The Minimal Dimension of Stable Faces Required to Guarantee Stability of a Matrix Polytope: D-Stability

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Abstract—**We consider the problem of determining whether a polytope $\mathcal{P}$ of $n \times n$ matrices is D-stable—i.e., whether each point in $\mathcal{P}$ has all its eigenvalues in a given nonempty, open, convex, conjugate-symmetric subset D of the complex plane. Our approach is to check D-stability of certain faces of $\mathcal{P}$. In particular, for each D and n we determine the smallest integer m such that D-stability of every m-dimensional face guarantees D-stability of $\mathcal{P}$.**

I. Introduction

Let $D \subset \mathbb{C}$ be nonempty, open, convex, and conjugate-symmetric (symmetric about the real axis), and define an $n \times n$ real matrix M to be *D-stable* if each eigenvalue λ of M satisfies $\lambda \in D$; otherwise, M is *D-unstable*. We consider the problem of determining whether certain subsets of $\mathbb{R}^{n \times n}$ consist entirely of D-stable matrices. To facilitate discussion, we begin with some definitions.

A (convex) *polytope* $\mathcal{P}$ in a vector space V is the convex hull conv(Ω) of any nonempty finite subset $\Omega \subset V$. The *dimension* of $\mathcal{P}$ is the dimension of the affine hull aff$(\mathcal{P})$ of $\mathcal{P}$. The *relative boundary* of $\mathcal{P}$ is the boundary of $\mathcal{P}$ as a subset of the topological space aff$(\mathcal{P})$. A *face* of $\mathcal{P}$ is any set of the form $\Pi \cap \mathcal{P}$, where Π is a supporting hyperplane of $\mathcal{P}$. Finally, a *k-dimensional half-plane* in V is any nonempty set of the form $\mathcal{H} = R \cap S$, where R is a closed half-space, S is a k-dimensional affine subspace, and $S \not\subset R$. (Note that this implies that aff$(\mathcal{H})$ is simply S.)

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In the robust control literature, considerable interest has been generated by the problem of determining whether a family of linear systems can be shown to consist entirely of D-stable systems by checking D-stability of certain representative members of that family. In many cases, such problems can be reduced to that of determining whether a polytope or other subset of $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ consists entirely of D-stable points [1], [2]. (D-stability of a vector $x \in \mathbb{R}^n$ means simply that the polynomial $s^n + x_n s^{n-1} + \cdots + x_1$ has all its roots in D, where x_i is the ith entry of x.) We are primarily interested in the technique of checking D-stability of lower dimensional faces of a polytope in order to guarantee D-stability of the entire set.

Most "facial" results pertain to continuous-time (CT) stability—i.e., where D is the open left half complex plane. The seminal result [3] for polynomial polytopes motivates the approach. In [3] it is shown that a polynomial polytope of a particular simple structure (an "interval polynomial") is CT stable whenever four specially constructed vertices are CT stable. A more recent result [1] demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee CT stability. With respect to polytopes in $\mathbb{R}^{n\times n}$, it has been shown [4] that 1) an arbitrary polytope is CT stable if all $(2n-4)$-dimensional faces are CT stable and 2) there exist CT unstable polytopes such that all $(2n-5)$-dimensional faces are CT stable; hence, the value $2n-4$ is minimal. In this note we extend the results of [4] to D-stability where D may be any nonempty, open, convex, conjugate-symmetric subset of $\mathbb{C}$.

We note that for the cases $n = 0$ and $n = 1$, our problem has a trivial solution: D-stability of vertices guarantees D-stability of the polytope. To handle $n \geq 2$, we need to partition the family of stability sets D according to the following two assumptions.

Assumption A: D is of the form $D = \{s \in \mathbb{C} | a < \text{Re}\, s < b\}$, where $-\infty \leq a < b \leq \infty$.

Assumption B: D is a nonempty, open, convex, conjugate-symmetric set not satisfying Assumption A.

In addition, we define

$$m_A(n) = \begin{cases} 1, & n = 2 \\ 2n-4, & n > 2 \end{cases} \quad \text{and } m_B(n) = 2n-2.$$

We intend to show that m_A and m_B are the values of m that we seek for cases A and B.

II. Sufficiency of m_A and m_B

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f: \mathbb{R}^k \to \mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P} \subset \mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, f sets up a one-to-one correspondence between the q-dimensional faces of $\mathcal{P}$ and the q-dimensional faces of $f(\mathcal{P})$. In addition, f maps each k-dimensional half-plane in $\mathbb{R}^k$ into another k-dimensional half-plane (e.g., see [5]). Finally, we note that every polytope is compact and that any set of the form $\{x \in \mathbb{R}^k \mid \|x\|_\infty \leq \gamma\}$, where $\gamma > 0$, is a polytope whose q-dimensional faces are generated by fixing $k-q$ entries of x at either $\pm\gamma$ and letting the remaining q entries vary independently over $\{-\gamma, \gamma\}$.

With these observations in mind, we prove a result characterizing the affine structure of the set of D-unstable points in $\mathbb{R}^{n\times n}$.

Lemma 2.1: If D satisfies Assumption A (respectively, Assumption B), then for each D-unstable $M \in \mathbb{R}^{n\times n}$, there exists an $(n^2 - m_A)$-dimensional (respectively, $(n^2 - m_B)$-dimensional) half-plane $\mathcal{H} \subset \mathbb{R}^{n\times n}$ such that a) $M \in \mathcal{H}$ and b) $N \in \mathcal{H}$ implies N is D-unstable.

Proof: Suppose Assumption A holds. If $a = -\infty$, $b = \infty$, the statement is vacuously true; otherwise, we need to consider two cases.

Case I—M has a real eigenvalue $\lambda_0 \notin D$: Let $T = [v \quad W]$, where v is an eigenvector corresponding to λ_0 and W is chosen to make T nonsingular. Clearly, the map $f: \mathbb{R}^{n^2-n+1} \to \mathbb{R}^{n\times n}$ determined by $f(\lambda, y, Z) = T \begin{bmatrix} \lambda & y \\ 0 & z \end{bmatrix} T^{-1}$ is affine and one-to-one. If $\lambda_0 < a$, let $I = (-\infty, \lambda_0]$ and let $\bar{\mathcal{H}}$ be the $(n^2 - n + 1)$-dimensional half-plane $\bar{\mathcal{H}} = \{f(\lambda, y, Z) | \lambda \in I,\ y \in \mathbb{R}^{1\times n-1},\ Z \in \mathbb{R}^{n-1\times n-1}\}$. If $a = -\infty$, then $\lambda_0 > b$ so we set $I = [\lambda_0, \infty)$ and construct $\bar{\mathcal{H}}$ in the same way. In either case, $M \in \bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is D-unstable. Since $n^2 - n + 1 \geq n^2 - m_A$, it remains to select any $(n^2 - m_A)$-dimensional half-plane $\mathcal{H}$ satisfying $M \in \mathcal{H} \subset \bar{\mathcal{H}}$.

Case I—M has a real eigenvalue $\lambda_0 \notin D$. Again, let $T = [v \quad W]$, where v is an eigenvector corresponding to λ_0. Since D is convex, either $(-\infty, \lambda_0] \cap D = \phi$ or $[\lambda_0, \infty) \cap D = \phi$. In the former case, let $\bar{\mathcal{H}}$ be the $(n^2 - n + 1)$-dimensional half-plane $\bar{\mathcal{H}} = \left\{T \begin{bmatrix} \lambda & y \\ 0 & Z \end{bmatrix} T^{-1} | \lambda \leq \lambda_0,\ y \in \mathbb{R}^{1\times n-1},\ Z \in \mathbb{R}^{n-1\times n-1}\right\}$. For the latter case, alter the definition of $\bar{\mathcal{H}}$ by substituting "$\lambda \geq \lambda_0$" for "$\lambda \leq \lambda_0$." In either case, $M \in \bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is D-unstable. Since $n^2 - n + 1 > n^2 - m_B$, it remains to select any $(n^2 - m_B)$-dimensional half-plane $\mathcal{H}$ satisfying $M \in \mathcal{H} \subset \bar{\mathcal{H}}$.

Now suppose Assumption B holds. We again consider two cases.

Case I—M has a real eigenvalue $\lambda_0 \notin D$. Again, let $T = [v \quad W]$, where v is an eigenvector corresponding to λ_0. Since D is convex, either $(-\infty, \lambda_0] \cap D = \phi$ or $[\lambda_0, \infty) \cap D = \phi$. In the former case, let $\bar{\mathcal{H}}$ be the $(n^2 - n + 1)$-dimensional half-plane $\bar{\mathcal{H}} = \left\{T \begin{bmatrix} \lambda & y \\ 0 & Z \end{bmatrix} T^{-1} | \lambda \leq \lambda_0,\ y \in \mathbb{R}^{1\times n-1},\ Z \in \mathbb{R}^{n-1\times n-1}\right\}$. For the latter case, alter the definition of $\bar{\mathcal{H}}$ by substituting "$\lambda \geq \lambda_0$" for "$\lambda \leq \lambda_0$." In either case, $M \in \bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is D-unstable. Since $n^2 - n + 1 > n^2 - m_B$, it remains to select any $(n^2 - m_B)$-dimensional half-plane $\mathcal{H}$ satisfying $M \in \mathcal{H} \subset \bar{\mathcal{H}}$.

Case II—M has a complex eigenvalue pair $\alpha_0 \pm i\beta_0 \notin D$. Let $T = [u \quad v \quad W]$, where $u + iv$ is an eigenvector corresponding to $\alpha_0 + i\beta_0$. Since D is convex, there exists a half-space $\Pi \subset \mathbb{C}$ such that $\alpha_0 + i\beta_0 \in \Pi$ and $\Pi \cap D = \phi$. Let $\mathcal{H}$ be the $(n^2 - 2n + 2)$-dimensional half-plane

$$\mathcal{H} = \left\{ T \begin{bmatrix} \alpha & \beta & x \\ -\beta & \alpha & y \\ 0 & 0 & Z \end{bmatrix} T^{-1} | \alpha + i\beta \in \Pi; \quad x, y \in \mathbb{R}^{1\times n-2}; \quad Z \in \mathbb{R}^{n-2\times n-2} \right\}.$$

Clearly, $\mathcal{H}$ contains only D-unstable points, and $M \in \mathcal{H}$. □

Next we prove an easy result concerning the intersection of affine sets.

Lemma 2.2: Let V be a p-dimensional Euclidean space, $\mathcal{H} \subset V$ a k-dimensional half-plane, and Γ a q-dimensional affine subspace with $k + q > p$. Consider any vector $x_0 \in \mathcal{H} \cap \Gamma$. There exists a $(k+q-p)$-dimensional half-plane $\bar{\mathcal{H}}$ such that $x_0 \in \bar{\mathcal{H}} \subset \mathcal{H} \cap \Gamma$.

Proof: By definition, $\mathcal{H} = R \cap S$, where R is a closed half-space and S is a k-dimensional affine subspace satisfying $S \not\subset R$. There exists an affine subspace $\bar{S} \subset S \cap \Gamma$ with $\dim \bar{S} = k + q - p$ and $x_0 \in \bar{S}$. If $\bar{S} \subset R$, let $\bar{\mathcal{H}} \subset R \cap \bar{S}$ be any $(k + q - p)$-dimensional half-space containing x_0. Then $\bar{\mathcal{H}} \subset R \cap S \cap \Gamma = \mathcal{H} \cap \Gamma$. If $\bar{S} \not\subset R$, let $\bar{\mathcal{H}} = R \cap \bar{S}$. Then $x_0 \in \bar{\mathcal{H}}$, since $x_0 \in \mathcal{H} \cap \Gamma \subset R$. Also, $\dim \bar{\mathcal{H}} = k + q - p$, since $\bar{\mathcal{H}}$ is nonempty. □

We are now in a position to prove our first main result.

Theorem 2.3: Under Assumption A (respectively, Assumption B), D-stability of every matrix in every m_A-dimensional (respectively, m_B-dimensional) face of $\mathcal{P}$ guarantees D-stability of every matrix in $\mathcal{P}$.

Proof: Suppose Assumption A holds. Our arguments here are similar to those used in [2, Lemma 1]. If $\mathcal{P}_k$ is a D-unstable polytope of dimension $k > m_A$, there exists a D-unstable matrix $M_1 \in \mathcal{P}_k$. From Lemma 2.1, there is an $(n^2 - m_A)$-dimensional half-plane $\mathcal{H}_1$, consisting entirely of D-unstable points and containing M_1. Since $\mathcal{H}_1$ is unbounded, there exists an $M_2 \in \mathcal{H}_1$ lying on the boundary of $\mathcal{P}_k$ and, hence, in one of its $(k-1)$-dimensional faces $\mathcal{P}_{k-1}$. From Lemma 2.2, the intersection $\mathcal{H}_1 \cap \text{aff}(\mathcal{P}_{k-1})$ contains a $(k - m_A - 1)$-dimensional half-plane $\mathcal{H}_2$ such that $M_2 \in \mathcal{H}_2$. Proceeding inductively, we find that there exists an m_A-dimensional face $\mathcal{P}_m$ and a point $M_{k-m} \in \mathcal{P}_m$ such that M_{k-m} is D-unstable.

Under Assumption B, the same proof holds if we replace m_A by m_B. □

III. Minimality of m_A and m_B

Our next task is to show that m_A and m_B are the smallest integers such that D-stability of all m_A-dimensional or m_B-dimensional faces of

$\mathcal{P}$ guarantees D-stability of $\mathcal{P}$ under Assumptions A and B, respectively. In order to prove this, we need a lemma which may be interpreted as a multivariable extension of L'Hospital's rule. For any $k \times k$ matrices Q and R, we use the notation $Q > 0$ and $R < 0$ to signify that Q is positive definite symmetric and R is negative definite symmetric, respectively.

Lemma 3.1: Let $0 \in U \subset \mathbb{R}^k$ with U open, and let $e_1, e_2: U \to \mathbb{R}^2$ be C^2 functions. In addition, suppose $e_1(0) = e_2(0) = 0$,

$$\left.\frac{\partial e_1}{\partial x}\right|_{x=0} = \left.\frac{\partial e_2}{\partial x}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 e_1}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 e_2}{\partial x^2}\right|_{x=0} < 0.$$

For every $\delta > 0$, there exists an $\epsilon > 0$ such that $0 \neq \|x\| < \epsilon$ implies $e_2(x) < -\frac{1}{\delta}|e_1(x)|$.

Proof: From [6, p. 340], for every $Q > 0$, there exists an $\epsilon > 0$ such that $\|x\| < \epsilon$ implies

$$\frac{\left| e_i(x) - e_i(0) - \left.\frac{\partial e_i}{\partial x}\right|_{x=0} x - \frac{1}{2}x^T \left.\frac{\partial^2 e_i}{\partial x^2}\right|_{x=0} x \right|}{x^T Q x} < \frac{1}{2}\left(\frac{\delta}{1+\delta}\right).$$

Setting $Q = -\left.\frac{\partial^2 e_2}{\partial x^2}\right|_{x=0}$ yields

$$\frac{|e_1(x)|}{x^T Q x} < \delta, \qquad \frac{\left|e_2(x) + \frac{1}{2}x^T Q x\right|}{x^T Q x} < \delta \tag{1}$$

and, from (1), $e_2(x) < \left(\delta - \frac{1}{2}\right) x^T Q x < 0$ for $x \neq 0$. Hence, for $x \neq 0$,

$$\left|\frac{e_1(x)}{e_2(x)}\right| = \frac{|e_1(x)|}{\left|\left(e_2(x) + \frac{1}{2}x^T Q x\right) - \frac{1}{2}x^T Q x\right|}$$

$$= \frac{\dfrac{|e_1(x)|}{x^T Q x}}{\left|\dfrac{e_2(x) + \frac{1}{2}x^T Q x}{x^T Q x} - \dfrac{1}{2}\right|} < \frac{\frac{1}{2}\left(\frac{\delta}{1+\delta}\right)}{\frac{1}{2} - \frac{1}{2}\left(\frac{\delta}{1+\delta}\right)} = \delta.$$

Thus, $e_2(x) < -\frac{1}{\delta}|e_1(x)|$. □

Now we can prove our second main result.

Theorem 3.2: Suppose D satisfies Assumption A (respectively, Assumption B). For each n, there exists an m_A-dimensional (respectively, m_B-dimensional) polytope $\mathcal{P} \subset \mathbb{R}^{n \times n}$ containing a D-unstable point and such that all $(m_A - 1)$-dimensional (respectively, $(m_B - 1)$-dimensional) faces of $\mathcal{P}$ are D-stable.

Proof: Under Assumption A, we need to consider nine cases.

Case I—$a > -\infty$, $b < \infty$, $n = 2$: Consider the affine, one-to-one map

$$f(x) = \begin{bmatrix} b & x \\ -x & \frac{a+b}{2} \end{bmatrix}$$

and the corresponding one-dimensional polytope $\mathcal{P} = \{f(x) \mid |x| \leq 1\}$. The point in $\mathcal{P}$ corresponding to $x = 0$ is clearly D-unstable. It suffices to prove that the characteristic polynomials Δ^+ and Δ^- of $f(x) - bI$ and $aI - f(x)$, respectively, are Hurwitz for all $x \neq 0$. This is in fact true, since $\Delta^+(s) = s^2 + \frac{1}{2}(b-a)s + x^2$ and $\Delta^-(s) = s^2 + \frac{3}{2}(b-a)s + \frac{1}{2}(b-a)^2 + x^2$ have positive coefficients for $x \neq 0$.

Case II—$a > -\infty$, $b < \infty$, $n = 3$: Let

$$f(x, y) = \begin{bmatrix} b & 1 & -x \\ -1 & b & -y \\ x & y & \frac{a+b}{2} \end{bmatrix}.$$

It is sufficient to show that the characteristic polynomials Δ^+ and Δ^- of $f(x, y) - bI$ and $aI - f(x, y)$, respectively, are Hurwitz for $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. A straightforward calculation yields $\Delta^+(s) = s^3 + \frac{1}{2}(b-a)s^2 + (1 + x^2 + y^2)s + \frac{1}{2}(b-a)$ and $\Delta^-(s) = s^3 + \frac{5}{2}(b-a)s^2 + (1 + x^2 + y^2 + 2(b-a)^2)s + \frac{1}{2}(b-a)(1 + 2x^2 + 2y^2 + (b-a)^2)$. Each polynomial has positive coefficients for $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. The fact that they are Hurwitz follows from positivity of the second-order leading principal minors $M_2^+ = \frac{1}{2}(b-a)(x^2 + y^2)$ and $M_2^- = \frac{1}{2}(b-a)(4 + 3x^2 + 3y^2 + 9(b-a)^2)$ of the corresponding 3×3 Hurwitz matrices.

Case III—$b > -\infty$, $a < \infty$, $n > 3$: Let

$$f(x, y) = \begin{bmatrix} b & 1 & -x^T \\ -1 & b & -y^T \\ x & y & \left(\frac{a+b}{2}\right) I \end{bmatrix}$$

where $x, y \in \mathbb{R}^{n-2}$. A tedious calculation shows that $\Delta^+(s) = (s + \frac{1}{2}(b-a)^{n-4}\hat{\Delta}^+(s)$ and $\Delta^-(s) = (s + \frac{1}{2}(b-a)^{n-4}\hat{\Delta}^-(s)$, where

$$\hat{\Delta}^+(s) = s^4 + (b-a)s^3 + \left(1 + x^T x + y^T y + \left(\frac{b-a}{2}\right)^2\right) s^2$$
$$+ \frac{b-a}{2}(2 + x^T x + y^T y)s + x^T x y^T y - (x^T y)^2 + \left(\frac{\beta - \alpha}{\alpha}\right)^2$$

$$\hat{\Delta}^-(s) = s^4 + 3(b-a)s^3 + \left(1 + x^T x + y^T y + \frac{13}{4}(b-a)^2\right) s^2$$
$$+ (b-a)\left(1 + \frac{3}{2}x^T x + \frac{3}{2}y^T y + \frac{3}{2}(b-a)^2\right) s$$
$$+ x^T x y^T y - (x^T y)^2 + \left(\frac{b-a}{2}\right)^2$$
$$\cdot (1 + 2x^T x + 2y^T y + (b-a)^2).$$

From the Schwartz inequality, $\hat{\Delta}^+$ and $\hat{\Delta}^-$ have positive coefficients when $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$. Furthermore, the third-order leading principal minors of the corresponding 4×4 Hurwitz matrices of $\hat{\Delta}^+$ and $\hat{\Delta}^-$ are

$$M_3^+(s) = \left(\frac{b-a}{2}\right)^2 \left(2\left(1 + \left(\frac{b-a}{2}\right)^2\right)\right.$$
$$\left.\cdot (x^T x + y^T y) + (x^T x - y^T y)^2 + 4(x^T y)^2\right)$$

$$M_3^-(s) = \left(2 + \frac{9}{2}x^T x + \frac{9}{2}y^T y + \frac{9}{4}(x^T x - y^T y)^2 + (x^T y)^2\right)(b-a)^2$$
$$+ \left(\frac{27}{4} + \frac{63}{8}x^T x + \frac{63}{8}y^T y\right)(b-a)^4 + \frac{81}{8}(b-a)^6.$$

Since M_3^+ and M_3^- are positive, $\hat{\Delta}^+$ and $\hat{\Delta}^-$ and, hence, Δ^+ and Δ^- are Hurwitz.

The remaining six cases are handled similarly by choosing all eigenvalues in the interior of D, except for one or two on the boundary of D. For example, for $a > -\infty$, $b = \infty$, $n > 3$, set

$$f(x, y) = \begin{bmatrix} a & 1 & -x^T \\ -1 & a & -y^T \\ x & y & (a+1)I \end{bmatrix}.$$

Adopting Assumption B, suppose D is not of the form $\{s \mid a < \operatorname{Re} s < b\}$. Since D is convex, there exists a real $\alpha_0 \in D$ such that the line $L = \{\alpha_0 + i\beta \mid \beta \in \mathbb{R}\}$ satisfies $L \not\subset D$. Since D is conjugate symmetric and open, there exists a $\beta_0 > 0$ such that $\alpha_0 \pm i\beta_0$ are boundary points of D, but $\alpha_0 \pm i\beta \in D$ when $|\beta| < \beta_0$. Furthermore, there exists a $\delta > 0$ such that $\alpha_0 \pm \delta \in D$. Again invoking convexity, the open diamond $\bar{d}_\delta = \operatorname{int conv}\{\alpha_0 \pm \delta\beta_0, \alpha_0 \pm i\beta_0\}$ is contained in D. To simplify the problem, consider the open diamond $d_\delta = (1/\beta_0)(\bar{d}_\delta - \alpha_0) = \operatorname{int conv}\{\pm i, \pm\delta\}$. We need only construct a single polytope $\mathcal{P}$ containing a matrix with a pair of eigenvalues at $\pm i$ and with all m_B-dimensional faces containing a matrix with a pair of eigenvalues at $\pm i$ and with all m_B-dimensional faces consisting of matrices with all eigenvalues in d_δ; then $\beta_0\mathcal{P} + \alpha_0 I$ satisfies the desired properties with respect to $\bar{d}_\delta$.

Consider the $(n^2 - 2n + 2)$-dimensional polytope

$$\mathcal{P}_\epsilon = \left\{ \begin{bmatrix} w & 1+x & y^T \\ -1+x & -w & z^T \\ y & z & 0 \end{bmatrix} \middle| \left\| \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \right\|_\infty \le \epsilon \right\}$$

where $y, z \in \mathbb{R}^{n-2}$. Clearly, $\mathcal{P}_\epsilon$ has a D-unstable point M at $w = x = 0$, $y = z = 0$. We will show that for sufficiently small ϵ, every point in $\mathcal{P}_\epsilon$ except M is D-stable. Hence, $\mathcal{P} = \mathcal{P}_\epsilon$ satisfies the desired properties.

Case I—n=2: Each point in $\mathcal{P}_\epsilon$ has characteristic polynomial $\Delta(w, x, s) = s^2 + 1 - w^2 - x^2$, and hence has eigenvalues $\pm i(1 - w^2 - x^2)^{1/2}$. Let $\epsilon < (((1+\delta)^2/2))^{1/2}$.

Case II—n=3: Each point in $\mathcal{P}_\epsilon$ has characteristic polynomial

$$\Delta(w, x, y, z, s) = s^3 + (1 - w^2 - x^2 - y^2 - z^2)s - (w(y^2 - z^2) + 2xyz).$$

Let

$$g(w, x, y, z, \alpha, \beta) = \begin{bmatrix} \mathrm{Re} & \Delta(w, x, y, z, \alpha + i\beta) \\ \mathrm{Im} & \Delta(w, x, y, z, \alpha + i\beta) \end{bmatrix}.$$

It is easy to see that g is a polynomial function and, hence, analytic. A straightforward calculation shows

$$\left.\frac{\partial g}{\partial(\alpha, \beta)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, from the implicit function theorem, there exists a unique analytic function $h: U \to \mathbb{R}^2$ such that $h(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $g(w, x, y, z, h(w, x, y, z)) = 0$ for every $[w \quad x \quad y \quad z]^T \in U$.

Next, let $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = h - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. A tedious computation shows

$$\left.\frac{\partial e_1}{\partial(w, x, y, z)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \left.\frac{\partial e_2}{\partial(w, x, y, z)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = 0,$$

$$\left.\frac{\partial^2 e_1}{\partial(w, x, y, z)^2}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = 0, \qquad \left.\frac{\partial^2 e_2}{\partial(w, x, y, z)^2}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = -I.$$

From Lemma 3.1, there exists an $\epsilon > 0$ such that $e_2(w, x, y, z) < -(1/\delta)|e_1(w, x, y, z)|$ whenever $0 \ne \|[w \quad x \quad y \quad z]^T\| < \epsilon$. Since e_1 and e_2 are continuous, we may also assume $|e_i| < 1$; $i = 1, 2$. Returning to $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$, it follows that $h_2(w, x, y, z) < 1 - (1/\delta)|h_1(w, x, y, z)|$, $|h_2(w, x, y, z)| < 1$, and $|h_1(w, x, y, z) - 1| < 1$ for all $[w \quad x \quad y \quad z]^T \ne 0$. Hence, $h \in d_\delta$.

Case III—n>3: We have

$$\Delta(w, x, y, z, s) = s^4 + (1 - w^2 - x^2 - x^T x - y^T y)s^2 - (wy^T y - 2xy^T z - wz^T z)s + y^T y z^T z - (y^T z)^2.$$

Let

$$g(w, x, y, z, \alpha, \beta) = \begin{bmatrix} \mathrm{Re} & \Delta(w, x, y, z, \alpha + i\beta) \\ \mathrm{Im} & \Delta(w, x, y, z, \alpha + i\beta) \end{bmatrix}.$$

Again, g is a polynomial function; in this case,

$$\left.\frac{\partial g}{\partial(\alpha, \beta)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

Thus, there exists an open $U \subset \mathbb{R}^{2n-2}$ with $0 \in U$ and $h: U \to \mathbb{R}^2$ such that $h(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $g(w, x, y, z, h(w, x, y, z)) = 0$ for every $[w \quad x \quad y \quad z]^T \in U$. Let $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = h - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$$\left.\frac{\partial e_1}{\partial(w, x, y, z)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \left.\frac{\partial e_2}{\partial(w, x, y, z)}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = 0,$$

$$\left.\frac{\partial^2 e_1}{\partial(w, x, y, z)^2}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = 0, \qquad \left.\frac{\partial^2 e_2}{\partial(w, x, y, z)^2}\right|_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = -I.$$

Applying Lemma 3.1 as in Case II, it follows that $h(w, x, y, z) \in d_\delta$ for every $[w \quad x \quad y \quad z]^T \ne 0$. □

Note that Theorem 3.2 also implies that the half-planes constructed in Lemma 2.1 are maximal in the sense that there exists a D-unstable matrix M in $\mathbb{R}^{n \times n}$ such that every half-plane containing M of dimension greater than $n^2 - m_A$ or $n^2 - m_B$ must also contain a D-unstable matrix. Indeed, if this were not the case, the arguments in Theorem 3.2 could be used to prove that m_A and m_B are not minimal.

IV. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [1] and [3] can be extended to the case of $n \times n$ matrices. We have shown that, without further information describing the particular structure of a polytope, either $(2n - 4)$-dimensional or $(2n - 2)$-dimensional faces need to be checked for D-stability, depending on the structure of D. Since testing even one such face can be a formidable task when n is large, and since the number of $(2n - 4)$-dimensional and $(2n - 2)$-dimensional faces grow exponentially with n, more work needs to be done before a computationally tractable algorithm can be devised for checking D-stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

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