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Feedback and pole placement in descriptor variable systems†

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The effects and uses of applying linear feedback to continuous time descriptor systems are studied. Structural changes resulting from feeding back the slow and fast parts of the trajectory separately are characterized. It is shown that under certain conditions related to controllability the poles of the slow subsystem may be shifted arbitrarily and the impulsive behaviour of the fast subsystem may be eliminated.

1. Introduction

In this paper we consider the continuous time descriptor variable system

$$E\dot{x} = Ax + Bu \tag{1}$$

where E and A are real $n \times n$ matrices and B is real $n \times m$. Such systems were first considered in the frequency domain by Rosenbrock (1974) and then in both discrete and continuous time by Luenberger (1977), Campbell *et al.* (1976), Campbell (1977), Yip and Manke (1978), Verghese (1978) and Verghese *et al.* (1979). Most results have centred around existence and uniqueness of solutions and modal decomposition. A theory of controllability was proposed by Yip and Manke (1978). Many examples of systems where descriptor modelling can be used to advantage were proposed by Luenberger (1977).

Our goal is to extend the theory into unexplored areas. In particular, we are interested in the effects of applying the linear feedback law

$$u = Kx + v \tag{2}$$

to (1) where K is an $m \times n$ matrix. It will be seen that the standard pole placement results concerning the application of (2) to state variable systems can be generalized. In fact, not only can finite pole shifting be accomplished, but (2) can also be used to influence the infinite poles of (1), as defined by Rosenbrock (1974).

Most of the results that we will obtain will be easier to conceptualize from a coordinate-free or geometric point of view. Thus (1) and (2) may be viewed as relations on real euclidean spaces X and U with dimensions n and m respectively taking E and A to be linear transformations on X, and B a linear transformation from U into X.

The canonical analytic decomposition of the pencil $Es - A$ (Gantmacher 1964) was first applied to (1) by Rosenbrock (1974). In keeping with our

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coordinate-free philosophy we introduce here the canonical *geometric* decomposition of (1). As done by Rosenbrock (1974) we henceforth make the standard assumption that there exists $\lambda \in R$ such that

$$\det(\lambda E - A) \neq 0 \tag{3}$$

Let

$$\det(Es - A) = \phi \prod_{i=1}^{k} (s - \lambda_i)^{n_i} \tag{4}$$

where $\phi \neq 0$ and $i \neq j$ implies $\lambda_i \neq \lambda_j$, let $\sigma(E, A) = \{\lambda_1, \ldots, \lambda_k\}$, and let

$$r = \sum_{i=1}^{k} n_i \tag{5}$$

Then n_i is simply the multiplicity of the root λ_i of the polynomial (4). If λ satisfies (3) then from (4) $\lambda \notin \sigma(E, A)$ and we may define

$$S = \bigoplus_{i=1}^{k} \operatorname{Ker} \left((\lambda E - A)^{-1} E - \frac{1}{\lambda - \lambda_i} I \right)^{n_i} \tag{6}$$

and

$$F = \operatorname{Ker} \, ((\lambda E - A)^{-1} E)^{n-r}. \tag{7}$$

Clearly S and F are $(\lambda E - A)^{-1}E$-invariant subspaces. The proof of the following decomposition theorem is presented in the Appendix.

Theorem 1

(i) $S \oplus F = X$ with $\dim S = r$.

(ii) Let $J_1 = (\lambda E - A)^{-1} E | S$, $J_2 = (\lambda E - A)^{-1} E | F$.

Let $\bar{M}$ be the linear transformation on X defined by

$$\bar{M}x = \begin{cases} J_1^{-1} x & \text{if } x \in S \\ (\lambda J_2 - I)^{-1} x & \text{if } x \in F \end{cases}$$

and $M = \bar{M}(\lambda E - A)^{-1}$. Then

(*a*) S and F are both ME- and MA-invariant,

(*b*) $ME|S = I$, $MA|F = I$,

(*c*) $L_f = ME|F$ is nilpotent,

(*d*) $\det(Is - L_s) = \prod_{i=1}^{k} (s - \lambda_i)^{n_i}$ where $L_s = MA|S$.

Let P and Q be the projections on S along F and on F along S respectively. Letting (1) have initial condition x_0, Theorem 1 allows us to decompose (1) by applying M to both sides yielding

$$\dot{x}_s = L_s x_s + B_s u, \quad \text{i.c. } x_{0s} \tag{8}$$

and

$$L_f \dot{x}_f = x_f + B_f u, \quad \text{i.c. } x_{0f} \tag{9}$$

where $x_{0s} = Px_0$, $x_{0f} = Qx_0$, $B_s = PMB$ and $B_f = QMB$.

The solution of (8) is well known. Controversy exists, however, regarding the solutions of (9). Some authors (Campbell *et al.* 1976, Campbell 1977, Yip and Manke 1978) allow only one initial condition x_{0f} in (9) and hence only one solution for each choice of u. The theory has been generalized by Verghese (1978) with formal justification by Cobb (1981) to allow arbitrary initial conditions. Following along these lines we adopt the generalized solution of (9)

$$x_f = -\sum_{1}^{q-1} \delta^{i-1} L_f^{\,i}\, x_{0f} - \sum_{i=0}^{i=1} L_f^{\,i}\, B_f u^i \tag{10}$$

where q is the index of nilpotency of L_f, δ is the Dirac delta, and δ^i, u^i denote the ith derivatives. As employed by Campbell *et al.* (1976), Campbell (1977), and Yip and Manke (1978) the restricted solution is the same as (10) but without the first term.

The form of (10) suggests that in any conventional sense the dynamics of the overall system are concentrated in (8). Hence the λ_i can be thought of as the (finite) eigenvalues of (1). The form of the unforced parts of the solutions of (8) and (9) suggest the labels ' slow ' and ' fast ' subsystems for (8) and (9). The decomposition will be valuable in developing the theory needed for choosing appropriate pole shifting feedback gains K.

A geometric theory of controllability was introduced by Yip and Manke (1978) using the restricted solutions of (9). The theory is based on the standard concept of reachable states with respect to a class of smooth controls. We may take this class to be the C^∞ mappings from $[0, \infty)$ into U. Since reachability is defined for $t>0$ and since the restricted solution of Campbell (1977) and generalized solution (10) differ only about the origin, the theory of Yip and Manke (1978) may be applied to our situation with only minor changes. For details see the work of Cobb (1980).

The main results concerning controllability that we will need are (i) that the sets of reachable states $\mathscr{R}_s \subset S$ and $\mathscr{R}_f \subset F$ of (8) and (9) respectively are subspaces and (ii) that $\mathscr{R} = \mathscr{R}_s \oplus \mathscr{R}_f$ is the set of reachable states of the overall system (1). Although $\mathscr{R}_s$ and $\mathscr{R}_f$ are defined with respect to the C^∞ controls, they are basically objects related to the structure of the system. In fact, it will be seen that these subspaces play an important role in questions related to linear feedback where, due to the nature of (10), controls which are not smooth may appear.

Before attacking the details of the closed loop structure of (2) combined with (1) we will state a useful result. It can be viewed as the geometric form of the tests for finite and infinite decoupling zeros as introduced by Rosenbrock (1970, 1974). The proof is given in the Appendix. Recall that an eigenvalue λ_i of subsystem (8) is said to be controllable if its eigenspace is reachable, i.e. $\operatorname{Ker}(\lambda_i I - L_s)^{n_i} \subset \mathscr{R}_s$.

Theorem 2

(i) An eigenvalue $\lambda_i \in \sigma(E, A)$ is controllable with respect to (8) if and only if

$$\operatorname{Im}(\lambda_i E - A) + \operatorname{Im} B = X$$

(ii) Subsystem (9) is controllable ($\mathscr{R}_f = F$) if and only if

$$\operatorname{Im} E + \operatorname{Im} B = X$$

2. Linear feedback

We are interested in properties of the closed loop system

$$E\dot{x}=(A+BK)x+Bv \tag{11}$$

which results when (2) is applied to (1). To ensure that (11) is a well defined system we must make the analogous assumption to (3) that there exists $\lambda \in R$ such that

$$\det(\lambda E-A-BK)\neq 0 \tag{12}$$

Henceforth we will denote by S_k, F_k, $\mathscr{R}_{sk}$, L_{sk}, etc. the corresponding subspaces and transformations of the closed loop system.

An important result in its own right and one that we will use later is the following.

Lemma 1

The controllable subspace $\mathscr{R}$ is invariant to linear feedback. That is, $\mathscr{R}=\mathscr{R}_k$ for all K satisfying (12).

Proof

We know from Campbell *et al.* (1976) that for each $x_{0s}\in S$ and each $u\in C^\infty$ there exists a unique C^∞ map $\Phi(x_{0s}, u, \cdot)$ from $[0, \infty)$ into X satisfying

$$E\frac{\partial \Phi}{\partial t}=A\Phi+Bu, \quad P\Phi(x_{0s}, u, 0)=x_{0s}$$

Let $w\in\mathscr{R}$. Then there exist $u\in C^\infty$ and $\tau>0$ such that $\Phi(x_{0s}, u, \tau)=w$. Let

$$v(t)=-K\Phi(x_0, u, t)+u(t)$$

Then, applying v to the closed loop system, we have

$$E\dot{x}=(A+BK)x+Bv=(A+BK)x-BK\Phi+Bu$$

Clearly, for initial condition x_{0s}, $x=\Phi$ so $x(\tau)=\Phi(x_{0s}, u, \tau)=w$, $w\in\mathscr{R}_k$ and $\mathscr{R}\subset\mathscr{R}_k$. Reversing the argument gives $\mathscr{R}_k\subset\mathscr{R}$. ■

Little else can be said in general about the effects of arbitrary linear feedback. However, for our purposes it will be sufficient to consider only feedback of the slow and fast trajectories x_s and x_f separately.

3. Slow feedback

Let K satisfy

$$\operatorname{Ker} K\subset F \tag{13}$$

and let

$$K_s=K|S \tag{14}$$

Then (2) becomes

$$u=K_s x_s+v \tag{15}$$

Choosing bases of S, F and U, and letting $\tilde{L}_s$, $\tilde{L}_f$, $\tilde{B}_s$, $\tilde{B}_f$ and $\tilde{K}_s$ be the matrix representations of the corresponding transformations gives

$$\text{Mat}\,(MEs-MA-MBK)=\begin{bmatrix} I_s-\tilde{L}_s-\tilde{B}_s\tilde{K}_s & 0 \\ -\tilde{B}_f\tilde{K}_s & \tilde{L}_f s-I \end{bmatrix} \tag{16}$$

Since $\det(L_f s-I)=(-1)^{n-r}$, (13) implies that (12) holds. The closed loop system

$$ME\dot{x}=(MA+MBK)x, \quad \text{i.e. } x_0 \tag{17}$$

has eigenvalues identical to those of $L_s+B_sK_s$. In (17) premultiplication by M or M^{-1} has no effect on the solutions. Since subsystem (8) is a state variable system with controllable subspace $\mathscr{R}_s$ we have the following result.

Theorem 3

An eigenvalue λ_i of the descriptor system (1) can be assigned arbitrarily by applying slow feedback without influencing the remaining eigenvalues if and only if λ_i is controllable.

For arbitrary linear feedback it is difficult to make general statements concerning the closed-loop structure of the overall system. However if only the slow part of the trajectory x_s is involved the resulting fast subsystem is essentially unchanged. This fact acts to simplify the calculations required to decompose the closed-loop system. The following theorem makes this precise.

Theorem 4

If $\text{Ker}\,K\supset F$ then $F_k=F$, $L_{fk}=L_f$ and $\mathscr{R}_{fk}=\mathscr{R}_f$.

Proof

Let $\lambda\in R-(\sigma(E,A)\cup\sigma(E,A+BK))$. Then

$$\text{Mat}\,(\lambda E-A-BK)^{-1}E)$$

$$=\begin{bmatrix} (\lambda I-\tilde{L}_s-\tilde{B}_s\tilde{K}_s)^{-1} & 0 \\ (\lambda\tilde{L}_f-I)^{-1}\tilde{B}_f\tilde{K}_s(\lambda I-\tilde{L}_s-\tilde{B}_s\tilde{K}_s)^{-1} & (\lambda\tilde{L}_f-I)^{-1}\tilde{L}_f \end{bmatrix}$$

Clearly,

$$F_k=\text{Ker}\,((\lambda E-A-BK)^{-1}E)^{n-r}=F$$

Also, $\text{Ker}\,K\supset F$ implies

$$(\lambda E-A-BK)^{-1}E\,|\,F=(\lambda E-A)^{-1}E\,|\,F$$

so from the construction in (ii) of Theorem 1,

$$\bar{M}_k\,|\,F=(\lambda J_{2k}-I)^{-1}=(\lambda J_2-I)^{-1}=\bar{M}\,|\,F$$

and

$$L_{fk}=\bar{M}_k(\lambda E-A-BK)^{-1}E\,|\,F=\bar{M}(\lambda E-A)^{-1}E\,|\,F=L_f$$

Finally, $\mathscr{R}_{fk}=\mathscr{R}_k\cap F_k$, $F_k=F$ and Lemma 1 together imply $\mathscr{R}_{fk}=\mathscr{R}_f$. ∎

4. Fast feedback

Let K satisfy

$$\text{Ker } K \supset S \tag{18}$$

and let

$$K_f = K|F \tag{19}$$

If $\tilde{K}_f$ is the matrix representation of K_f then

$$\text{Mat}\,(MEs - MA - MBK) = \begin{bmatrix} Is - \tilde{L}_s & -\tilde{B}_s\tilde{K}_f \\ 0 & \tilde{L}_f s - I - \tilde{B}_f\tilde{K}_f \end{bmatrix} \tag{20}$$

Clearly, the eigenvalues of the open-loop system are also eigenvalues of the closed-loop system. But $\det\,(L_f s - I - B_f K_f)$ may not be a constant polynomial so fast feedback may induce additional eigenvalues in the system. In this case assumption (12) is equivalent to

$$\det\,(\lambda L_f - I - B_f K_f) \neq 0 \tag{21}$$

which we will adopt.

Let

$$\det\,(L_f s - I - B_f K_f) = \psi \prod_{i=1}^{p} (s - \beta_i)^{m_i} \tag{22}$$

with $\psi \neq 0$. An additional assumption that we will need is that none of the induced eigenvalues is equal to any of the other eigenvalues of the closed loop system. That is,

$$\beta_i \neq \lambda_j \quad \forall i, j \tag{23}$$

This assumption allows the following decomposition result.

Theorem 5

Let

$$D_k = \bigoplus_{i=1}^{p} \text{Ker}\left((\lambda E - A - BK)^{-1}E - \frac{1}{\lambda - \beta_i} I\right)^{m_i}$$

Then

(i) $S_k = S \oplus D_k$,

(ii) S and D_k and both $M_k E$- and $M_k(A+BK)$-invariant with

$$M_k(A+BK)|S + L_s$$

(iii) $\mathscr{R}_{sk} = \mathscr{R}_s \oplus D_k$.

Proof

We have

$$\text{Mat}\,((\lambda E - A - BK)^{-1}E)$$
$$= \begin{bmatrix} (\lambda I - \tilde{L}_s)^{-1} & (\lambda I - \tilde{L}_s)^{-1}\tilde{B}_s\tilde{K}_f(\lambda\tilde{L}_f - I - \tilde{B}_f\tilde{K}_f)^{-1}\tilde{L}_f \\ 0 & (\lambda\tilde{L}_f - I - \tilde{B}_f\tilde{K}_f)^{-1}\tilde{L}_f \end{bmatrix}$$

From (22) and (23) $(\lambda L_f - I - B_f K_f)^{-1} L_f - (\lambda - \lambda_i)^{-1} I$ is invertible. (That the eigenvalues of $(\lambda L_f - I - B_f K_f)^{-1} L_f$ are $1/(\lambda - \beta_i)$ can be seen from the proof of Theorem 1 in the Appendix.) Hence

$$S = \bigoplus_{i=1}^{k} \operatorname{Ker}\left((\lambda E - A - BK)^{-1} E - \frac{1}{\lambda - \lambda_i} I\right)^{n_i}$$

and (i) follows from the definition of D_k.

Appealing to the algorithm in (ii) of Theorem 1 we have that S and D_k are $(\lambda E - A - BK)^{-1} E$-invariant, J_{1k}- and $\bar{M}_k$-invariant, and hence $M_k E$-invariant. From

$$(\lambda E - A - BK)^{-1}(A + BK) = \lambda(\lambda E - A - BK)^{-1} E - I$$

$M_k(A + BK)$-invariance of S and D_k follows. From (18) if $x \in S$ then

$$Ex = (\lambda E - A - BK)(\lambda E - A - BK)^{-1} Ex = (\lambda E - A)(\lambda E - A - BK)^{-1} Ex$$

so

$$(\lambda E - A)^{-1} E | S = (\lambda E - A - BK)^{-1} E | S$$

and

$$\bar{M}_k | S = J_{1k}{}^{-1} | S = J_1{}^{-1} | S = \bar{M} | S$$

Hence

$$\begin{aligned} M_k(A + BK) | S &= \bar{M}_k(\lambda E - A - BK)^{-1}(A + BK) | S = \bar{M}(\lambda(\lambda E - A)^{-1} E - I) | S \\ &= \bar{M}(\lambda E - A)^{-1} A | S = L_s \end{aligned}$$

Finally, to prove (iii) observe that

$$\operatorname{Im}(\beta_i ME - MA - MBK) + \operatorname{Im} MB = S \oplus (\operatorname{Im}(\beta_i L_f - I - B_f K_f) + \operatorname{Im} B_f)$$

For $x \in F$ let $x_1 = (\beta_i L_f - I)^{-1} x$ and $x_2 = K_f x_1$. Then

$$(\beta_i L_f - I - B_f K_f) x_1 + B_f x_2 = x$$

so

$$\operatorname{Im}(\beta_i L_f - I - B_f K_f) + \operatorname{Im} B_f = F$$

Hence, from Theorem 2, part (i), $D_k \subset \mathscr{R}$. Also, $D_k \subset S_k$ so

$$D_k \subset \mathscr{R} \cap S_k = \mathscr{R}_{sk}$$

Furthermore, by Lemma 1

$$\mathscr{R}_s = \mathscr{R} \cap S = \mathscr{R}_k \cap S \subset \mathscr{R}_k \cap S_k = \mathscr{R}_{sk}$$

so

$$\mathscr{R}_s \oplus D_k \subset \mathscr{R}_{sk}$$

To prove the converse let

$$x \in \mathscr{R}_{sk} = \mathscr{R}_k \cap S_k = \mathscr{R} \cap (S \oplus D_k)$$

Then $x \in \mathscr{R}$ and there exist $y \in S$, $z \in D_k$ such that $x = y + z$. But

$$y = x - z \in \mathscr{R} + D_k = \mathscr{R}$$

so

$$x \in (\mathscr{R} \cap S) \oplus D_k = \mathscr{R}_s \oplus D_k$$

and

$$\mathscr{R}_{sk} \subset \mathscr{R}_s \oplus D_k \qquad \blacksquare$$

We thus have a three-fold decomposition of the closed-loop system. One subsystem is essentially the open-loop slow subsystem (8) with possibly a different input transformation but with the same controllable subspace $\mathscr{R}_s$. The second subsystem acts on D_k with the induced eigenvalues β_i. Part (iii) states that this subsystem is controllable. Together the first two subsystems comprise the closed-loop slow subsystem. The third subsystem determines the fast trajectory. Its structure depends heavily on the feedback gain K_f.

5. Elimination of impulses by fast feedback

In this section we consider the problem of eliminating the impulsive portion of (10) by applying linear feedback. We would like to eliminate the impulses in (10) for arbitrary initial conditions. Clearly this is achieved if and only if $L_{fk} = 0$. First we need a lemma.

Lemma 2

Let Y and Z be euclidean spaces with $\dim Y = \dim Z$ and let $N : Y \to Z$ and $G : U \to Z$ be linear transformations. There exists a linear transformation $H : Y \to U$ such that $N + GH$ is invertible if and only if $\operatorname{Im} N + \operatorname{Im} G = Z$.

Proof

If N is invertible let $H = 0$. If N is singular the existence of an appropriate H is equivalent to controllability of the zero eigenvalue of N with respect to the pair (N, G). This is equivalent to

$$\operatorname{Im} N + \operatorname{Im} G = Z$$

Theorem 6

The following statements are equivalent :

(i) there exists K satisfying (12) such that $L_{fk} = 0$,

(ii) there exists K satisfying (12) and (18) such that $L_{fk} = 0$,

(iii) $\operatorname{Im} L_f + \operatorname{Im} B_f + \operatorname{Ker} L_f = F$.

Proof

That (ii) implies (i) is obvious. Applying the decomposition of Theorem 1 to the closed-loop system corresponding to some K gives that $L_{fk} = 0$ if and only if $r_k = \operatorname{rank} E$ where r_k is the degree of $\det(Es - A - BK)$. From elementary matrix arguments it follows that the (rank E)th coefficient of

$$\det(MEs - MA - MBK) = \det \begin{bmatrix} I_s - \tilde{L}_s - \tilde{B}_s \tilde{K}_s & -\tilde{B}_s \tilde{K}_f \\ -\tilde{B}_f \tilde{K}_s & \tilde{L}_f s - I - \tilde{B}_f \tilde{K}_f \end{bmatrix} \tag{24}$$

is equal to the (rank L_f)th coefficient of $\det(L_f s - I - B_f K_f)$. If a transformation K exists as in (i) then $r_k = \operatorname{rank} E$ and the (rank E)th coefficient of (24) is non-zero. Let the linear transformation $\bar{K}$ be defined by

$$\bar{K}x = \begin{cases} 0 & \text{if } x \in S \\ Kx & \text{if } x \in F \end{cases}$$

Then $\bar{K}$ satisfies (12) and (18) and the degree $r_{\bar{k}}$ of

$$\det(MEs - A - B\bar{K}) = \det \begin{bmatrix} I_s - \tilde{L}_s & -\tilde{B}_s \tilde{K}_f \\ 0 & \tilde{L}_f s - I - \tilde{B}_f \tilde{K}_f \end{bmatrix}$$

is $r + \operatorname{rank} L_f = \operatorname{rank} E$ so $L_{f\bar{k}} = 0$.

To show the equivalence of (ii) and (iii) choose a basis $(e_1, \ldots, e_{p_1};\ e_{p_1+1}, \ldots, e_{p_2};\ \ldots;\ e_{p_{d-1}+1}, \ldots, e_{p_d})$ of F so that $\tilde{L}_f$ is in Jordan form with d blocks of sizes $p_{i+1} - p_i$. Let $I + \tilde{B}_f \tilde{K}_f = [h_{ij}]$, $p_0 = 0$, $\theta_{ij} = h_{p_i,\, p_{j-1}+1}$ for $j = 1, \ldots, d$, and $\Theta = [\theta_{ij}]$. A straightforward calculation yields that the (rank L_f)th coefficient of $\det(L_f s - I - B_f K_f)$ is just $\det \Theta$. Hence (ii) is equivalent to

$$\det \Theta \neq 0$$

Note that

$$\operatorname{Im} L_f = \operatorname{span}\{e_j \mid j = p_{i-1} + 1, \ldots, p_i - 1;\ i = 1, \ldots, d\}$$

and

$$\operatorname{Ker} L_f = \operatorname{span}\{e_1, e_{p_1+1}, \ldots, e_{p_{d-1}+1}\}$$

Let

$$T = \operatorname{span}\{e_{p_1}, \ldots, e_{p_d}\}$$

and let V be the projection of T along $\operatorname{Im} L_f$. Then

$$\operatorname{Mat}(V(I + B_f K_f) | \operatorname{Ker} L_f) = \Theta$$

But

$$V(I + B_f K_f) | \operatorname{Ker} L_f = V | \operatorname{Ker} L_f + (VB_f)(K_f | \operatorname{Ker} L_f)$$

so from Lemma 2 an appropriate K_f may be found if and only if

$$V(\operatorname{Ker} L_f + \operatorname{Im} B_f) = \operatorname{Im}(V | \operatorname{Ker} L_f) + \operatorname{Im}(VB_f) = T$$

This is equivalent to

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f + F$$

since $T \oplus \operatorname{Im} L_f = F$. ∎

Theorem 6 may be interpreted as a pole-placement theorem concerned with shifting poles at infinity into the finite portion of the complex plane. Theorem 5, part (iii), says that the shifted poles correspond to controllable eigenvalues and can thus be placed arbitrarily.

Comparing the subspace condition (iii) to Theorem 2, part (ii), we see that impulses can be eliminated under assumptions somewhat weaker than controllability. In fact, condition (iii) is equivalent to controllability of the quotient system of (9) modulo Ker L_f. It is not surprising that controllability is related to our ability to eliminate impulsive transients.

6. Conclusions

Pole placement of the over-all descriptor system can be accomplished in two stages. First, the given system must be decomposed as in Theorem 1. If any impulsive behaviour is present it can be eliminated under the conditions and according to the procedure of Theorem 6. It is not clear from the construction in Theorem 6 whether or not assumption (23) holds for the closed-loop system. If it does not then Theorem 5 cannot be applied in the decomposition of the system. However, it seems reasonable to expect that this is a pathological case. A topic for further research might be to see if condition (23) is in fact generic.

The second stage involves the decomposition of the closed-loop system after fast feedback and the calculation of the appropriate feedback matrix to place the finite eigenvalues. As shown in Theorem 3 this can be accomplished using standard pole-placement procedures from state-variable theory.

Appendix

Proof of Theorem 1

Let

$$\det(Is-(\lambda E-A)^{-1}E)=s^{n-d}\prod_{i=1}^{\delta}(s-\eta_i)^{p_i}$$

where

$$d=\sum_{i=1}^{\delta}p_i,\quad \eta_i\neq 0\quad \text{for } i=1,\ldots,\delta \text{ and } i\neq j \text{ implies } \eta_i\neq\eta_j$$

$n-d$ is the multiplicity of the zero eigenvalue of $(\lambda E-A)^{-1}E$. Define

$$R_1=\bigoplus_{i=1}^{\delta}\operatorname{Ker}((\lambda E-A)^{-1}E-\eta_i I)^{p_i}$$

and

$$R_2=\operatorname{Ker}((\lambda E-A)^{-1}E)^{n-d}$$

Then $R_1\oplus R_2=X$, dim $R_1=d$, and R_1 and R_2 are $(\lambda E-A)^{-1}E$-invariant.

Let $H_1=(\lambda E-A)^{-1}E\,|\,R_1$ and $H_2=(\lambda E-A)^{-1}E\,|\,R_2$. Then

$$\det(Is-H_1)=\prod_{i=1}^{\delta}(s-\eta_i)^{p_i}$$

and H_2 is nilpotent. Since

$$(\lambda E-A)^{-1}A=\lambda(\lambda E-A)^{-1}E-I,\quad (\lambda E-A)^{-1}A\,|\,R_1=\lambda H_1-I$$

and

$$(\lambda E - A)^{-1}A \,|\, R_2 = \lambda H_2 - I$$

Define the linear transformation $\bar{N}$ on X according to

$$\bar{N}x = \begin{cases} H_1^{-1}\,x & \text{if } x \in R_1 \\ (\lambda H_2 - I)^{-1}x & \text{if } x \in R_2 \end{cases}$$

H_1 and $\lambda H_2 - I$ are invertible since H_1 has no zero eigenvalues and H_2 is nilpotent. Let $N = \bar{N}(\lambda E - A)^{-1}$. Then R_1 and R_2 are both NE- and NA-invariant with

$$NE \,|\, R_1 = \bar{N}(\lambda E - A)^{-1}E \,|\, R_1 = H_1^{-1}\,H_1 = I$$

and

$$NA \,|\, R_2 = \bar{N}(\lambda E - A)^{-1}A \,|\, R_2 = (\lambda H_2 - I)^{-1}(\lambda H_2 - I) = I$$

Also,

$$NE \,|\, R_2 = (\lambda H_2 - I)^{-1}H_2$$

which is nilpotent and

$$NA \,|\, R_1 = H_1^{-1}(\lambda H_1 - I) = \lambda I - H_1^{-1}$$

Next, observe that

$$\begin{aligned} \det(Es - A) &= \frac{\det(NEs - NA)}{\det N} \\ &= \frac{\det(Is - NA \,|\, R_1)\,\det(NE \,|\, R_2 s - I)}{\det N} \end{aligned}$$

But $\det(NE \,|\, R_2 s - I) = (-1)^{n-d}$ so

$$\det(Is - NA \,|\, R_1) = \prod_{i=1}^{k} (s - \lambda_i)^{n_i}$$

Also,

$$\det(Is - NA \,|\, R_1) = \det(Is - (\lambda I - H_1^{-1})) = \prod_{i=1}^{\delta} (s - (\lambda - 1/\eta_i))^{p_i}$$

Thus, if the η_i are indixed properly, we have $\delta = k$, $p_i = n_i$, $d = r$ and $\lambda - 1/\eta_i = \lambda_i$ so $\eta_i = 1/(\lambda - \lambda_i)$. Hence $R_1 = S$, $R_2 = F$, $H_1 = J_1$, $H_2 = J_2$, $\bar{N} = \bar{M}$ and $N = M$. ∎

Proof of Theorem 2

(i) Let M be as in Theorem 1. Then

$$\begin{aligned} M(\operatorname{Im}(\lambda_i E - A) + \operatorname{Im} B) &= \operatorname{Im}(\lambda_i ME - MA) + \operatorname{Im}(MB) \\ &= \operatorname{Im}(\lambda_i I - L_s) + \operatorname{Im}(\lambda_i L_f - I) + \operatorname{Im}(MB) \\ &= (\operatorname{Im}(\lambda_i I - L_s) + \operatorname{Im} B_s) \oplus F \end{aligned}$$

since $\lambda_i L_f - I$ is invertible. From state-variable theory,

$$\operatorname{Im}(\lambda_i I - L_s) + \operatorname{Im} B_s = S$$

if and only if λ_i is a controllable eigenvalue. Since M is invertible the result follows.

(ii) From Yip and Manke (1978) we know that

$$\mathscr{R}_f = \operatorname{Im} B_f + \operatorname{Im} L_f B_f + \ldots + \operatorname{Im} L_f^{q-1} B_f$$

so $\mathscr{R}_f = F$ if and only if (L_f, B_f) is a controllable pair or equivalently,

$$\operatorname{Im} L_f + \operatorname{Im} B_f = F$$

But

$$M(\operatorname{Im} E + \operatorname{Im} B) = \operatorname{Im} (ME) + \operatorname{Im} (MB) = S \oplus (\operatorname{Im} L_f + \operatorname{Im} B_f)$$

and the result follows. ■

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