The Minimal Dimension of Stable Faces Required to Guarantee Stability of a Matrix Polytope

J. DANIEL COBB AND CHRISTOPHER L. DEMARCO

Abstract—**We consider the problem of determining whether each point in a polytope of $n \times n$ matrices is stable. Our approach is to check stability of certain faces of the polytope. For $n \geq 3$, we show that stability of each point in every $(2n - 4)$-dimensional face guarantees stability of the entire polytope. Furthermore, we prove that, for any $k \leq n^2$, there exists a k-dimensional polytope containing a strictly unstable point and such that all its subpolytopes of dimension $\min \{k - 1, 2n - 5\}$ are stable.**

Manuscript received July 11, 1988; revised September 21, 1988. This work was supported by the National Science Foundation under Grants ECS-8612948, ECS-8611728, and ECS-8857019.

The authors are with the Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, WI 53706-1691.

IEEE Log Number 8929491

I. Background and Introduction

In this note we consider the problem of ascertaining whether certain subsets of $\mathbb{R}^{n \times n}$ consist entirely of stable matrices. (Here we take stability of a matrix to mean that all its eigenvalues are in the open left-half plane.) First we need some definitions. A *(convex) polytope* $\mathcal{P}$ in vector space V is the convex hull of any nonempty finite subset of V. The *dimension* of $\mathcal{P}$ is the dimension of the affine hull aff $(\mathcal{P})$ of $\mathcal{P}$. The *relative boundary* of $\mathcal{P}$ is the boundary of $\mathcal{P}$ as a subset of the topological space aff $\mathcal{P}$. A *face* of $\mathcal{P}$ is any set of the form $\Pi \cap \mathcal{P}$, where Π is a supporting hyperplane of $\mathcal{P}$. A *vertex* of $\mathcal{P}$ is a zero-dimensional face. An *edge* of $\mathcal{P}$ is a one-dimensional face. A *subpolytope* of $\mathcal{P}$ is the convex hull of any set of vertices of $\mathcal{P}$. Finally, a *k-dimensional half-plane* in V is any nonempty set of the form $\mathcal{H} = R \cap S$, where K is a closed half-space, S is a k-dimensional affine subspace, and $S \not\subset R$. (Note that this implies that the affine hull of $\mathcal{H}$ is simply S.)

In the robust control literature, considerable interest has been generated by the problem of determining whether stability of a polytope in either $\mathbb{R}^n$ or $\mathbb{R}^{n \times n}$ can be guaranteed simply by checking stability of low-dimensional faces. (Stability of a vector $x \in \mathbb{R}^n$ means simply that the polynomial $s^n + x_n s^{n-1} + \cdots + x_1$ is Hurwitz.) We first note that the cases $n = 0$ and $n = 1$ are trivial; stability of the vertices always guarantees stability of the polytope. Several recent papers consider the case $n \geq 2$. For example, polynomial polytopes of a particularly simple structure ("interval polynomials") were addressed by Kharitonov [1]; he showed that only four specially constructed vertices need be checked. A more recent result of Bartlett, Hollot, and Lin [2] demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee stability of $\mathcal{P}$. With respect to polytopes in $\mathbb{R}^{n \times n}$, Fu and Barmish [3] have shown that stability of all one-dimensional subpolytopes is insufficient to guarantee stability of $\mathcal{P}$. DeMarco [4] has shown that, for $n \geq 3$, $(n - 2)$-dimensional faces are insufficient, but $2n$-dimensional faces are sufficient.

In this note we refine the bounds of [4] and show that stability of all m-dimensional faces is sufficient to guarantee stability of $\mathcal{P}$, where

$$m(n) = \begin{cases} 1, & n = 2 \\ 2n - 4, & n > 2 \end{cases}.$$

Furthermore, we show that for any n and $k \leq n^2$ there exists a polytope of dimension k, containing a strictly unstable point (a matrix with an eigenvalue λ satisfying $\operatorname{Re} \lambda > 0$), and such that all its $\min \{k - 1, m - 1\}$-dimensional subpolytopes are stable; hence, in this sense, m is minimal.

II. Sufficiency of m

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f: \mathbb{R}^k \rightarrow \mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P} \subset \mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, f sets up a one-to-one correspondence between q-dimensional faces of $\mathcal{P}$ and q-dimensional faces of $f(\mathcal{P})$. In addition, f maps each k-dimensional half-plane in $\mathbb{R}^k$ into another k-dimensional half-plane (e.g., see [5]). Finally, we note that every polytope is compact and that the set $\{x \in \mathbb{R}^k \mid \|x\|_\infty \leq 1\}$ is a polytope whose q-dimensional faces are generated by fixing $k - q$ entries of x at either ± 1 and letting the remaining q entries vary independently over $[-1, 1]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of unstable points in $\mathbb{R}^{n \times n}$.

Lemma 2.1: For each unstable $A \in \mathbb{R}^{n \times n}$ there exists an $(n^2 - m)$-dimensional half-plane $\mathcal{H} \subset \mathbb{R}^{n \times n}$ such that: 1) $A \in \mathcal{H}$; and 2) $B \in \mathcal{H}$ implies B is unstable.

0018-9286/89/0900-0990$01.00 © 1989 IEEE

Proof:

Case I—A Has a Real Eigenvalue $\lambda_0 \geq 0$: Let $T = [v\ W]$, where v is an eigenvector corresponding to λ_0 and W is chosen to make T nonsingular. Clearly, the map $f: \mathbb{R}^{n^2-n+1} \to \mathbb{R}^{n\times n}$ determined by

$$f(\lambda, y, Z) = T \begin{bmatrix} \lambda & y \\ 0 & Z \end{bmatrix} T^{-1}$$

is affine and one-to-one. Let $\bar{\mathcal{H}}$ be the $(n^2 - n + 1)$-dimensional half-plane

$$\bar{\mathcal{H}} = \{f(\lambda, y, Z) | \lambda \geq \lambda_0, y \in \mathbb{R}^{1\times n-1}, Z \in \mathbb{R}^{n-1\times n-1}\}.$$

Then $A \in \bar{\mathcal{H}}$ and every matrix in $\bar{\mathcal{H}}$ is unstable. Since $n^2 - n + 1 \geq n^2 - m$, we need only select any $(n^2 - m)$-dimensional half-plane $\mathcal{H}$ satisfying $A \in \mathcal{H} \subset \bar{\mathcal{H}}$.

Case II—A Has a Complex Eigenvalue Pair $\alpha_0 \pm i\beta_0$ *with* $\alpha_0 > 0$: Let $T = [v\ w\ X]$, where $v + iw$ is an eigenvector corresponding to $\alpha_0 + i\beta_0$ and X is chosen to make T nonsingular. Let $\bar{\mathcal{H}}$ be the $(n^2 - 2n + 4)$-dimensional half-plane

$$\bar{\mathcal{H}} = \left\{ T \begin{bmatrix} U & Y \\ 0 & Z \end{bmatrix} T^{-1} \middle| \operatorname{tr} U \geq 2\alpha_0,\ Y \in \mathbb{R}^{2\times n-2},\ Z \in \mathbb{R}^{n-2\times n-2} \right\}.$$

($\operatorname{tr} U \geq 2\alpha_0$ describes a four-dimensional half-plane, since $\operatorname{tr} U = \langle U, I \rangle$.) $\bar{\mathcal{H}}$ contains only unstable points, since $\operatorname{tr} U \geq 2\alpha_0$ implies U has at least one eigenvalue λ with $\operatorname{Re} \lambda \geq \alpha_0$. Also, $A \in \mathcal{H}$, since our choice of T guarantees that A has

$$U = \begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}.$$

Finally, $n^2 - 2n + 4 \geq n^2 - m$, so the desired $\mathcal{H} \subset \bar{\mathcal{H}}$ exists. □

Next we prove an easy result concerning the intersection of affine sets.

Lemma 2.2: Let V be a p-dimensional Euclidean space, $\mathcal{H} \subset V$ a k-dimensional half-plane, and Γ a q-dimensional affine subspace with $k + q > p$. Consider any vector $x_0 \in \mathcal{H} \cap \Gamma$. There exists a $(k + q - p)$-dimensional half-plane $\bar{\mathcal{H}}$ such that $x_0 \in \bar{\mathcal{H}} \subset \mathcal{H} \cap \Gamma$.

Proof: By definition, $\mathcal{H} = R \cap S$, where R is a closed half-space and S is a k-dimensional affine subspace satisfying $S \not\subset R$. There exists an affine subspace $\bar{S} \subset S \cap \Gamma$ with $\dim \bar{S} = k + q - p$ and $x_0 \in \bar{S}$. If $\bar{S} \subset R$, let $\bar{\mathcal{H}} \subset R \cap \bar{S}$ be any $(k + q - p)$-dimensional half-space containing x_0. Then $\bar{\mathcal{H}} \subset R \cap S \cap \Gamma = \mathcal{H} \cap \Gamma$. If $\bar{S} \not\subset R$, let $\bar{\mathcal{H}} = R \cap \bar{S}$. Then $x_0 \in \bar{\mathcal{H}}$, since $x_0 \in \mathcal{H} \cap \Gamma \subset R$. Also, $\dim \bar{\mathcal{H}} = k + q - p$, since $\bar{\mathcal{H}}$ is nonempty. □

We are now in a position to prove our first main result.

Theorem 2.3: Stability of every matrix in every m-dimensional face of $\mathcal{P}$ guarantees stability of every matrix in $\mathcal{P}$.

Proof: Our arguments here are similiar to those used in [2, Lemma 1]. Suppose $\mathcal{P}_k$ is an unstable polytope of dimension $k > m$. Then there exists an unstable matrix $A_1 \in \mathcal{P}_k$. From Lemma 2.1, there is an $(n^2 - m)$-dimensional half plane $\mathcal{H}_1$, consisting entirely of unstable points and containing A_1. Since $\mathcal{H}_1$ is unbounded, there exists an $A_2 \in \mathcal{H}_1$ lying on the boundary of $\mathcal{P}_k$ and, hence, in one of its $(k - 1)$-dimensional faces $\mathcal{P}_{k-1}$. From Lemma 2.2, the intersection $\mathcal{H}_1 \cap \operatorname{aff}(\mathcal{P}_{k-1})$ contains a $(k - m - 1)$-dimensional half-plane $\mathcal{H}_2$ such that $A_2 \in \mathcal{H}_2$. Proceeding inductively, we find there exists an m-dimensioanl face $\mathcal{P}_m$ and a point $A_{k-m} \in \mathcal{P}_m$ such that A_{k-m} is unstable. □

III. Minimality of m

Our next task is to show that m is the smallest integer such that stability of all m-dimensional faces of $\mathcal{P}$ guarantees stability of $\mathcal{P}$.

Theorem 3.1: For each integer $n \geq 2$ there exists an m-dimensional polytope $\mathcal{P} \subset \mathbb{R}^{n\times n}$ containing an unstable point and such that all its $(m - 1)$-dimensional faces are stable.

Proof:

Case I—$n = 2$: Consider the affine, one-to-one map

$$f(x) = \begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix}$$

and the corresponding one-dimensional polytope $\mathcal{P} = \{f(x) | |x| \leq 1\}$. Each matrix in $\mathcal{P}$ has characteristic polynomial $\Delta(s) = s^2 + s + x^2$; hence, each vertex of $\mathcal{P}$(i.e., $x = \pm 1$) is stable, but the point corresponding to $x = 0$ is unstable.

Case II—$n = 3$: The two-dimensional polytope

$$\mathcal{P} = \left\{ \begin{bmatrix} 0 & 1 & -x \\ -1 & 0 & -y \\ x & y & -1 \end{bmatrix} \middle| \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\infty \leq 1 \right\}$$

has characteristic polynomial $\Delta(s) = s^3 + s^2 + (1 + x^2 + y^2)s + 1$. Each coefficient of $\Delta(s)$ is positive, and the corresponding 2×2 Hurwitz matrix has its leading principal second-order minor equal to $M_2(x, y) = x^2 + y^2$. Thus, each edge is stable, but the matrix corresponding to $x = y = 0$ is unstable.

Case III—$n \geq 4$: Consider the $(2n - 4)$-dimensional polytope

$$\mathcal{P} = \left\{ \begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix} \middle| \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\infty \leq 1 \right\}.$$

A routine calculation shows that $\mathcal{P}$ has characteristic polynomial $p(s) = (s + 1)^{n-4}\Delta(s)$, where

$$\Delta(s) = s^4 + 2s^3 + (2 + x^Tx + y^Ty)s^2 + (2 + x^Tx + y^Ty)s + 1 + x^Txy^Ty - (x^Ty)^2.$$

From the Schwartz inequality, it is clear that all coefficients of Δ are strictly positive. The corresponding 4×4 Hurwitz matrix has its leading principal third-order minor equal to

$$M_3(x, y) = 4x^Tx + 4y^Ty + 4(x^Ty)^2 + (x^Tx - y^Ty)^2.$$

Clearly, $M_3 \geq 0$ with equality if and only if $x = y = 0$. Thus, the $(2n - 5)$-dimensional faces of $\mathcal{P}$ are stable, but the point corresponding to $x = y = 0$ is unstable. □

It is interesting to note that Theorem 3.1 also implies that the half-planes considered in Lemma 2.1 are maximal in the sense that there exists an unstable matrix A in $\mathbb{R}^{n\times n}$ such that every half-plane of dimension greater than $n^2 - m$ containing A must also contain a stable matrix. Indeed, if this were not the case, the arguments in Theorem 2.3 could be used to prove that m is not minimal.

IV. A Stronger Version of the Minimality Theorem

The construction in the proof of Theorem 3.1 is weak in the following three respects: 1) The polytope $\mathcal{P}$ contains only a single marginally unstable matrix (i.e., a matrix having all eigenvalues λ satisfying $\operatorname{Re} \lambda \leq 0$ and at least one with $\operatorname{Re} \lambda = 0$). 2) The construction yields only a polytope of dimension m. 3) Arbitrary subpolytopes are not considered; thus it is not clear that checking all subpolytopes of dimension, say $m - 1$, would not guarantee stability. The minimality proof would be more convincing if it could be extended to give a family of polytopes, each: 1) containing a strictly unstable point (and hence, infinitely many unstable points); 2) having arbitrary dimension k; and 3) having all $\min\{k - 1, m - 1\}$-dimensional subpolytopes stable.

Theorem 4.2 shows that such improvements over Theorem 3.1 can be made. The proof requires a simple lemma. For any normed linear space V, subset $\Omega \subset V$, and point $\gamma \in V$ consider the distance function

$$d(\gamma, \Omega) = \inf_{\omega \in \Omega} \|\gamma - \omega\|.$$

Let $\operatorname{conv}(\Omega)$ denote the convex hull of Ω.

Lemma 4.1: Suppose $\Omega \subset V$ is convex, $\epsilon > 0$, and $\Gamma \subset V$ is any set such that $d(\gamma, \Omega) < \epsilon$ for every $\gamma \in \Gamma$. Then $d(\eta, \Omega) < \epsilon$ for every $\eta \in \operatorname{conv}(\Omega \cup \Gamma)$.

Proof: Each $\eta \in \operatorname{conv}(\Omega \cup \Gamma)$ is of the form $\eta = \alpha\eta_1 + (1 - \alpha)\eta_2$, where $\alpha \in [0, 1]$ and $\eta_1, \eta_2 \in \Omega \cup \Gamma$. There exist $\omega_1, \omega_2 \in \Omega$ such that $\|\eta_1 - \omega_1\| < \epsilon$ and $\|\eta_2 - \omega_2\| < \epsilon$. Let $\omega = \alpha\omega_1 + (1 - \alpha)\omega_2$.

Then $\omega \in \Omega$, and

$$\begin{aligned} d(\eta, \Omega) &\le \|\eta-\omega\| \\ &= \|\alpha(\eta_1-\omega_1)+(1-\alpha)(\eta_2-\omega_2)\| \\ &\le \alpha\|\eta_1-\omega_1\|+(1-\alpha)\|\eta_2-\omega_2\| \\ &< \epsilon. \end{aligned}$$

□

Theorem 4.2: For any integers n, k with $n \ge 2$ and $1 \le k \le n^2$, there exists a polytope $\mathcal{P}_k$ of dimension k containing a strictly unstable point and such that each min $\{k-1, m-1\}$-dimensional subpolytope is stable.

Proof: Suppose a marginally unstable polytope $\bar{\mathcal{P}}_k$ of dimension k is constructed such that all its min $\{k-1, m-1\}$ subpolytopes are stable. Then, since the set of stable points in $\mathbb{R}^{n\times n}$ is open and the union of all min $\{k-1, m-1\}$-dimensional subpolytopes of $\bar{\mathcal{P}}_k$ is compact, the subpolytopes of $\mathcal{P}_k = \bar{\mathcal{P}}_k + \epsilon I$ of the same dimension are stable for sufficiently small ϵ, but $\mathcal{P}_k$ is strictly unstable. Thus, it suffices to construct any k-dimensional unstable $\mathcal{P}_k$ with stable subpolytopes.

If $n = 2$, let

$$f(x, y)=\begin{bmatrix} 0 & x \\ -x & -1 \end{bmatrix},$$

where x, y range over $\mathbb{R}$; otherwise, let

$$f(x, y)=\begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix}$$

where $x, y \in \mathbb{R}^{n-2}$. We consider two cases; first, assume $k < m$. Define $f_k: \mathbb{R}^k \to \mathbb{R}^{n\times n}$ according to $f_k([{}^{W}_{0}], [{}^{Z}_{0}])$, where the vectors x and y are partitioned in any way such that $[{}^{W}_{Z}] \in \mathbb{R}^k$. Since each f_k is affine and one-to-one, the set

$$\mathcal{Q}_k=\left\{f_k(w, z) \,\middle|\, \left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|_\infty \le 1\right\}$$

is a k-dimensional polytope. As in the proof of Theorem 3.1, each matrix in $\mathcal{Q}_k$ is stable except for the point corresponding to $w = z = 0$. The union of the $(k-1)$-dimensional subpolytopes of $\mathcal{Q}_k$ is compact and nowhere dense in $f(\mathbb{R}^k)$; hence, there exist vectors $w_0, z_0 \in f(\mathbb{R}^k)$ such that

$$\mathcal{P}_k=\left\{f_k(w+w_0, z+z_0) \,\middle|\, \left\|\begin{bmatrix} w \\ z \end{bmatrix}\right\|_\infty \le 1\right\}$$

is unstable, but has all its $(k-1)$-dimensional subpolytopes stable.

Next consider the case $k > m$. The union of the $(m-1)$-dimensional subpolytopes of the m-dimensional polytopes $\mathcal{P}$ (defined in Theorem 3.1) are compact and nowhere dense in $f(\mathbb{R}^m)$; hence, there exist x_0, y_0 such that

$$\mathcal{Q}=\left\{f(x+x_0, y+y_0) \,\middle|\, \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1\right\}$$

has all its $(m-1)$-dimensional subpolytopes stable, but $\mathcal{Q}$ is unstable. If $n = 2$, let

$$g(x, y, z)=f(x, y)+\begin{bmatrix} z_1 & z_2 \\ 0 & z_3 \end{bmatrix}.$$

Otherwise, define

$$g(x, y, z)=f(x, y)+\begin{bmatrix} z_1 & z_2 & z_3 & \cdots & z_n \\ z_{n+1} & z_{n+2} & z_{n+3} & \cdots & z_{2n} \\ 0 & 0 & z_{2n+1} & \cdots & z_{3n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & z_{n^2-3n+7} & \cdots & z_{n^2-2n+4} \end{bmatrix}$$

In each case, $z \in \mathbb{R}^{n^2-m}$. Also let $g_{k\epsilon}(x, y, w) = g(x, y, [{}^{(\epsilon/2)w}_{0}])$, where $w \in \mathbb{R}^{k-2n+4}$ and $\epsilon > 0$. Note that each $g_{k\epsilon}$ is affine and one-to-one.

Now consider the k-dimensional polytope

$$\mathcal{P}_{k\epsilon}=\left\{g_{k\epsilon}(x, y, w) \,\middle|\, \left\|\begin{bmatrix} x \\ y \\ w \end{bmatrix}\right\|_\infty \le 1\right\}.$$

If we choose the matrix norm $\|M\| = \max |m_{ij}|$, it follows that for every vertex A of $\mathcal{P}_{k\epsilon}$ there exists a vertex $\bar{A}$ of $\mathcal{Q}$ such that $\|A - \bar{A}\| < \epsilon$. Furthermore, every $(m-1)$-dimensional subpolytope of $\mathcal{P}_{k\epsilon}$ can be expressed as a disjoint finite union $\cup \Lambda_\nu$, where each Λ_ν is the convex hull of $m-1$ vertices $A_1, \cdots, A_{m-1}$ of $\mathcal{P}_{k\epsilon}$. Suppose $A_1, \cdots, A_q \in \mathcal{Q}$ and $A_{q+1}, \cdots, A_{m-1} \notin \mathcal{Q}$; let $\Omega = \text{conv}\{A_1, \cdots, A_q, \bar{A}_{q+1}, \cdots, \bar{A}_{m-1}\}$, where each $\bar{A}_i$ is a vertex of $\mathcal{Q}$ satisfying $\|A_i - \bar{A}_i\| < \epsilon$, and let $\Gamma = \{A_{q+1}, \cdots, A_{m-1}\}$. From Lemma 4.1, every $B \in \text{conv}\{A_1, \cdots, A_{m-1}\} \subset \text{conv}(\Omega \cup \Gamma)$ satisfies $d(B, \Omega) < \epsilon$. Hence, for sufficiently small ϵ, each $(m-1)$-dimensional subpolytope of $\mathcal{P}_{k\epsilon}$ is stable. □

V. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [2] can be extended to the case of $n \times n$ matrices. We have shown that, without further information describing the particular structure of a polytope, $(2n-4)$-dimensional faces must be checked for stability. Since testing even one such face can be a formidable task when n is large, and since the number of $(2n-4)$-dimensional faces grows exponentially with n, more work needs to be done before a computationally tractable algorithm can be devised for checking stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

References

[1] V. L. Kharitonov, "Asymptotic stability of an equilibrium position of a family of systems of linear differential equations," *Differential'nye Uravneniya*, vol. 14, no. 11, pp. 1483–1485, 1978.

[2] A. C. Bartlett, C. V. Hollot, and H. Lin, "Root locations of an entire polytope of polynomials: It suffices to check the edges," in *Proc. Amer. Contr. Conf.*. 1987, pp. 1611–1616.

[3] M. Fu and B. R. Barmish, "Stability of polytope of matrices: Counterexamples," *IEEE Trans. Automat. Contr.*, to be published.

[4] C. L. DeMarco, "Necessary and sufficient conditions for stability of polytopes of matrices through tests of lower dimensional subsets," in *Proc. 25th Annual Allerton Conf. Commun., Contr., Comput.*, 1987, pp. 72–77.

[5] A. Bronsted, *An Introduction to Convex Polytopes*. New York: Springer-Verlag, 1983.