

$$\bar{N} x = \begin{cases} 
H_1^{-1} x & \text{if } x \in R_1 \\
(\lambda H_2 - I)^{-1} x & \text{if } x \in R_2
\end{cases}$$

$H_1$ and $\lambda H_2 - I$ are invertible since $H_1$ has no zero eigenvalues and $H_2$ is nilpotent. Let $N = \bar{N} (\lambda E - A)^{-1}$. Then $R_1$ and $R_2$ are both $NE$- and $\bar{N} A$-invariant with

$$NE| R_1 = \bar{N} (\lambda E - A)^{-1} E | R_1 = H_1^{-1} H_1 = I$$

and

$$NA| R_2 = \bar{N} (\lambda E - A)^{-1} A | R_2 = (\lambda H_2 - I)^{-1} (\lambda H_2 - I) = I$$

Also,

$$NE| R_2 = (\lambda H_2 - I)^{-1} H_2$$

which is nilpotent and

$$NA| R_1 = H_1^{-1} (\lambda H_1 - I) = M - H_1^{-1}$$

Next, observe that

$$\det(E_\delta - A) = \frac{\det(N E_\delta - N A)}{\det N}$$

$$= \frac{\det(I_\delta - N A| R_1) \det(NE| R_2 E_\delta - I)}{\det N}$$

But $\det(NE| R_2 E_\delta - I) = (-1)^{n-d}$ so

$$\det(I_\delta - N A| R_1) = \prod_{i=1}^{k} (\delta - \lambda_i)^{n_i}$$

Also,

$$\det(I_\delta - N A| R_1) = \det(I_\delta - (\lambda I - H_1^{-1})) = \prod_{i=1}^{k} (\delta - (\lambda - \eta_i))^{p_i}$$

Thus, if the $\eta_i$ are indexed properly, we have $\delta = k$, $p_i = n_i$, $d - r$ and $\lambda - 1/\eta_i = \lambda_i$ so $\eta_i = 1/(\lambda - \lambda_i)$. Hence $R_1 = S$, $R_2 = F$, $H_1 = J_1$, $H_2 = J_2$, $N = M$ and $N = M$.

**Proof of Theorem 2**

(i) Let $M$ be as in Theorem 1. Then

$$M(\text{Im} (\lambda E - A) + \text{Im} B) = \text{Im} \left( (\lambda I E - M A) + \text{Im} (MB) \right)$$

$$- \text{Im} (\lambda I - L_2) \left( \text{Im} (\lambda I - L_2) \right) + \text{Im} (MB)$$

$$= (\text{Im} (\lambda I - L_2) + \text{Im} B) \oplus F$$

since $\lambda I - L_2$ is invertible. From state-variable theory,

$$\text{Im} (\lambda I - L_2) + \text{Im} B = S$$
if and only if $\lambda_i$ is a controllable eigenvalue. Since $M$ is invertible the result follows.

(ii) From Yip and Manke (1978) we know that

$$\mathcal{Y}_i = \text{Im } B_i + \text{Im } L_i B_i + \cdots + \text{Im } L_i^{i-1} B_i$$

so $\mathcal{Y}_i = F$ if and only if $(L_i, B_i)$ is a controllable pair or equivalently,

$$\text{Im } L_i + \text{Im } B_i = F$$

But

$$M(\text{Im } E + \text{Im } B) - \text{Im } (ME) + \text{Im } (MB) = S \oplus (\text{Im } L_i + \text{Im } B_i)$$

and the result follows.

References


