Technical Notes and Correspondence

Linear Compensator Designs Based Exclusively on Input-Output Information are Never Robust with Respect to Unmodeled Dynamics

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Abstract—**We investigate the effects of unmodeled, higher order dynamics or parasitics on the stability of linear control systems. We first describe a class of perturbations of a given state equation which cannot be distinguished from the original on the basis of input-output measurements alone. Then it is shown that, given any plant-compensator pair, such perturbations of each system can always be found which destabilize the closed-loop configuration. Finally, the effect of destabilizing perturbations on output behavior is explored.**

I. Introduction

The effects of high-frequency or parasitic phenomena on closed-loop system performance have long been studied. A popular framework for addressing this issue has been that of singular perturbation theory (see, e.g., [1], [2]). The point of view that parasitics are ultimately connected with unmodeled plant dynamics has become quite popular in recent years, sometimes with surprising consequences. For example, it was shown by Rohrs *et al.* [8] and Ioannou and Kokotovic [3] that high-frequency phenomena can lead to instability in adaptive control schemes. Adaptive controllers being highly nonlinear, a natural question to ask is whether parasitics could have a similar destabilizing effect on control systems which are based on linear compensators. This was answered in the affirmative by Khalil in [4] and [5]. A notable effort to circumvent these difficulties in the case of linear, time-invariant systems was made by Vidyasagar, culminating in the results of [6] and [7].

Our work is most similar to [7], but differs primarily in that we investigate the stability of a closed-loop system when *both* the plant and compensator are perturbed. The idea of perturbing both systems has been largely neglected in the literature (with the notable exception of [6]), even though one can easily make a strong case for considering such perturbations. Indeed, one need only recognize that a compensator, like the plant, is a physical system governed by a mathematical model which is inherently subject to uncertainty.

In light of examples such as those contained in [4] and [5], even arbitrarily small model errors are to be feared since such effects have the capability of destabilizing a system just as certainly as larger errors do. In fact, those examples illustrate that in some cases, small errors can cause greater instability than do larger ones.

In this paper, we intend to show that, when uncertainties in both plant and compensator are taken into account, even strictly proper compensators are subject to parasitic destabilization. Hence, properness of the compensator is really not the pivotal issue here as it is in [7]. We will show that, if only input-output information concerning the plant and compensator is available, robust compensation can never be achieved.

The results of this paper are by nature primarily negative. We do not claim to have a clear understanding yet of exactly what constitutes sufficient information for robust compensation, although we do mention a possible approach to finding an answer in Section V. It is hoped that our results will stimulate further discussion in an area which has been neglected by all but a handful of researchers.

Manuscript received August 1, 1987; revised August 27, 1987. Paper recommended by Past Associate Editor, S. P. Bhattacharyya. This work was supported by NSF Grant ECS-8612948.

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IEEE Log Number 8718553.

II. Preliminaries

We study systems characterized by the linear, time-invariant state equations

$$\dot{x} = Ax + Bu, \; y = Cx + Du \tag{1}$$

and perturbations of (1) given by

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \; y = C_1 x + C_2 \xi \tag{2}$$

where the submatrices in (2) satisfy

$$A_{11} - A_{12} A_{22}^{-1} A_{21} = A, \; B_1 - A_{12} A_{22}^{-1} B_2 = B \tag{3}$$

$$C_1 - C_2 A_{22}^{-1} A_{21} = C, \; -C_2 A_{22}^{-1} B_2 = D \tag{4}$$

and A_{22} is nonsingular. If we set $\epsilon = 0$ in (2) and eliminate ξ, (1) is obtained; hence, (2) with $\epsilon = 0$ may be thought of as a state augmentation of (1). Setting $\epsilon > 0$ in (2) constitutes a perturbation of that augmentation. For the moment, we allow A_{22} to be either stable or unstable.

To aid our analysis, we will use the decomposition for singularly perturbed systems developed in [10] where it is shown that there exist real matrix-valued analytic maps $\epsilon \mapsto M_\epsilon$ and $\epsilon \mapsto N_\epsilon$, defined on some interval $[0, \beta)$, such that M_ϵ and N_ϵ are square and nonsingular for every ϵ and

$$M_\epsilon \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} N_\epsilon = \begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix}, \; M_\epsilon \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N_\epsilon = \begin{bmatrix} A_{s\epsilon} & 0 \\ 0 & I \end{bmatrix} \tag{5}$$

with $A_{f\epsilon}$ and $A_{s\epsilon}$ analytic and A_{f0} nilpotent. According to [10], the matrices M_ϵ and N_ϵ are unique up to change of bases; hence, we may take M_0 and N_0 to be any matrices which achieve the decomposition (5) at $\epsilon = 0$. For example, let

$$M_0 = \begin{bmatrix} I & -A_{12} A_{22}^{-1} \\ 0 & I \end{bmatrix}, \; N_0 = \begin{bmatrix} I & 0 \\ -A_{22}^{-1} A_{21} & A_{22}^{-1} \end{bmatrix}.$$

Next, define

$$\begin{bmatrix} B_{s\epsilon} \\ B_{f\epsilon} \end{bmatrix} = M_\epsilon \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; [C_{s\epsilon} \;\; C_{f\epsilon}] = [C_1 \;\; C_2] N_\epsilon. \tag{6}$$

Equations (5) and (6) yield the decoupled state equations

$$\begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_{s\epsilon} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_s \\ x_f \end{bmatrix} + \begin{bmatrix} B_{s\epsilon} \\ B_{f\epsilon} \end{bmatrix} u$$

$$y = C_{s\epsilon} x_s + C_{f\epsilon} x_f \tag{7}$$

where

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = N_\epsilon^{-1} \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

0018-9286/88/0600-0559$01.00

We now present a series of technical results which will be useful in Sections III and IV.

Lemma 1: $A_{s0} = A$, $B_{s0} = B$, $C_{s0} = C$, $B_{f0} = B_2$, $C_{f0} = C_2A_{22}^{-1}$, and $A_{f\epsilon} = \epsilon F_\epsilon$ for every $\epsilon \in [0, \beta)$ where $F_0 = A_{22}^{-1}$.

Proof: From (5) and (6), we have

$$\begin{bmatrix} A_{s0} & 0 \\ 0 & I \end{bmatrix} = M_0 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} N_0 = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{bmatrix} B_{s0} \\ B_{f0} \end{bmatrix} = M_0 \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B \\ B_2 \end{bmatrix}$$

$$[C_{s0} \quad C_{f0}] = [C_1 \quad C_2]N_0 = [C \quad C_2A_{22}^{-1}].$$

Let

$$M_\epsilon = \begin{bmatrix} M_{11\epsilon} & M_{12\epsilon} \\ M_{21\epsilon} & M_{22\epsilon} \end{bmatrix}, \quad N_\epsilon^{-1} = \begin{bmatrix} \tilde{N}_{11\epsilon} & \tilde{N}_{12\epsilon} \\ \tilde{N}_{21\epsilon} & \tilde{N}_{22\epsilon} \end{bmatrix}$$

and note that

$$N_0^{-1} = \begin{bmatrix} I & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad M_\epsilon \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A_{f\epsilon} \end{bmatrix} N_\epsilon^{-1}.$$

We thus have $\epsilon M_{22\epsilon} = A_{f\epsilon}\tilde{N}_{22\epsilon}$ so $A_{f\epsilon} = \epsilon F_\epsilon$ where $F_0 = M_{220}\tilde{N}_{220}^{-1} = A_{22}^{-1}$. □

From $A_{f\epsilon} = \epsilon F_\epsilon$, we immediately obtain the well-known result that the eigenvalues of (2) which tend to infinity as $\epsilon \to 0^+$ are "close" to those of $(1/\epsilon)A_{22}$ (see, e.g., [2, Corollary 2.1]). One useful way of stating this result is the following.

Lemma 2: If μ is an eigenvalue of A_{22}, $\gamma > 0$, and $R < \infty$, then there exists $\epsilon_0 > 0$ such that (2) has an eigenvalue λ_ϵ satisfying $|\lambda_\epsilon| > R$ and $|\arg \lambda_\epsilon - \arg (1/\epsilon)\mu| < \gamma$ whenever $0 < \epsilon < \epsilon_0$.

Proof: From (7), the eigenvalues of $(1/\epsilon)F_\epsilon^{-1}$ are also eigenvalues of (2). Since $F_0^{-1} = A_{22}$ and F_ϵ^{-1} is continuous in ϵ, each F_ϵ^{-1} has an eigenvalue μ_ϵ with $\mu_\epsilon \to \mu$ as $\epsilon \to 0^+$. Choose ϵ_0 so that $(1/\epsilon)|\mu_\epsilon| > R$ and $|\arg \mu_\epsilon - \arg \mu| < \gamma$ whenever $0 < \epsilon < \epsilon_0$, and let $\lambda_\epsilon = (1/\epsilon)\mu_\epsilon$. Then λ_ϵ is an eigenvalue of (2), $|\lambda_\epsilon| > R$, and $|\arg \lambda_\epsilon - \arg (1/\epsilon)\mu| = |\arg \mu_\epsilon - \arg \mu| < \gamma$. □

Suppose the transfer matrices of (1) and (2) are P and P_ϵ, respectively. We will need conditions under which an eigenvalue of (2) is also a pole of P_ϵ.

Lemma 3: If (A_{22}, B_2, C_2) is controllable and observable, there exists $\epsilon_0 > 0$ and $R < \infty$ such that every eigenvalue λ_ϵ of (2) satisfying $|\lambda_\epsilon| > R$ is also a pole of P_ϵ whenever $0 < \epsilon < \epsilon_0$.

Proof: An eigenvalue λ_ϵ of (2) is a pole of P_ϵ if

$$\begin{bmatrix} \lambda_\epsilon I - A_{11} & -A_{12} & B_1 \\ -A_{21} & \epsilon\lambda_\epsilon I - A_{22} & B_2 \end{bmatrix} = M_\epsilon^{-1} \begin{bmatrix} \lambda_\epsilon I - A_{s\epsilon} & 0 & B_{s\epsilon} \\ 0 & \epsilon\lambda_\epsilon I - F_\epsilon^{-1} & B_{f\epsilon} \end{bmatrix} \cdot \begin{bmatrix} \begin{bmatrix} I & 0 \\ 0 & F_\epsilon \end{bmatrix} N_\epsilon^{-1} & 0 \\ 0 & I \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} \lambda_\epsilon I - A_{11} & -A_{12} \\ -A_{21} & \lambda_\epsilon I - A_{22} \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} M_\epsilon^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda_\epsilon I - A_{s\epsilon} & 0 \\ 0 & \epsilon\lambda_\epsilon I - F_\epsilon^{-1} \\ C_{s\epsilon} & C_{f\epsilon}F_\epsilon^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & F_\epsilon \end{bmatrix} N_\epsilon^{-1} \tag{9}$$

have full rank. Choose $R > \max \{|\lambda| \mid \lambda \text{ is an eigenvalue of } A\}$. From Lemma 1, $(F_0^{-1}, B_{f0}, C_{f0}F_0^{-1}) = (A_{22}, B_2, C_2)$. Hence, there exists $\epsilon_0 > 0$ such that, whenever $0 < \epsilon < \epsilon_0$, $(F_\epsilon^{-1}, B_{f\epsilon}, C_{f\epsilon}F_\epsilon^{-1})$ is controllable and observable and $|\lambda_\epsilon| > R$ implies that λ_ϵ is not an eigenvalue of $A_{s\epsilon}$. It follows immediately that all matrices on the right-hand sides of (8) and (9) have full rank. □

III. Input-Output Equivalence

In this section, we explore the relationship between the nominal and perturbed systems (1) and (2) and discuss the conditions under which they are indistinguishable if only input-output information is available. Consider the process of obtaining or verifying an input-output model of a physical system. We are allowed to take measurements by applying an input signal starting at $t = 0$ and by observing the output; it is assumed that no direct access to internal states is possible. Once a nominal model is obtained, a controllable and observable realization can be chosen, yielding the state equation (1). Since we have no direct control over initial states except through the input ports, and since $t = 0$ presumably occurs long after the system was built, the system may be assumed initially at rest. Hence, we choose $x(0) = 0$ and $\xi(0) = 0$ in (1) and (2).

We define the class of admissible input signals $\mathcal{U}$ to be all C^1 functions $u:[0, \tau] \to \mathbb{R}^m$ satisfying $\max \|u(t)\| < K_0$, $\max \|\dot{u}(t)\| < K_1$, and $u(0) = 0$ where the constants $\tau < \infty$, $K_0 < \infty$, and $K_1 < \infty$ are independent of u. From an engineering standpoint, it is not unreasonable to place such restrictions on u. Indeed, in any real-world scenario, there is a maximum length of time one would be willing to invest in collecting data, as well as a maximum amplitude of voltage, force, or other input quantity that could possibly be generated using available technology. Furthermore, there is always an upper bound on the rate at which $u(t)$ can be made to vary (e.g., every amplifier has a maximum slew rate). Thus, the constants τ, K_0, and K_1, although possibly very large, must be finite. Since no input is applied prior to $t = 0$ and since $K_1 < \infty$, we must have $u(0) = 0$. We would surely be in serious trouble if, in order to design a robust compensator, we needed the capability of generating inputs over arbitrarily large intervals of time or with arbitrarily large amplitudes or rates of change.

Associated with any real-world measuring device is a minimum error which can be detected. For example, if a function y represents an output voltage, velocity, or other physical quantity of interest, there must exist a number $\delta > 0$, characteristic of the measuring device alone, such that another output $\tilde{y}$ cannot be distinguished from y if

$$\sup \{\|y(t) - \tilde{y}(t)\| \mid 0 \le t \le \tau\} < \delta. \tag{10}$$

For the remainder of the paper, we assume a fixed source of input signals and measurements and, consequently, a fixed set $\mathcal{U}$ and number $\delta > 0$.

The quantities $\mathcal{U}$ and δ together determine an equivalence between systems: two systems are indistinguishable under input-output measurement if for every $u \in \mathcal{U}$, the output functions y and $\tilde{y}$ of the two systems satisfy (10). The next result applies this idea to the nominal and perturbed models (1) and (2).

Theorem 1: If A_{22} is strictly stable, there exists $\epsilon_0 > 0$ such that, whenever $u \in \mathcal{U}$ and $0 \le \epsilon < \epsilon_0$, the respective outputs y and y_ϵ of (1) and (2) satisfy $\max \{\|y(t) - y_\epsilon(t)\| \mid 0 \le t \le \tau\} < \delta$.

Proof: We first note that $y_0(t) = \int_0^t C_{s0} \exp(\eta A_{s0})B_{s0}u(t - n)\, d\eta - C_{f0}B_{f0}u(t) = y(t)$. Hence, we need only show that there exists ϵ_0 such that $\|y_\epsilon(t) - y_0(t)\| < \delta$ whenever $0 \le t \le \tau$ and $0 \le \epsilon < \epsilon_0$. Decomposing $y_\epsilon = y_{s\epsilon} + y_{f\epsilon}$ in the obvious way, we have $\|y_{s\epsilon}(t) - y_{s0}(t)\| \le K_0 \int_0^\tau \|C_{s\epsilon} \exp(\eta A_{s\epsilon})B_{s\epsilon} - C_{s0} \exp(\eta A_{s0})B_{s0}\|\, d\eta$. Choose $\epsilon_1 > 0$ such that $0 \le \epsilon < \epsilon_1$ implies $\max \{\|C_{s\epsilon} \exp(\eta A_{s\epsilon})B_{s\epsilon} - C_{s0} \exp(\eta A_{s0})B_{s0}\| \mid 0 \le \eta \le \tau\} < \delta/(2K_0\tau)$. Integrating by parts, we obtain

$$\|y_{f\epsilon}(t) - y_{f0}(t)\| \le K_1\|C_{f\epsilon}\| \int_0^\tau \left\| \exp\left(\frac{\eta}{\epsilon} F_\epsilon^{-1}\right) \right\| d\eta \cdot \|B_{f\epsilon}\| + K_0\|C_{f\epsilon}B_{f\epsilon} - C_{f0}B_{f0}\|.$$

There exist $\epsilon_2 > 0$ and $K < \infty$ such that $\|\exp(tF_\epsilon^{-1})\| < K$, $\|C_{f\epsilon}\| < K$, and $\|B_{f\epsilon}\| < K$ whenever $t \ge 0$ and $0 \le \epsilon < \epsilon_2$. Let $\tilde{\delta} = \delta/(4K_1K^2(K + \tau))$. We know that there exists $\epsilon_3 > 0$ such that $\|\exp((\eta/\epsilon)F_\epsilon^{-1})\| < \tilde{\delta}$ whenever $\tilde{\delta} \le \eta \le \tau$ and $0 < \epsilon < \epsilon_3$ (see, e.g., [13]). Finally, there exists $\epsilon_4 > 0$ such that $\|C_{f\epsilon}B_{f\epsilon} - C_{f0}B_{f0}\| < \delta/4K_0$ when $\epsilon < \epsilon_4$. Let $\epsilon_0 = \min \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$. Then $0 \le \epsilon < \epsilon_0$ implies $\|y_\epsilon(t) - y_0(t)\| < \delta/2 + K_1K^2(K\tilde{\delta} + \tau\tilde{\delta}) + \delta/4 = \delta$. □

We have thus established that, for sufficiently small ϵ, (1) and (2) are indistinguishable on the basis of input-output information. Hence, although the physical system is nominally described by (1), an equally valid model from an input–output perspective is given by (2) with ϵ sufficiently small and A_{22} strictly stable.

IV. Closed-Loop Destabilization

We are now ready to investigate the effects that the system perturbations in Section II have on a closed-loop configuration. Consider the feedback compensator governed by

$$\dot{z}=Fz+Gy,\ u=Hz+v. \tag{11}$$

We consider only compensators with strictly proper transfer matrices since the results of [7] indicate that nonstrictly proper compensators are never robust with respect to unmodeled dynamics. Perturbations of (11) are of the form

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{\zeta} \end{bmatrix}=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\begin{bmatrix} z \\ \zeta \end{bmatrix}+\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} y$$

$$u=H_1 z+H_2\zeta+v \tag{12}$$

where

$$F_{11}-F_{12}F_{22}^{-1}F_{21}=F,\ G_1-F_{12}F_{22}^{-1}G_2=G \tag{13}$$

$$H_1-H_2F_{22}^{-1}F_{21}=H,\ -H_2F_{22}^{-1}G_2=0 \tag{14}$$

and F_{22} is nonsingular. The discussion of Section III applies equally well to both plant and compensator.

Combining (1) and (13) in a standard feedback configuration yields

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix}=\begin{bmatrix} A & BH \\ GC & F+GDH \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}+\begin{bmatrix} B \\ GD \end{bmatrix} v$$

$$y=Cx+DHz. \tag{15}$$

Combining the perturbed systems (2) and (12) gives

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \epsilon I & 0 \\ 0 & 0 & 0 & \epsilon I \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\xi} \\ \dot{\zeta} \end{bmatrix}$$

$$=\begin{bmatrix} A_{11} & B_1H_1 & A_{12} & B_1H_2 \\ G_1C_1 & F_{11} & G_1C_2 & F_{12} \\ A_{21} & B_2H_1 & A_{22} & B_2H_2 \\ G_2C_1 & F_{21} & G_2C_2 & F_{22} \end{bmatrix}\begin{bmatrix} x \\ z \\ \xi \\ \zeta \end{bmatrix}+\begin{bmatrix} B_1 \\ 0 \\ B_2 \\ 0 \end{bmatrix} v$$

$$y=C_1x+C_2z. \tag{16}$$

Let (15) and (16) have transfer matrices H and H_ϵ, respectively.

From this point on, we assume that A_{22} and F_{22} are strictly stable matrices. Thus, according to Theorem 1, (2) and (12) are equivalent to (1) and (11) for sufficiently small ϵ in an input–output sense. The perturbed closed-loop system (16) is also of the form (2); no obvious conclusions can be drawn, however, concerning stability of either (16) or the matrix

$$X=\begin{bmatrix} A_{22} & B_2H_2 \\ G_2C_2 & F_{22} \end{bmatrix}.$$

In view of Lemmas 2–4 as related to (16), we see that the properties of X as well as those of the matrices

$$Y=\begin{bmatrix} B_2 \\ 0 \end{bmatrix},\ z=[C_2\ \ 0]$$

are crucial for understanding the behavior of (16).

We are ultimately interested not only in the eigenvalues of the closed-loop system, but also in the poles of H_ϵ and the behavior of the system output $y(t)$. The next two results treat first the closed-loop poles and then output behavior. As a means of quantifying instability, let $\alpha \in (0, \pi/2)$ and consider the open sector $S=\{s \in \mathbb{C}-\{0\} \mid |\arg s| < \alpha\}$.

Theorem 2: Suppose $R < \infty$, (X, Y, Z) is controllable and observable, and X is nonsingular with an eigenvalue in the sector S. Then there exists $\epsilon_0 > 0$ such that H_ϵ has a pole $p_\epsilon \in S$ satisfying $|p_\epsilon| > R$ whenever $0 < \epsilon < \epsilon_0$.

Proof: Since X is nonsingular, the closed-loop system (16) is of the form (2). Let $\mu \in S$ be an eigenvalue of X. There exists $\gamma > 0$ such that $s \in S$ whenever $|\arg s - \arg (1/\epsilon)\mu| < \gamma$. The result then follows from Lemmas 2 and 3. □

Now consider behavior of the output $y(t)$ in the closed-loop system (16). Theorem 3 shows that under certain conditions, the instability described in Theorem 2 also has a pronounced effect on $y(t)$. Let m denote Lebesgue measure.

Theorem 3: Suppose $R < \infty$, $\delta_1, \delta_2 > 0$, (X, Y, Z) is controllable and observable, and X is nonsingular with an eigenvalue in the sector S.

1) There exists $\epsilon_0 > 0$ such that corresponding to each $\epsilon \in (0, \epsilon_0)$, there exist vectors $x_{0\epsilon} \in \mathbb{R}^n$, $z_{0\epsilon} \in R^k$, $\xi_{0\epsilon} \in R^{\tilde{n}}$, $\zeta_{0\epsilon} \in \mathbb{R}^{\tilde{n}}$ with $\|x_{0\epsilon}\|$, $\|z_{0\epsilon}\|$, $\|\xi_{0\epsilon}\|$, $\|\zeta_{0\epsilon}\| < \delta_1$ and a set $\Omega_\epsilon \subset [0, \tau]$ with $m\Omega_\epsilon < \delta_2$ such that the output y_ϵ of (20), subject to $x(0)=x_{0\epsilon}$, $z(0)=z_{0\epsilon}$, $\xi(0)=\xi_{0\epsilon}$, $\zeta(0)=\zeta_{0\epsilon}$, and $u \equiv 0$, satisfies $\|y_\epsilon(t)\| > R$ for every $t \in [0, \tau]-\Omega_\epsilon$.

2) There exists $\epsilon_0 > 0$ such that corresponding to each $\epsilon \in (0, \epsilon_0)$, there exist a continuous function $u_\epsilon:[0, \tau] \to \mathbb{R}^m$ with $\|u_\epsilon(t)\| < \delta_1$ for all $t \in [0, \tau]$ and a set $\Omega_\epsilon \subset [0, \tau]$ with $m\Omega_\epsilon < \delta_2$ such that the output of (20), subject to $x(0)=z(0)=\xi(0)=\zeta(0)=0$ and $u \equiv u_\epsilon$, satisfies $\|y_\epsilon(t)\| > R$ for every $t \in [0, \tau]-\Omega_\epsilon$.

Proof: 1) Since R is arbitrary and the system (16) is linear, we need only prove the result for a single vector norm, say, the Euclidean norm. The decomposition (7) may be applied to (16), yielding real-valued analytic matrix functions $M_\epsilon, N_\epsilon, A_{s\epsilon}, B_{s\epsilon}, \cdots, F_\epsilon$ defined on an interval $[0, \beta]$. Since $F_0 = X^{-1}$ is nonsingular, F_ϵ^{-1} is analytic. It is shown in [15] that there exists a continuous complex unitary matrix-valued function $\epsilon \mapsto U_\epsilon$ defined for sufficiently small values of ϵ that puts F_ϵ^{-1} into continuous upper triangular form—i.e.,

$$U_\epsilon^{-1}F_\epsilon^{-1}U_\epsilon=\begin{bmatrix} \mu_{1\epsilon} & \alpha_{12\epsilon} & \cdots & \alpha_{1,\tilde{n}+\tilde{k},\epsilon} \\ 0 & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & \alpha_{\tilde{n}+\tilde{k}-1,\tilde{n}+\tilde{k},\epsilon} \\ 0 & \cdots & 0 & \mu_{\tilde{n}+\tilde{k},\epsilon} \end{bmatrix}$$

where each of the maps $\epsilon \to \mu_{i\epsilon}$ and $\epsilon \to \alpha_{ij\epsilon}$ is continuous. Additional row and column interchanges can be used to reindex the $\mu_{i\epsilon}$; equivalently, U_0 may be chosen so that $\mu_{10} \in S$.

Let

$$w_\epsilon=\frac{\delta_1}{2\|N_\epsilon\|}N_\epsilon\begin{bmatrix} 0 \\ U_\epsilon\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{bmatrix}.$$

Since N_ϵ is nonsingular on $[0, \beta)$, $\|N_\epsilon\|$ is nonzero. Standard norm inequalitites reveal that $\|W_\epsilon\| < \delta_1$. From (7), it follows that the natural response of (16) due to the initial condition w_ϵ is

$$\tilde{y}_\epsilon(t)=\frac{\delta_1}{2\|N_\epsilon\|}(C_{f\epsilon}U_\epsilon)\exp\left(\frac{t}{\epsilon}U_\epsilon^{-1}F_\epsilon^{-1}U_\epsilon\right)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{17}$$

From Lemma 1, $(F_0^{-1}, C_{f0})=(X, ZX^{-1})$. This pair is observable since X is nonsingular; the corresponding observability matrix is

$$\begin{bmatrix} ZX^{-1} \\ Z \\ ZX \\ \vdots \\ ZX^{\tilde{n}+\tilde{k}-2} \end{bmatrix}=\begin{bmatrix} Z \\ ZX \\ \vdots \\ ZX^{\tilde{n}+\tilde{k}-1} \end{bmatrix}X^{-1}$$

and the pair (X, Z) is observable. Thus, $(U_0^{-1}F_0^{-1}U_0, C_{f0}U_0)$ is observable. Since $U_0^{-1}F_0^{-1}U_0$ is upper triangular, the first column of $C_{f0}U_0$ is nonzero. Suppose $\alpha_0 \neq 0$ is the ith entry of the first column of $C_{f0}U_0$. Then the same entry α_ϵ of $C_{f\epsilon}U_\epsilon$ is nonzero for sufficiently small

ϵ. From (17), it follows that $\tilde{y}_\epsilon$ has ith entry $\tilde{y}_{i\epsilon}(t) = (\delta_1/(2\|N_\epsilon\|))\alpha_\epsilon \exp((1/_\epsilon)\mu_{1_\epsilon}t)$. Thus, Re $\tilde{y}_\epsilon$ has the ith entry

$$\text{Re } \tilde{y}_{i\epsilon}(t) = (\delta_1/(2\|N_\epsilon\|))|\alpha_\epsilon| \exp\left(\frac{1}{\epsilon}\text{ Re } \mu_{1\epsilon}t\right) \cos\left(\frac{1}{\epsilon}\text{ Im } \mu_{1\epsilon}t + \arg \alpha_\epsilon\right).$$

Since $\mu_{10} \in S$, Re $\mu_{1\epsilon} > 0$ for small ϵ. From elementary analysis, there exists $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$ implies the existence of a set Ω_ϵ with $m\Omega_\epsilon < \delta_2$ and $\|\text{Re } \tilde{y}_{i\epsilon}(t)\| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$.

We note that the initial condition w_ϵ may be complex. In general, the natural response of (16) is of the form $y(t) = \Gamma_\epsilon(t)w$ where $\Gamma_\epsilon(t)$ is a real-valued matrix. Hence, Re $y(t) = \Gamma_\epsilon(t)$ Re w, and if we set $[x_{0\epsilon} z_{0\epsilon} \xi_{0\epsilon} \zeta_{0\epsilon}]^T$ = Re w_ϵ^T, we obtain an output y_ϵ with ith entry $y_{i\epsilon}(t)$ = Re $\tilde{y}_{i\epsilon}(t)$. Therefore, $\|y_\epsilon(t)\| \geq |\tilde{y}_{i\epsilon}(t)| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$. Finally, we note that $\|x_{0\epsilon}\|, \|z_{0\epsilon}\|, \|\xi_{0\epsilon}\|, \|\zeta_{0\epsilon}\| \leq \|\text{Re } w_\epsilon\| \leq \|w_\epsilon\| < \delta_1$.

2) Our approach is to construct an input function u_ϵ which steers the system (20) from the origin to some state w_ϵ satisfying the conditions of part 1), the transfer occurring on an arbitrarily small t interval; then the system will be allowed to evolve from w_ϵ with zero input. We first consider the pair $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$. From Lemma 1, $(F_0^{-1}, F_0^{-1}B_{f0}) = (X, XY)$. This pair is controllable since X is nonsingular; the corresponding controllability matrix is

$$[XY \quad X^2Y \quad \cdots \quad X^{\bar{n}+\bar{k}}Y] = X[Y \quad XY \quad \cdots \quad X^{\bar{n}+\bar{k}-1}Y]$$

and (X, Y) is controllable. Hence, $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$ is controllable for sufficiently small ϵ. Let

$$\psi_\epsilon(t) = B_{f\epsilon}^T F_\epsilon^{-T} \exp(-tF_\epsilon^{-T}) W_\epsilon(\tau)^{-1} \exp(-\tau F_\epsilon^{-T}) \tag{18}$$

where the Gramian $W_\epsilon(\tau)$ is given by $W_\epsilon(\tau) = \int_0^\tau \exp(-\eta F_\epsilon^{-1}) F_\epsilon^{-1}B_{f\epsilon}B_{f\epsilon}^T F_\epsilon^{-T} \exp(-\eta F_\epsilon^{-T})\, d\eta$. W_ϵ is nonsingular for small ϵ since $(F_\epsilon^{-1}, F_\epsilon^{-1}B_{f\epsilon})$ is controllable (see [11, p. 184]). All matrices in (18) converge and $\exp(-tF_\epsilon^{-T})$ converges uniformly on $[0, \tau]$ as $\epsilon \to 0^+$; hence, ψ_ϵ converges uniformly to ψ_0. Thus, there exists a number $M_1 < \infty$ such that $\|\psi_\epsilon(t)\| < M_1$ for all $t \in [0, \tau]$ and ϵ sufficiently small.

Choose $M_2 < \infty$ such that $\|C_{s\epsilon} \exp(tA_{s\epsilon})\| < M_2$ for small ϵ and all $t \in [0, \tau]$ where $C_{s\epsilon}$ and $A_{s\epsilon}$ are given by (7). Since N_ϵ^{-1} is continuous, we know from part 1) that for sufficiently small ϵ, there exist real vectors $x_{0\epsilon}$, $z_{0\epsilon}$, $\xi_{0\epsilon}$, and $\zeta_{0\epsilon}$ with $\|x_{0\epsilon}\|, \|z_{0\epsilon}\|, \|\xi_{0\epsilon}\|, \|\zeta_{0\epsilon}\| < \delta_1/(2M_1\|N_\epsilon^{-1}\|)$ and a set $\tilde{\Omega}_\epsilon$ with $m\tilde{\Omega}_\epsilon < \delta_2/2$ such that the corresponding output $\hat{y}_\epsilon$ of (16) satisfies $\|\hat{y}_\epsilon(t)\| > R + (M_2/M_1)\delta_1$ for every $t \in [0, \tau] - \tilde{\Omega}_\epsilon$. Let

$$\begin{bmatrix} x_{0s\epsilon} \\ x_{0f\epsilon} \end{bmatrix} = N_\epsilon^{-1} \begin{bmatrix} x_{0\epsilon} \\ z_{0\epsilon} \\ \xi_{0\epsilon} \\ \zeta_{0\epsilon} \end{bmatrix}. \tag{19}$$

Then the output $\hat{y}_\epsilon$ may be written $\hat{y}_\epsilon = y_{s_\epsilon} + y_{f_\epsilon}$ where $y_{s_\epsilon}(t) = C_{s_\epsilon} \exp(tA_{s_\epsilon})\, x_{0s_\epsilon}$ and $y_{f_\epsilon}(t) = C_{f_\epsilon} \exp((t/\epsilon)\, F_\epsilon^{-1})\, x_{0f_\epsilon}$. From (19), $\|x_{0s_\epsilon}\| < \delta_1/M_1$; therefore, $\|y_{s_\epsilon}(t)\| < (M_2/M_1)\, \delta_1$ for every $t \in [0, \tau]$. It follows that $\|y_{f_\epsilon}(t)\| > \|\hat{y}_\epsilon(t)\| - \|y_{s_\epsilon}(t)\| > R$ for each $t \in [0, \tau] - \tilde{\Omega}_\epsilon$.

Next, define $\tilde{u}_\epsilon(t) = \psi_\epsilon(t)x_{0f_\epsilon}$. Then $\|x_{0f\epsilon}\| < \delta_1/M_1$ guarantees that $\|\tilde{u}_\epsilon(t)\| < \delta_1$, and $\tilde{u}_\epsilon$ steers the system $\dot{x} = F_\epsilon^{-1}x + F_\epsilon^{-1}B_{f\epsilon}u$ from the origin as $t = 0$ to $x_{0f\epsilon}$ at $t = \tau$. (See, e.g., [11, p. 556].) Let

$$u_\epsilon(t) = \begin{cases} \tilde{u}_\epsilon\left(\frac{t}{\epsilon}\right), & 0 \leq t \leq \epsilon\tau \\ 0, & \epsilon\tau < t \leq \tau. \end{cases}$$

Then $\|u_\epsilon(t)\| < \delta_1$ and u_ϵ steers the second subsystem in (7) from the origin at $t = 0$ to $x_{0f\epsilon}$ at $t = \epsilon\tau$. u_ϵ also steers the first subsystem in (7) from the origin to some state $\tilde{x}_{0s\epsilon}$ at $t = \epsilon\tau$. Since $\tilde{x}_{0s\epsilon}$ is given by the convolution integral $\tilde{x}_{0s\epsilon} = \int_0^{\epsilon\tau} \exp(tA_{s\epsilon})B_{s\epsilon}u_\epsilon(t)\, dt$, the construction of u_ϵ and uniform convergence of $\exp(tA_{s\epsilon})$ guarantee that $\tilde{x}_{0s\epsilon} \to 0$ as $\epsilon \to 0^+$. Hence, $\tilde{y}_{s\epsilon} \to 0$ uniformly on $[0, \tau]$ as $\epsilon \to 0^+$ where $\tilde{y}_{s\epsilon}(t) = C_{s\epsilon} \exp(tA_{s\epsilon})\tilde{x}_{0s\epsilon}$. Applying the input u_ϵ steers the system (20) to $w_\epsilon = N_\epsilon[\tilde{x}_{0s\epsilon}x_{0f\epsilon}]^T$ at $t = \epsilon\tau$. For $t \in [\epsilon\tau, \tau]$, the corresponding output is $y_\epsilon(t) = \tilde{y}_{s\epsilon}(t - \epsilon\tau) + y_{f\epsilon}(t - \epsilon\tau)$, so $\|y_\epsilon(t)\| > R - \|\tilde{y}_{s\epsilon}(t - \epsilon\tau)\| > R$ for small ϵ and all $t \in [\epsilon\tau, \tau] - (\epsilon\tau + \tilde{\Omega}_\epsilon)$. Thus, if we choose ϵ_0 sufficiently small with $\epsilon_0 < \delta_2/2_\tau$ and $\Omega_\epsilon = [0, \epsilon\tau] \cup (\epsilon\tau + \tilde{\Omega}_\epsilon)$, we obtain $m\Omega_\epsilon < \delta_2$ and $\|y_\epsilon(t)\| > R$ for all $t \in [0, \tau] - \Omega_\epsilon$ whenever $0 < \epsilon < \epsilon_0$. □

The divergence of the output of the closed-loop system described in Theorem 3 is referred to in analysis texts as "almost uniform convergence to infinity." In view of the arbitrarily tight bounds that may be placed on an input or initial condition which generate this divergent behavior, we conclude that, if the assumptions of the theorem are met, unbounded instability at the output of a closed-loop configuration can result from arbitrarily small noise impinging on the system.

So far we have demonstrated that the existence of destabilizing perturbations of the plant and compensator is guaranteed if a certain linear algebra problem admits a solution. Indeed, if any A_{22}, B_2, and C_2 are chosen, (3) and (4) may be satisfied by simply selecting A_{12} and A_{21} arbitrarily and solving for A_{11}, B_1, and C_1. A similar remark applies to (13) and (14). It is sufficient, therefore, to find A_{22}, B_2, C_2, F_{22}, G_2, and H_2 such that 1) A_{22} and F_{22} are strictly stable, 2) (X, Y, Z) is controllable and observable, 3) X is nonsingular with an eigenvalue in S, and 4) (4) and (14) are satisfied. Theorems 2 and 3 further indicate that, if 1)–4) are met, the resulting instability in (16) becomes progressively worse as $\epsilon \to 0$ since R may be chosen arbitrarily large. *Thus, arbitrarily small uncertainty can lead to arbitrarily large instability.*

We now address the linear algebra problem 1)–4). We really need to find only one solution in order to demonstrate the existence of destabilizing perturbations; however, it is possible to do better. To obtain an understanding of just how many destabilizing perturbations actually exist, let (1), (2), (11), and (12) have orders n, $n + \bar{n}$, k, and $k + \bar{k}$, respectively; define $q = (n + \bar{n})(n + \bar{n} + m + p) + (k + \bar{k})(k + \bar{k} + p + m)$. Also, consider the variety in $\mathbb{R}^q$ consisting of all $(A_{11}, \cdots, C_2, F_{11}, \cdots, H_2)$ such that (3)–(6) and (13) and (14) are satisfied, and let $V \subset \mathbb{R}^q$ denote the intersection of that variety with the subset in which A_{22} and F_{22} are strictly stable. V may be interpreted as the set of all possible state augmentations of (1) and (11) of order $\bar{n}$ and $\bar{k}$, respectively. Finally, let $\Gamma \subset \mathbb{R}^q$ be the set of all points for which (X, Y, Z) is controllable and observable and X is nonsingular with an eigenvalue in S. We are interested in properties of the set $V \cap \Gamma$.

Theorem 4:

1) $V \cap \Gamma$ is relatively open in V.

2) $V \subset \Gamma$ is nonempty if $\bar{k} \geq 2$ and either a) $D = 0$ and $\bar{n} \geq 2$ or b) $D \neq 0$ and $\bar{n} \geq$ rank D.

Proof:

1) This is obvious since Γ is open in $\mathbb{R}^q$.

2) Suppose $D = 0$ and consider

$$T(s) = \begin{bmatrix} 2^{\bar{n}}s/(s+1)^{\bar{n}} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & \cdot & \cdots & 0 \end{bmatrix}, \quad U(s) = \begin{bmatrix} 2^{\bar{k}}s/(s+1)^{\bar{k}} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & \cdot & \cdots & 0 \end{bmatrix}.$$

Let (A_{22}, B_2, C_2) and (F_{22}, G_2, H_2) be controllable and observable realizations of T and U, respectively. Then A_{22} and F_{22} are strictly stable, $-C_2A_{22}^{-1}B_2 = T(0) = D$, and $-H_2F_{22}^{-1}G_2 = U(0) = 0$. Note that T and U have degrees $\bar{n}$ and $\bar{k}$. Since (X, Y, Z) has transfer function

$$V(s) = (I - T(s)U(s))^{-1}T(s)$$

$$= \begin{bmatrix} 2^{\bar{k}}s(s+1)^{\bar{n}}/((s+1)^{\bar{n}+\bar{k}} - 2^{\bar{n}+\bar{k}}s) & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & & \cdots & 0 \end{bmatrix}$$

and V has characteristic polynomial $\Delta(s) = (s + 1)^{\bar{n}+\bar{k}} - 2^{\bar{n}+\bar{k}}s$, it follows that (X, Y, Z) is controllable and observable and X is nonsingular with a unit eigenvalue.

Now suppose $D \neq 0$. There exist nonsingular matrices M and N such that

$$MDN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where r = rank D. Let

$$T(s)=M^{-1}\left[\begin{array}{cccc|c} 1/(s+1)^{\bar{n}-r+1} & & & & \\ & 1/(s+1) & & & 0 \\ & & \ddots & & \\ & & & 1/(s+1) & \\ \hline & & 0 & & 0_{p-r\times m-r} \end{array}\right]N^{-1},$$

$$U(s)=N\begin{bmatrix} 2^{\bar{n}+\bar{k}-r+1}s/(s+1)^{\bar{k}} & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \\ 0 & & \cdots & 0 \end{bmatrix}M.$$

Then

$$V(s)=(I-T(s)V(s))^{-1}T(s)=M^{-1}$$

$$\left[\begin{array}{cccc|c} (s+1)^{\bar{k}}/((s+1)^{\bar{n}+\bar{k}-r+1}-2^{\bar{n}+\bar{k}}s) & & & & \\ & 1/(s+1) & & & 0 \\ & & \ddots & & \\ & & & 1/(s+1) & \\ \hline & & 0 & & 0_{p-r\times m-r} \end{array}\right]N^{-1}$$

has characteristic polynomial $\Delta(s) = (s+1)^{r-1}((s+1)^{\bar{n}+\bar{k}-r+1} - 2^{\bar{n}+\bar{k}-r+1}s)$. Reasoning similarly as for part a), we conclude that A_{22} and F_{22} are strictly stable, (4) and (14) hold, (X, Y, Z) is controllable and observable, and X is nonsingular with a unit eigenvalue.

To complete the proof, we need only choose A_{12}, A_{21}, F_{12}, and F_{21} arbitrarily and solve for the remaining matrices from (3), (4) and (13), (14). □

Part 1) of Theorem 4 demonstrates that, in a certain sense, the high-frequency effects which bring about closed-loop instability do not correspond to the complement of a generic set, and hence cannot be dismissed as merely a pathological case.

V. Conclusions

We have shown that input–output information alone is insufficient for designing robust linear compensators. This conclusion leads one immediately to ask what further information is actually required to allow a robust design. Although we cannot give a clear answer yet, we can offer some insight. The development of our results indicates the high-frequency behavior in (2) and (12) plays a role in destabilization. Such behavior is closely related to the infinite-frequency structure of (2) and (12) with $\epsilon = 0$ (see, e.g., [14]). One might therefore suspect that some knowledge of the poles and zeros at infinity in either the plant or compensator is essential. The exact form of such information and whether it can be easily measured are important topics for further research.

References

[1] P. V. Kokotovic, R. E. O'Malley, and P. Sannuti, "Singular perturbations and order reduction in control theory—An overview," *Automatica*, vol. 12, pp. 123–132, 1976.
[2] P. V. Kokotovic, "Applications of singular perturbation techniques to control problems," *SIAM Rev.*, vol. 26, pp. 501–550, 1984.
[3] P. Ioannou and P. V. Kokotovic, "Singular perturbations and robust redesign of adaptive control," in *Proc. 21st IEEE Conf. Decision Contr.*, Dec. 1982, pp. 24–29.
[4] H. K. Khalil, "On the robustness of output feedback control methods to modeling errors," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 524–526, 1981.
[5] ——, "A further note on the robustness of feedback control methods to modeling errors," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 861–862, 1984.
[6] M. Vidyasagar, "The graph metric for unstable plants and robustness estimates for feedback stability," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 403–418, 1984.
[7] ——, "Robust stabilization of singularly perturbed systems," *Sys. Contr. Lett.*, vol. 5, pp. 413–418, 1985.
[8] C. E. Rohrs, L. Valvani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 881–889, 1985.
[9] J. D. Cobb, "Robust stabilization relative to the unweighted $H\infty$ norm is generically unattainable in the presence of singular plant perturbations," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 51–53, 1987.
[10] ——, "Global analyticity of a geometric decomposition for linear singularly perturbed systems," *Circuits, Syst., Signal Processing*, vol. 5, pp. 139–152, 1986.
[11] C. T. Chen, *Linear System Theory and Design*. New York: Holt, Rinehart, and Winston, 1984.
[12] F. Hoppensteadt, "On systems of ordinary differential equations with several parameters multiplying the derivatives," *J. Differential Eq.*, vol. 5, pp. 106–116, 1969.
[13] S. L. Campbell and N. J. Rose, "Singular perturbation of autonomous linear systems," *SIAM J. Math. Anal.*, vol. 10, pp. 542–551, 1979.
[14] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 811–831, 1981.
[15] H. Gingold, "On continuous triangularization of matrix functions," *SIAM J. Math. Anal.*, vol. 10, pp. 709–720, 1979.