Toward a Theory of Robust Compensation for Systems with Unknown Parasitics

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***Abstract*—We consider the problem of designing a robust compensator based on a plant model with order uncertainty. The uncertainty is characterized mathematically as a class of generalized singular perturbations of the plant. This paper considers the case of static compensation. A necessary and sufficient condition is established under which actual closed-loop behavior is close to that predicted by the plant model under sufficiently small singular perturbations. The condition is shown to be generic.**

I. INTRODUCTION

THE problem of robust compensation may be roughly stated as that of designing a good controller for a given physical system on the basis of a model which contains less than complete information about that system. The resulting closed-loop configuration should exhibit reasonable performance in spite of the uncertain aspects of the system. In the strictest sense, every model contains uncertainty; hence, any good controller design should address the issue of robustness.

Among the many types of robust control theories appearing in the literature is the asymptotic approach. Typical results in this area guarantee reasonable closed-loop performance under sufficiently small perturbations of a nomial model (e.g., variations in the coefficients of a single differential equation). Although only local in nature, such results are often a first step in developing a global theory where an explicit characterization is attained for classes of systems which can be simultaneously compensated. The results of this paper fall into the asymptotic category.

It is possible to view most asymptotic robustness theories within a common mathematical framework. Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$ be topological spaces, and let $\mathcal{R} \subset \mathcal{P} \times \mathcal{Q}$ inherit subset topology. $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$ correspond to the sets of all possible models of plants, compensators, and closed-loop systems, respectively. The topologies on $\mathcal{P}$ and $\mathcal{Q}$ are chosen so that small perturbations characterize measurement error inherent in developing each model; small perturbations in the topology of $\mathfrak{I}$ reflect tolerable closed-loop performance error. If $\mathcal{R}$ is interpreted as the class of all plant–compensator pairs which lead to closed-loop systems that are well-defined and which satisfy any additional constraints present in the design problem, we may naturally define the *loop-closing map* $\mathcal{C}: \mathcal{R} \rightarrow \mathfrak{I}$ which takes each plant and compensator into their corresponding closed-loop configuration. Many robustness questions then reduce to that of finding the points of continuity of $\mathcal{C}$. In other words, we wish to characterize the class of all plant–compensator pairs such that small perturbations of each pair result in small perturbations in the closed-loop system.

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We now examine various existing theories which lie within the asymptotic framework. The most obvious body of such results centers around the well-known fact that, for state-space models, the parameters of the closed-loop system are continuous functions of the open-loop plant and compensator parameters. For example, if we let $\mathcal{P}$ be the set of all matrix triples $\xi = (A, B, C)$ and $\mathcal{Q}$ consist of all feedback matrices K, and if we combine ξ and K in a standard way, then $\mathcal{R} = \mathcal{P} \times \mathcal{Q}$ and $\mathfrak{I}$ consists of triples $\mathcal{C}(\xi, K) = (A + BKC, B, C)$. Adopting Euclidean topology on $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$, it follows that $\mathcal{C}$ is continuous everywhere, i.e., every compensator is robust relative to every plant. One immediate consequence of this observation is that closed-loop eigenvalues are continuous functions of plant and compensator parameters; hence, every stable closed-loop configuration remains stable under sufficiently small parameter variations. These facts are used routinely in many control system analyses without explicit mention. It should be noted, however, that the perturbations considered here do not alter either plant or compensator order. Therefore, this approach alone is inadequate when dealing with order uncertainty.

The main body of existing results that does deal with order uncertainty in an asymptotic setting can be broadly termed singular perturbation theory (see [1]–[3]). Here a typical analysis treats a parametrized system of the form

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [C_1 \ \ C_2] x \tag{1}$$

with A_{22} stable and seeks to achieve some closed-loop performance criteria for all sufficiently small $\epsilon \geq 0$. (In this case, we might take $\mathcal{P} = [0, \infty)$.) A major drawback with this approach is that explicit knowledge of the parasitic structure giving rise to order uncertainty is assumed. If more than one perturbation (1) need to be considered, serious problems may develop. For example, the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [-1 \ \ 1 \ \ 0] x \tag{2}$$

is nominally ($\epsilon = 0$) unstable, but can be stabilized with the static compensator $u = 2y$. The perturbed system ($\epsilon > 0$) is also stabilized by the same compensator for sufficiently small ϵ. Setting $\epsilon = 0$, premultiplication of (2) by the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

yields an equivalent system equation which may in turn be perturbed according to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} u$$

$$y = [-1 \ 1 \ 0] x. \tag{3}$$

In this case the compensator $u = 2y$ yields a perturbed closed-loop system having a pair of eigenvalues $\lambda_{1\epsilon}$ and $\lambda_{2\epsilon}$ with $\text{Re}\,\lambda_{i\epsilon} \to +\infty$ as $\epsilon \to 0^+$. Such divergent behavior does not coincide with any reasonable definition of small perturbations in $\mathfrak{I}$. We may therefore conclude that examination of a single parasitic effect is in general not sufficient to guarantee robustness of a compensator with respect to other order uncertainties.

Additional singular perturbation results include the multiple time-scale extensions [7] and [8] and the robust compensation theorems of [5]. Multiple time-scale techniques suffer from the same drawback as single time-scale analyses based on (1) in that they assume an explicit knowledge of parasitic structure. Also, much less is known about the ϵ-dependence of the time response of multiple time-scale systems than in the single time-scale case.

In [5] it is shown that any compensator having a strictly proper transfer function matrix, which stabilized (1) with $\epsilon = 0$, also stabilizes (1) when $\epsilon > 0$ is sufficiently small. Furthermore, it is shown that the corresponding family of closed-loop transfer matrices converges uniformly on compact subsets of the right-half complex plane as $\epsilon \to 0^+$. These results thus provide a means for robustly compensating a system in the presence of a large class of possible perturbations. One drawback to this theory is that only single time-scale systems (1) are treated. In practice, a much larger class of perturbations may be required to model all relevant effects. Additional problems are that the results of [5] do not take into account uncertainties in the compensator model and that uniform convergence on compact sets in $\mathfrak{C}$ is difficult to relate to time-domain performance of the system.

Another notable asymptotic robustness theory is that of [6] where the graph topology is introduced. Let $\mathcal{P}$ and $\mathcal{Q}$ each be the space of all rational matrices, $\mathfrak{I}$ the space of strictly proper and stable rational matrices, equipped with the H_∞ norm, and $\mathfrak{R} = \mathfrak{C}^{-1}(\mathfrak{I})$. The graph topology is the weakest topology on $\mathcal{P}$ and $\mathcal{Q}$ under which $\mathfrak{C}$ is continuous. We have shown in [9], however, that singularly perturbed systems generically do not converge in the graph topology; hence, in this sense, robust compensation in the presence of order uncertainty is unattainable.

In view of the shortcomings of the existing asymptotic techniques, we wish to propose a framework as well as some preliminary results for an alternative robustness theory which will be taken into account: 1) multirate and other relatively unexplored classes of singular perturbations; 2) the necessity of dealing simultaneously with a large class of system perturbations, each corresponding to a possible higher order model; and 3) time-domain behavior of the closed-loop system. Although treatment of 1) and 2) seems on the surface to be a formidable task, we will see that it is possible to approach the problem in a roundabout way, thus avoiding having to explicity characterize all possible parasitic phenomena. We feel that the inclusion of 3) is a desirable feature for any good robustness theory, since the goal of system design must ultimately be satisfactory closed-loop time response. In view of this fact, a time-domain approach has certain advantages over frequency domain techniques, since the relationship between time response and frequency-domain behavior can be rather complex.

Before becoming too engrossed in technicalities, we will briefly describe (in rough terms) the problem we wish to address. Consider the system

$$\xi: \quad \begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{4}$$

where E, A, B, are real matrices with E and A square. We assume that (4) exhibits existence and uniqueness of solutions for each initial condition x_0 and each input function u; from [17] we know that this is equivalent to

$$|sE - A| \not\equiv 0.$$

Such systems have been studied extensively (e.g., see [14]–[16]), and are referred to as singular when E is singular and regular otherwise. The polynomial

$$\Delta(s) = |sE - A| \tag{5}$$

may be considered the characteristic polynomial of (4) and its roots the eigenvalues of ξ. An important property of singular systems is that small perturbations in the entries of E and A can change the system order; one example of this phenomenon is (1).

Suppose we wish to find a compensator of the form $u = Ky - v$ which is robust with respect to perturbations in E, A, B and C. Since we are inevitably interested in time response, we might ask which compensators result in a closed-loop system whose time response varies continuously with E, A, B, and C, regardless of the perturbation. Unfortunately, it is easy to show that for any K there exist perturbations in the system matrices that yield divergent behavior in the closed-loop system trajectories for some initial conditions. A more meaningful problem can be formulated by first observing that not necessarily all perturbations in the matrix entries of (4) are physically realistic. For example, a simple RC circuit consisting of a single resistor, capacitor, and voltage source may be modeled as

$$\begin{aligned} \epsilon\dot{x} &= -x + u \\ y &= x \end{aligned} \tag{6}$$

where x is the capacitor voltage, $R = 1$, and $C = \epsilon$. Positive ϵ makes perfect physical sense, and it seems reasonable to try to design a compensator based on the low-order model corresponding to $\epsilon = 0$. On the other hand, if ϵ is negative, the system engineer could not expect to produce a robust compensator without first being aware of the negative capacitance and then using an appropriate higher order (in this case, first-order) model.

A simple way to characterize physically meaningful perturbations in the plant is to look at their effect on plant trajectories for various inputs and initial conditions. For example, in (6) an initial condition $x_0 = 1$ yields $x(t) = e^{-t/\epsilon}$ which converges on compact subintervals of $(0, \infty)$ as $\epsilon \to 0^+$, but diverges as $\epsilon \to 0^-$. Strictly speaking, we are really not saying as much about perturbations of (4) which can occur in the physical world as we are about those perturbations which are consistent with the measurements taken while formulating our plant model; a system model is good only if it is capable of predicting the behavior of the actual physical system.

We may now state our definition of asymptotic robustness more precisely. For a given plant of the form (4), a compensator is robust if all perturbations in both the plant and compensator, which bring about only small variations in the trajectories of each system individually under all inputs and initial conditions, result in only small variations in the closed-loop system trajectories. The meaning of the phrase small variations will be precisely defined in Section III. In the same section we will see that our approach implicitly incorporates the idea that small system variations should correspond to only small changes in system parameters.

II. Preliminaries

In this section we summarize the constructions of [10], [11], [13], and [14] which are pertinent to subsequent developments. Let

$$\Sigma(n, m, p) = \{(E, A, B, C) \in \mathbb{R}^{n(2n+m+p)} \mid |sE - A| \not\equiv 0\}$$

and let $\mathcal{L}(n, m, p)$ be the corresponding quotient manifold (see [18]) determined by the equivalence

$$(E_1, A_1, B_1, C_1) \approx (E_2, A_2, B_2, C_2) \text{ iff } C_1 = C_2 \text{ and } \exists \text{ nonsingular } M \text{ s.t. } ME_1 = E_2, MA_1 = A_2, \text{ and } MB_1 = B_2. \quad (7)$$

(The arguments n, m, and p will be dropped when clear from context.) We choose the equivalence relation (7) because premultiplication by M has no significant effect on the system representation. Indeed, premultiplication by M merely performs elementary row operations on the system of scalar equations (7). Hence, we are merely identifying systems formed from each other by reshuffling the equations. We do not wish to identify systems which are related by a coordinate change on the state variable x, since this would reduce the system space to one consisting of input-output descriptions. Our intention is to produce results which exploit internal information.

The equivalence class containing $\sigma = (E, A, B, C)$ is denoted $\xi = [E, A, B, C]$. In this case, we say σ represents ξ. Let

$$r = \text{ord } \sigma = \text{ord } \xi = \deg \Delta$$

where Δ is the characteristic polynomial (5) of ξ, and note that a unique matrix C is determined by each $\xi \in \mathcal{L}$. A sequence $\xi_k \in \mathcal{L}$ converges weakly to $\xi \in \mathcal{L}$ ($\xi_k \overset{w}{\to} \xi$) if $\xi_k \to \xi$ in manifold topology. Since $\mathcal{L}$ is a quotient manifold, the natural projection $(E, A, B, C) \mapsto [E, A, B, C]$ is continuous with respect to weak convergence. Conversely, we have shown in [10] that, for each convergent sequence $\xi_k \overset{w}{\to} \xi$ in $\mathcal{L}$, there exists a sequence $(E_k, A_k, B_k, C_k) \to (E, A, B, C) \in \Sigma$ such that $[E, A, B, C] = \xi$ and $[E_k, A_k, B_k, C_k] = \xi_k$ for every k.

Let $\xi_k \overset{w}{\to} \xi = [E, A, B, C]$ with E singular. In [11] is shown that there exist nonsingular matrix sequences $M_k \to M$ and $N_k \to N$ such that

$$M_kE_kN_k = \begin{bmatrix} I_r & 0 \\ 0 & A_{fk} \end{bmatrix}, \quad M_kA_kN_k = \begin{bmatrix} A_{sk} & 0 \\ 0 & I_{n-r} \end{bmatrix} \quad (8)$$

where $r = \text{ord } \xi$, $A_{sk} \to A_s$, and $A_{fk} \to A_f$ with A_f nilpotent. For sufficiently large k, the matrices A_{fk} and A_{sk} are unique up to a similarity transformation. For a constant sequence, the decomposition (8) reduces to the Weierstrass decomposition for matrix pencils (see [17])

$$\text{MEN} = \begin{bmatrix} I_r & 0 \\ 0 & A_f \end{bmatrix}, \quad \text{MAN} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n-r} \end{bmatrix}. \quad (9)$$

The matrices M and N may also be used to decompose (4), yielding

$$MB = \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \quad CN = [C_s \ \ C_f].$$

Referring to [14], we say that (4) is *slow controllable* if and only if

$$\text{rank } [\lambda E - A \ \ B] = n \quad (10)$$

for every $\lambda \in \mathbb{C}$ and *fast controllable* if and only if

$$\text{rank } [E \ \ B] = n. \quad (11)$$

The system is *controllable* if and only if both (10) and (11) hold. In addition, we say that (4) is *impulse controllable* if and only if

$$\text{Im } A_f + \text{Ker } A_f + \text{Im } B_f = \mathbb{R}^{n-r}. \quad (12)$$

(All four system properties can also be defined directly in terms of the solutions of the differential equation (4), but we find the linear algebraic characterizations more useful in the context of this paper.) Controllability and observability imply impulse controllability and impulse observability, respectively. The corresponding definitions for observability are dual to (10), (11), and (12) (see [14]). Since each of these definitions is invariant under the equivalence transformation (7), we may also consider the subsets $\mathcal{L}_{sc}$, $\mathcal{L}_{fc}$, $\mathcal{L}_c$, $\mathcal{L}_{ic}$, $\mathcal{L}_{so}$, $\mathcal{L}_{fo}$, $\mathcal{L}_o$, $\mathcal{L}_{io} \subset \mathcal{L}$ determined by (10), (11), (12), and their duals, as well as the controllable and observable systems $\mathcal{L}_{co} = \mathcal{L}_c \cap \mathcal{L}_o$. Various properties of these spaces are studied in [13]; for example, $\mathcal{L}_{fc}$ and $\mathcal{L}_{fo}$ are open, and $\mathcal{L}_{ic}$ and $\mathcal{L}_{io}$ are dense in $\mathcal{L}$.

Other important subsets of $\mathcal{L}$ are the singular subspace $\mathcal{L}^s$, consisting of all points $[E, A, B, C]$ with E singular, the regular subspace $\mathcal{L}^n = \mathcal{L} - \mathcal{L}^s$, and the subspace of unit index systems

$$\mathcal{L}_1 = \{\xi \in \mathcal{L} \mid \deg |sE - A| = \text{rank } E\}.$$

In [10] it is shown that $\mathcal{L}^n$ is open and dense in $\mathcal{L}$; from [13], $\mathcal{L}_{co} \cap \mathcal{L}^s$ is dense in $\mathcal{L}^s$.

Let $\mathfrak{D}$ be the set of all C^∞ functions $\varphi: \mathbb{R} \to \mathbb{R}$ with compact support and let $\mathfrak{D}_+$ be the space of distributions with support in $[0, \infty)$ (see [19]). To define convergence in $\mathfrak{D}_+$, we adopt the weak* topology: A sequence $f_k \in \mathfrak{D}_+$ is said to converge to f if $\langle f, \phi \rangle$ for every $\phi \in \mathfrak{D}$, where $\langle f_k, \phi \rangle$ denotes the functional f_k evaluated at the point ϕ.

Associated with each initial condition $x_0 \in \mathbb{R}^n$ and each piecewise continuous input u there exists a unique solution $\Psi_{x_0u}(\xi_k) \in \mathfrak{D}_+^n$ of the system ξ_k (see [17]). From linearity it follows that the solution can be decomposed into natural and forced response

$$\Psi_{x_0u}(\xi_k) = \Psi_{x_00}(\xi_k) + \Psi_{0u}(\xi_k).$$

Letting

$$\begin{bmatrix} B_{sk} \\ B_{fk} \end{bmatrix} = M_kB_k, \ [C_{sk} \ \ C_{fk}] = C_kN_k, \ \begin{bmatrix} x_{0sk} \\ x_{0fk} \end{bmatrix} = N_k^{-1}x_0$$

$$\begin{bmatrix} \Psi^s_{x_0u}(\xi_k) \\ \Psi^f_{x_0u}(\xi_k) \end{bmatrix} = N_k^{-1}\Psi_{x_0u}(\xi_k) \quad (13)$$

we have from [17] that

$$\Psi^s_{x_0u}(\xi_k) = \exp(A_{sk})x_0 + \exp(A_{sk}) * B_{sk}u \quad (14)$$

where $\exp(A) \in \mathfrak{D}_+^{n^2}$ is defined by

$$\exp(A)(t) = e^{tA}$$

and "*" denotes convolution. Each $\Psi^s_{x_0u}$ satisfies several properties of continuity. Indeed, convergence of A_{sk} guarantees uniform convergence of $\exp(A_{sk})$ on compact intervals and, hence, weak* convergence. Continuity of convolution with respect to both types of convergence assures that each sequence $\Psi^s_{x_0u}(\xi_k)$ converges weak* and uniformly on compact intervals whenever $\xi_k \overset{w}{\to} \xi$. Furthermore, since $\Psi_{x_0u}(\xi) = \Psi^s_{x_0u}(\xi)$ for $\xi \in \mathcal{L}^n$, Ψ_{x_0u} satisfies the same properties when restricted to $\mathcal{L}^n$.

To aid in writing a general expression for $\Psi^f_{x_0u}(\xi_k)$, we note that there exists a nonsingular matrix sequence T_k (not necessarily convergent) such that

$$T_k^{-1}A_{fk}T_k = \begin{bmatrix} \bar{A}_{fk} & 0 \\ 0 & \tilde{A}_{fk} \end{bmatrix} \quad (15)$$

where $\bar{A}_{fk}$ is nonsingular and $\tilde{A}_{fk}$ is nilpotent. Then from [17],

$$\Psi^f_{x_0u}(\xi_k) = T_k \begin{bmatrix} \exp(\bar{A}_{fk}^{-1}) & 0 \\ 0 & \sum_{i=1}^{q_k-1} \delta^i \tilde{A}^i_{fk} \end{bmatrix} T_k^{-1}x_{0fk} + T_k \begin{bmatrix} \exp(\bar{A}_{fk}^{-1}) * \bar{A}_{fk}^{-1}\bar{B}_{fk}u \\ -\sum_{i=0}^{q_k-1} \tilde{A}^i_{fk}\tilde{B}_{fk}u^i \end{bmatrix} \quad (16)$$

where

$$\begin{bmatrix} \tilde{B}_{fk} \\ \bar{B}_{fk} \end{bmatrix} = T_k^{-1} B_{fk}$$

and δ^i and u^i denote the ith distribution derivatives.

III. Problem Formulation

We are now in a position to precisely state the basic problem under consideration. For two reasons we are forced to select a rather abstract mathematical framework for our constructions. First, since perturbations leading to changes in order require the use of systems of the form (4) and since such systems can have impulsive solutions, the space $\mathfrak{D}_+$ of distributions and its associated weak* topology underlie all analyses. Second, it will be seen that in order to meaningfully incorporate the idea that small system perturbations should lead to only small changes in the entries of the matrices E, A, B, and C, it is necessary to identify systems according to the equivalence relation (7). Hence, we must work with the non-Euclidean system spaces $\mathcal{L}(n, m, p)$.

We consider the problem of compensating the plant model (4) with a static system of the form

$$u = Ky + v \tag{17}$$

where K is a matrix and v is an external input. Let $\mathcal{P} = \mathcal{L}(n, m, p)$ and $\mathcal{Q} = \mathbb{R}^{mp}$, and note that the closed-loop system takes the form

$$\mathcal{C}(\xi, K): \quad \begin{aligned} E\dot{x} &= (A + BKC)x + Bv \\ y &= Cx. \end{aligned} \tag{18}$$

In general, the system (18) may not exhibit existence and uniqueness of solutions or may respond to certain initial conditions with impulsive transients (see [15], [17]). Since we are only interested in choosing a compensator such that the resulting closed-loop system does not suffer from either of these defects, we restrict attention to

$$\mathcal{R} = \{(\xi, K) \in \mathcal{P} \times \mathcal{Q} \mid \deg |sE - (A + BKC)| = \operatorname{rank} E\}. \tag{19}$$

Adopting (19) is equivalent to assuming that (18) has unit index; hence, we may set $\mathfrak{I} = \mathcal{L}_1(n, m, p)$. Note that the loop-closing map C is continuous with respect to manifold topology on $\mathcal{L}$ (weak convergence); i.e., small changes in the entries of E, A, B, C, and K bring about only small changes in the closed-loop system matrices.

We say that a sequence ξ_k in $\mathcal{P}$ *converges weakly* to $\xi \in \mathcal{P}$ ($\xi_k \xrightarrow{s} \xi$) if $\Psi_{x_0 u}(\xi_k) \to \Psi_{x_0 u}(\xi)$ weak* for every x_0 and u and if $C_k \to C$. On the other hand, we say that a sequence $\xi_k \in \mathfrak{I}$ *converges strongly* in $\mathfrak{I}$ if each $\Psi_{x_0 u}(\xi_k)$ converges uniformly on compact subintervals of $(0, \infty)$ and $C \to C_k$. Uniform convergence of solutions is meaningful for systems in $\mathfrak{I} = \mathcal{L}_1$ only because unit index systems have no impulsive components in their solutions (see [17]). We have shown in [10] that strong convergence in $\mathcal{P}$ implies weak convergence in $\mathcal{P}$.[1] It is easy to verify that strong convergence in $\mathfrak{I}$ implies convergence of each $\Psi_{x_0 u}(\xi_k)$ in the weak* sense (see [19]); hence, strong convergence implies weak convergence in $\mathfrak{I}$ as well.

Although strong convergence of ξ_k does not necessarily imply that the entries of the system matrices E_k, A_k, B_k, and C_k converge regardless of the representation (4) of ξ_k, such a strict requirement would not be particularly meaningful, since premultiplication of (4) by any nonsingular matrix produces an entirely equivalent representation. From [10] it does follow that strong convergence of ξ_k implies convergence of some representative sequence $(E_k, A_k, B_k, C_k) \in \Sigma(n, m, p)$. We are therefore justified in interpreting weak convergence in $\mathcal{R}$ and $\mathfrak{I}$ as convergence of system parameters and stating that a perturbation of a system which yields only small changes in system trajectories also results in only small variations in system parameters.

A plant-compensator pair $(\xi, K) \in \mathcal{R}$ is asymptotically robust (or K is a robust compensator for ξ) if $\mathcal{C}(\xi_k, K_k) \xrightarrow{s} \mathcal{C}(\xi, K)$ for every x_0, u whenever $\xi_k \xrightarrow{s} \xi$ and $K_k \to K$. This is equivalent to continuity of the loop-closing map $\mathcal{C}$ at (ξ, K) with respect to strong convergence in $\mathcal{R}$ and $\mathfrak{I}$. It is routine to verify that our definition of robustness can be couched in terms of topologies on $\mathcal{P}$, $\mathcal{Q}$, and $\mathfrak{I}$ simply by imposing on each set the weakest topology that makes each map $\Psi_{x_0 u}$ continuous (see [20]). Our main problem of interest is to characterize the class of all robust plant-compensator pairs of $(\xi, K) \in \mathcal{R}$ for any given values of n, m, and p. Equivalently, we seek to describe the class of all compensators K which are robust with respect to a given plant model ξ.

IV. The Class of Asymptotically Robust Compensators

We begin by presenting a result which formalizes the intuitive idea that robustness can fail to hold only when the plant model (4) is singular. Note that when (4) is regular ($\xi \in \mathcal{L}^n$), $\mathcal{R} = \mathbb{R}^{mp}$.

Proposition 3.1: If $\xi \in \mathcal{L}^n$, every $K \in \mathbb{R}^{mp}$ is robust.

Proof: Choose K, x_0, u, and sequences $K_k \to K$ and $\xi_k \xrightarrow{s} \xi$. Then $C_k \to C$. Since $\mathcal{L}^n$ is open in manifold topology, $\xi_k \in \mathcal{L}^n$ for sufficiently large k. From [10], $\xi_k \xrightarrow{w} \xi$ so continuity of $\mathcal{C}$ with respect to weak convergence implies $\mathcal{C}(\xi_k, K_k) \xrightarrow{w} \mathcal{C}(\xi, K)$. Since each $\Psi_{x_0 u}$ is continuous on $\mathcal{L}^n$ and $\mathcal{C}(\mathcal{L}^n \times \mathbb{R}^{mp}) \subset \mathcal{L}^n$, we have

$$\Psi_{x_0 u}(\mathcal{C}(\xi_k, K_k)) \to \Psi_{x_0 u}(\mathcal{C}(\xi, K)).$$

□

Before starting our main result on robustness, we need to consider one more algebraic system property of (4). We say that a system (4) is fast cyclic if, in the Weierstrass decomposition (8), the nilpotent matrix A_f is cyclic. If A_f is in Jordan form, fast cyclicity is equivalent to

$$A_f = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}.$$

Hence, from (9), a system (4) is fast cyclic if and only if rank $E = n - 1$. Note that fast cyclicity is independent of the choice of representation for ξ.

In order to prove that a certain algebraic condition on the compensator K is well defined, we next present a pair of lemmas. It will eventually be proven that this condition is necessary and sufficient for robustness.

Lemma 4.1: Let N and T be any $n \times n$ matrices with N nilpotent and having index q. Then Ker N is $N^{q-1}T$-invariant.

Proof: Since $N(N^{q-1}T) = N^q T = 0$, Im $N^{q-1}T \subset$ Ker N. Hence,

$$(N^{q-1}T) \operatorname{Ker} N \subset \operatorname{Ker} N.$$

□

Note that, if N is cyclic, Ker N is one-dimensional.

Lemma 4.2: If $\xi \in \mathcal{L}^s$ is fast cyclic, impulse controllable, and impulse observable, and (A_f, B_f, C_f) is obtained from the

[1] Actually, it is shown in [10] that convergence of $\Psi_{x_0 u}(\xi_k)$ for every x_0, u guarantees convergence of ξ_k in manifold topology when u ranges over $\mathfrak{D}_+^m$. It is easy to show, however, that the same result holds when u is restricted to be piecewise continuous.

Weierstrass decomposition of any representation of ξ, then

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)\ |\text{Ker } A_f>0 \tag{20}$$

determines a nonempty open affine half-space in $\mathbb{R}^{mp}$ which is independent of the representation. (The vertical bar denotes the restriction of the linear operator to the subspace Ker A_f.)

Proof: In view of Lemma 4.1, (20) is well-defined. For the case $r = n - 1$, $A_f = 0$ and

$$A_f^{n-r-1}(I+B_fKC_f)=1+B_fKC_f. \tag{21}$$

For $r < n - 1$, choose a nonsingular T so that $T^{-1}A_fT$ is in Jordan form. Letting

$$\begin{bmatrix} b_1 \\ \vdots \\ b_{n-r} \end{bmatrix} = T^{-1}B_f,\ [c_1 \cdots c_{n-r}]=C_fT \tag{22}$$

we know from [14] that impulse controllability and impulse observability guarantee $b_{n-r} \neq 0$ and $c_1 \neq 0$. Also,

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f) = T\begin{bmatrix} b_{n-r}Kc_1 & \cdots & b_{n-r}Kc_{n-r-1} & 1+b_{n-r}Kc_{n-r} \\ 0 & \cdot & \cdot\ \cdot\ \cdot & 0 \end{bmatrix} T^{-1}.$$

Hence,

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)\ |\text{Ker } A_f=b_{n-r}Kc_1. \tag{23}$$

Setting (21) and (23) positive determines nonempty open affine half-spaces.

From [11], (A_f, B_f, C_f) is unique up to similarity transformation for a given ξ. Clearly, similarity transformation does not alter (21), so the resulting half-space is unchanged. To see how (23) is affected by similarity transformation, note that (23) means

$$A_f^{n-r-1}(I_{n-r}+B_fKC_f)x=b_{n-r}Kc_1x$$

for any $x \in$ Ker A_f. Let $z = T^{-1}x$. Then

$$T^{-1}A_f^{n-r-1}(I_{n-r}+B_fKC_f)Tz=b_{n-r}Kc_1z$$

so

$$(T^{-1}A_fT)^{n-r-1}(I_{n-r}+(T^{-1}B_f) \cdot K(C_fT))|(T^{-1}\text{ Ker } A_f)=b_{n-r}Kc_1.$$

But T^{-1} Ker A_f = Ker $(T^{-1}AT)$ so the resulting half-space is again unchanged. □

A final technical lemma is needed to prove our main robustness theorem.

Lemma 4.3: Let a_{ik}; $i = 0, \cdots, \mu$ be convergent sequences in $\mathbb{R}$ with a $a_{\mu k} \neq 0$ for every k, and let f_{ik}: $\mathbb{R} \to \mathbb{R}$; $i = 1, \cdots, \nu - 1$; $k = 1, 2, \cdots$ be continuous at the origin and satisfy $f_{ik}(0) = 0$, where $\nu > \mu$. Then there exists a sequence ϵ_k in $\mathbb{R}$ such that for each k:

1) $0 < |\epsilon_k| < 1/k$
2) $\text{sgn } \epsilon_k = -\text{sgn } a_{\mu k}$
3) the polynomial $\epsilon_k s^\nu + f_{\nu-1,k}(\epsilon_k)s^{\nu-1} + \cdots + f_{\mu+1,k}(\epsilon_k)s^{\mu+1} + (a_{\mu k} + f_{\mu k}(\epsilon_k))s^\mu + \cdots + (a_{1k} + f_{1k}(\epsilon_k))s + a_{0k}$ has at least one real root λ_k with $\lambda_k > k$.

Proof: Fix k, let $\alpha_j = -1/j \text{ sgn } a_{\mu k}$, and consider the sequence (in j)

$$p_j(s)=\alpha_js^\nu+f_{\nu-1,k}(\alpha_j)s^{\nu-1}+\cdots +f_{\mu+1,k}(\alpha_j)s^{\mu+1}+(a_{\mu k}+f_{\mu k}(\alpha_j))s^\mu+\cdots +(a_{1k}+f_{1k}(\alpha_j))s+a_{0k}. \tag{24}$$

From [12, Lemma 4.3], p_j can be factored as

$$p_j(s)=\varphi_j(s^\mu+b_{\mu-1,j}s^{\mu-1}+\cdots+b_{0j})\prod_{i=1}^{\nu-\mu}(\sigma_{ij}s-1) \tag{25}$$

where each $\sigma_{ij} \to 0$ and φ_j, b_{ij}; $i = 0, \cdots, \mu - 1$ all converge. Equating the coefficients in (24) and (25) of s^ν and s^μ yields

$$\alpha_j=\varphi_j\prod_{i=1}^{\nu-\mu}\sigma_{ij}$$

$$a_{\mu k}=(-1)^{\nu-\mu}\lim \varphi_j.$$

For sufficiently large j it follows that

$$\begin{aligned}\text{sgn}\prod_{i=1}^{\nu-\mu}\sigma_{ij}&=\text{sgn } \alpha_j \text{ sgn } \varphi_j\\ &=-\text{sgn } a_{\mu k}\text{ sgn lim } \varphi_j\\ &=(-1)^{\nu-\mu+1}.\end{aligned}$$

Hence, for each sufficiently large j there must exist an i such that

$$\sigma_{ij}\in\mathbb{R},\ \sigma_{ij}>0.$$

Since $\sigma_{ij} \to 0$, there exists $j > k$ such that $1/\sigma_{ij} > k$. Set $\lambda_k = 1/\sigma_{ij}$ and $\epsilon_k = \alpha_j$. □

Our main result, Theorem 4.4 completely characterizes the robust static compensator gains K.

Theorem 4.4: Let $\xi \in \mathfrak{L}^s$.

1) A robust $K \in \mathbb{R}^{pm}$ exists iff ξ is fast cyclic, impulse controllable, and impulse observable.

2) Under the conditions of part 1), K is robust iff

$$A_f^{n-r-1}(I+B_fKC_f)\ |\text{Ker } A_f>0. \tag{26}$$

Proof:

1) (Necessary): Let r = ord ξ. We need only consider the case $r < n - 1$, since $r = n - 1$ implies ξ is fast cyclic, impulse controllable, and impulse observable (see [14]). Suppose $r < n - 1$ and choose a representation (E, A, B, C) for ξ. Invoke the Weierstrass decomposition (9), select a similarity transformation to put A_f in Jordan form, use the notation (22), and let

$$\begin{bmatrix} 0 & \gamma_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1} \\ & & & 0 \end{bmatrix}=T^{-1}A_fT. \tag{27}$$

(Each γ_i is either 0 or 1.) If ξ is not fast cyclic, impulse controllable, or impulse observable, then either $\gamma_i = 0$ for some i, $b_{n-r} = 0$, or $c_1 = 0$ (see [14]). Choose nonzero sequences $\gamma_{ik} \to \gamma_i$, $b_{n-r,k} \to b_{n-r}$, $c_{1k} \to c_1$, and $K_k \to K$ such that

$$b_{n-r,k}K_kc_{1k}\prod_{i=1}^{n-r-1}\gamma_{ik}<0 \tag{28}$$

for every k, and define

$$B_{fk}=T\begin{bmatrix} b_1 \\ \vdots \\ b_{n-r-1} \\ b_{n-r,k}\end{bmatrix},\quad C_{fk}=[c_{1k}c_2 \cdots c_{n-r}]T^{-1} \tag{29}$$

$$A_{fk}(x)=T\begin{bmatrix} x & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & x \end{bmatrix}T^{-1}. \tag{30}$$

Now, we may uniquely define sequences a_{ik}; $i = 0, \cdots, n-1$ and polynomials p_{ik}; $i = 1, \cdots, n-1$, with $p_{ik}(0) = 0$, according to

$$x^{n-r}s^n+(a_{n-1,k}+p_{n-1,k}(x))s^{n-1}+\cdots+(a_{1k}+p_{1k}(x))s+a_{0k}$$
$$=\begin{vmatrix} sI-(A_s+B_sK_kC_s) & -B_sK_kC_{fk} \\ -B_{fk}K_kC_s & sA_{fk}(x)-(I+B_{fk}K_kC_{fk}) \end{vmatrix}.$$

By elementary matrix arguments,

$$a_{n-1,k}=(-1)^{n-r}b_{n-r,k}K_kc_{1k}\prod_{i=1}^{n-r-1}\gamma_{ik}. \tag{31}$$

Letting $f_{ik}(x) = p_{ik}(x^{1/(n-r)})$, $\nu = n$, and $\mu = n-1$, we may select a sequence ϵ_k satisfying the properties in Lemma 4.3 and define

$$\alpha_k=-\left(\frac{1}{\epsilon_k}\right)^{1/(n-r)}.$$

Since $\epsilon_k \to 0$ and $\operatorname{sgn}\epsilon_k = -\operatorname{sgn} a_{n-1,k}$, (28) and (31) guarantee that $\alpha_k \to +\infty$. If we set

$$E_k=M^{-1}\begin{bmatrix} I_{n-r} & 0 \\ 0 & A_{fk}\left(-\frac{1}{\alpha_k}\right)\end{bmatrix}N^{-1},\ A_k=A$$

$$B_k=M^{-1}\begin{bmatrix} B_s \\ B_{fk}\end{bmatrix},\ C_k=[C_s\ \ C_{fk}]N^{-1} \tag{32}$$

we have

$$\det(sE_k-(A_k+B_kK_kC_k))$$
$$=\beta(\epsilon_ks^n+(a_{n-1,k}+f_{n-1,k}(\epsilon_k))s^{n-1}+\cdots$$
$$+(a_{1k}+f_1(\epsilon_k))s+a_{0k}) \tag{33}$$

for some constant β. From Lemma 4.3, (33) has at least one real root $\lambda_k > k$ for every k. Since (33) is just the characteristic polynomial of the closed-loop system $\mathcal{C}(\xi, K)$, (15) shows that λ_k must be an eigenvalue of the closed-loop $\bar{A}_{fk}^{-1}$ for sufficiently large k. Thus, $\exp(\bar{A}_{fk}^{-1})$ cannot converge uniformly on compact subintervals of $(0, \infty)$, since this would imply uniform convergence of its eigenvalue $\exp(\lambda_k)$. Letting $u = 0$, it follows from (13) and (16) that $\Psi_{x_0u}(\mathcal{C}(\xi_k, K))$ does not converge for every x_0.

In order to prove that K is not robust, we have only left to show that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ in the weak* sense, where $\xi_k = [E_k, A_k, B_k, C_k]$. To do so, we note that

$$\Psi^f_{x_0uk}=\exp\left(A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)x_0+A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}$$
$$\cdot\exp\left(A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)*B_{fk}u$$

where

$$\exp\left(tA_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)=\sum_{i=0}^{n-1}\frac{1}{i!}\frac{d^i}{ds^i}e^{t/s}\bigg|_{s=-1/\alpha_k}$$
$$\cdot\begin{bmatrix} 0 & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & 0 \end{bmatrix}^i$$

and

$$A_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\exp\left(tA_{fk}\left(-\frac{1}{\alpha_k}\right)^{-1}\right)$$
$$=\sum_{i=0}^{n-1}\frac{1}{i!}\frac{d^i}{ds^i}\frac{1}{s}e^{t/s}\bigg|_{s=-1/\alpha_k}$$
$$\cdot\begin{bmatrix} 0 & \gamma_{1k} & & \\ & \ddots & \ddots & \\ & & \ddots & \gamma_{n-r-1,k} \\ & & & 0 \end{bmatrix}^i*B_{fk}u.$$

Consider the matrix

$$\Sigma_k=\begin{bmatrix} -\frac{1}{\alpha_k} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -\frac{1}{\alpha_k}\end{bmatrix}.$$

A routine calculation shows that the (i, j)th entry of $\Sigma_k^{-1}\exp(t\,\Sigma_k^{-1})$ is

$$\sigma_{ijk}=\frac{1}{(j-i)!}\frac{d^{j-i}}{ds^{j-i}}\left(\frac{1}{s}e^{t/s}\right)\bigg|_{s=-1/\alpha_k},\qquad j\geq i.$$

It was shown in the proof of [10, Theorem 4] that

$$\Sigma_k^{-1}\exp(\Sigma_k^{-1})\to-\begin{bmatrix} 1 & \delta & \cdots & \delta^{n-r-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \delta \\ & & & 1\end{bmatrix}.$$

Therefore, $\sigma_{ijk} \to -\delta^{j-i}$ for each $j \geq i$, and $\Psi^f_{x_0uk} \to \Psi^f_{x_0u}$ for any x_0u. It follows from (13), weak* continuity of convolution, and continuity of $\Psi^s_{x_0u}$ that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$.

(Sufficient): Let ξ_k be any sequence in $\mathcal{L}$ such that $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ for every x_0u and with $C_k \to C$. Then, from [10], $\xi_k \to \xi$ and the decomposition (8) may be invoked. It follows that

$$|M_k|\ |sE_k-A_k|\ |N_k|=|sI_r-A_{sk}|\ |sA_{fk}-I_{n-r}|$$

for any convergent representation (E_k, A_k, B_k, C_k). Suppose $(-1)^{n-r}|A_{fk}| < 0$ for infinitely many k. Since $|A_{fk}|$ is just the product of the eigenvalues of A_{fk}, there exists a subsequence of A_{fk_j} with at least one real, positive eigenvalue for each k_j. It follows that $A_{fk_j}^{-1}$ has an eigenvalue $\lambda_j \to \infty$. Let $u = 0$ and observe that

$$\langle\Psi^f_{x_0u}(\xi_{k_j}),\ \phi\rangle=\Gamma_{k_j}x_{0fk_j}$$

where

$$\Gamma_{k_j}=\int_0^\infty \phi(t)e(tA_{fk_j}^{-1})\,dt.$$

A function $\phi \in \mathfrak{D}$ can always be chosen such that an eigenvalue of Γ_{k_j} satisfies

$$\int_0^\infty \exp\,(\lambda_{k_j}t)\phi(t)\,dt\to\infty.$$

Hence, $\|\Gamma_{k_j}\| \to \infty$. It follows that $\Gamma_k x_{0fk}$ is unbounded for an appropriate choice of x_0 and that $\Psi^f_{x_0u}(\xi_k)$ and [from (13)] $\Psi_{x_0u}(\xi_k)$ are not convergent. This contradiction leads us to conclude that $(-1)^{n-r}|A_{fk}| \geq 0$ for sufficiently large k.

Appealing to the notation of (22) and (27), we have b_{n-r}, $c_1 \neq 0$ and, if $r < n-1$, $\gamma_1 = \cdots = \gamma_{n-r-1} = 1$. Choose K to satisfy the condition (26). For $r = n-1$, (21) indicates that $1 + b_1Kc_1 > 0$; for $r < n-1$, (23) implies that $b_{n-r}Kc_1 > 0$. Defining

$$\Delta_k(s)=|sE_k-(A_k+B_kK_kC_k)|$$

we have

$$\begin{aligned}|M_k|\Delta_k(s)|N_k| &= \begin{vmatrix} sI_r-(A_{sk}+B_{sk}K_kC_{sk}) & -B_{sk}K_kC_{fk}\\ -B_{fk}K_kC_{sk} & sA_{fk}-(I_{n-r}+B_{fk}K_kC_{fk})\end{vmatrix}\\ &=|A_{fk}|s^n+(\alpha_k-|A_{fk}|\,tr(A_{sk}+B_{sk}K_kC_{sk}))s^{n-1}+\cdots\end{aligned}\tag{34}$$

where α_k is defined by

$$|sA_{fk}-(I_{n-r}+B_{fk}K_kC_{fk})|=|A_{fk}|s^{n-r}+\alpha_ks^{n-r-1}+\cdots.$$

From elementary matrix arguments we have, for $r < n-1$,

$$|sA_f-(I_{n-r}+B_fKC_f)|=(-1)^{n-r}b_{n-r}Kc_1s^{n-r-1}+\cdots$$

so

$$\lim\,\alpha_k=\begin{cases}-(1+b_1Kc_1) & \text{if } n-r=1\\ (-1)^{n-r}b_{n-r}Kc_1 & \text{if } n-r>1.\end{cases}\tag{35}$$

From our choice of K it follows that the closed-loop system $\mathcal{C}(\xi, K)$ exhibits no impulsive behavior in its natural response, i.e., ord $\mathcal{C}(\xi, K) = \text{rank}\,E = n-1$. Hence, from [12, Lemma 4.3] we know that

$$|M_k|\Delta_k(s)|N_k|=\zeta_k(\sigma_ks-1)\prod_{i=1}^{n-1}(s-\lambda_{ik})\tag{36}$$

where φ_k, σ_k, and λ_k all converge and $\lim\sigma_k = 0$. Matching coefficients in (34) and (36) yields

$$\varphi_k\sigma_k=|A_{fk}|\tag{37}$$

$$\lim\,\alpha_k=-\lim\,\varphi_k.\tag{38}$$

From (35) and our choice of K, $(-1)^{n-r}\lim\alpha_k < 0$. Hence, from (38),

$$(-1)^{n-r}\lim\,\varphi_k=(-1)^{n-r+1}\lim\,\alpha_k<0.$$

Thus, $(-1)^{n-r}\varphi_k < 0$ for sufficiently large k, and

$$\sigma_k=\frac{(-1)^{n-r}|A_{fk}|}{(-1)^{n-r}\varphi_k}<0.$$

Applying the decomposition (8) to the closed-loop system yields nonsingular transformations $\bar{M}_k$ and $\bar{N}_k$ such that

$$\bar{M}_kE_k\bar{A}_k=\begin{bmatrix}I_{n-1} & 0\\ 0 & \sigma_k\end{bmatrix},\ \bar{M}_k(A_k+B_kK_kC_k)\bar{N}_k=\begin{bmatrix}\bar{A}_{sk} & 0\\ 0 & 1\end{bmatrix}$$

$$\bar{M}_kB_k=\begin{bmatrix}\bar{B}_{sk}\\ \bar{B}_{fk}\end{bmatrix},\ C_k\bar{N}_k=[\bar{C}_{sk}\ \bar{C}_{fk}]$$

where all sequences converge. The decomposition (13) may also be applied to the closed-loop system yielding

$$\Psi_{x_0u}(\mathcal{C}(\xi_k))=\bar{N}_k\begin{bmatrix}\bar{\Psi}^s_{x_0u}(\xi_k)\\ \bar{\Psi}^f_{x_0u}(\xi_k)\end{bmatrix}.$$

From Lemma 4.3, $\bar{\Psi}^s_{x_0u}(\xi_k) \to \bar{\Psi}^s_{x_0u}(\xi)$, and

$$\bar{\Psi}^f_{x_0u}(\xi_k)=\begin{cases}\exp\left(\dfrac{1}{\sigma_k}\right)x_{0fk}+\dfrac{1}{\sigma_k}\exp\left(\dfrac{1}{\sigma_k}\right)*\bar{B}_{fk}u & \text{if } \sigma_k<0\\ -\bar{B}_{fk}u & \end{cases}$$

so, as in the necessity proof of part 1), $\bar{\Psi}^f_{x_0u}(\xi_k) \to \bar{\Psi}^f_{x_0u}(\xi)$ for any x_0, u. Hence, $\Psi_{x_0u}(\mathcal{C}(\xi_k)) \to \Psi_{x_0u}(\mathcal{C}(\xi))$ and K is robust.

2) (Sufficient): This part has already been treated in the Sufficiency section of 1).

(Necessary): Invoke the Weierstrass decomposition (9). If (26) fails and $r = n-1$, we have $1 + B_fKC_f \leq 0$ so $B_f \neq 0$ and $C_f \neq 0$; hence, there exists a sequence $K_k \to K$ such that

$$1+B_fK_kC_f<0\tag{39}$$

for every k. Now define $a_{1k}, \cdots, a_{n-1,k}$; $\beta_{1k}, \cdots, \beta_{n-1,k}$ according to

$$\begin{aligned}xs^n+(a_{n-1,k}+\beta_{n-1,k}x)s^{n-1}+\cdots+(a_{1k}+\beta_{1k}x)s+a_{0k}\\ =\begin{vmatrix}sI_{n-1}-(A_s+B_sK_kC_s) & -B_sK_kC_f\\ -B_fK_kC_s & xs-(1+B_fK_kC_f)\end{vmatrix}.\end{aligned}$$

Then

$$a_{n-1,k}=-(1+B_fK_kC_f).\tag{40}$$

Letting $f_{ik}(x) = \beta_{ik}x$, we can find a sequence ϵ_k satisfying the properties in Lemma 4.3; define $\alpha_{nk} = -1/\epsilon_k$. Since $\epsilon_k \to 0$ and $\text{sgn}\,\epsilon_k = -\text{sgn}\,a_{n-1,k}$, (39) and (40) guarantee that $\alpha_k \to +\infty$. If we set

$$E_k=M^{-1}\begin{bmatrix}I_{n-1} & 0\\ 0 & -\dfrac{1}{\alpha_k}\end{bmatrix}N^{-1},\ A_k=A$$

$$B_k=B,\ C_k=C$$

we have that $\det\,(sE_k - (A_k + B_kK_kC_k))$ has at least one real root $\lambda_k > k$ for each k. As in the sufficiency proof of part 1), $\Psi_{x_0u}(\mathcal{C}(\xi_k, K_k))$ does not converge for some x_0, u. Since

$$\Psi^f_{x_0u}(\xi_k)=\exp\,(-\alpha_k)x_0-\alpha_k\exp\,(-\alpha_k)*B_{fk}u\tag{41}$$

in the open-loop system, $\Psi_{x_0u}(\xi_k) \to \Psi_{x_0u}(\xi)$ and K is not robust.

If (26) fails and $r < n-1$, we may adopt the notation (29) and (30) and observe that $b_{n-r}Kc_1 \leq 0$. Since fast cyclicity, impulse controllability, and impulse observability guarantee that $b_{n-r} \neq 0$ and $c_1 \neq 0$, a sequence $K_k \to K$ may be chosen so that $b_{n-r}K_kc_1 < 0$ for every K. The remaining arguments are the same as in the necessity part of 1) with $b_{n-r,k} = b_{n-r}$, $c_{1k} = c_1$, and $\gamma_{ik} = 1$.□

Theorem 4.4 is somewhat pessimistic in that, in the strictest theoretical sense, robustness can only be guaranteed when at most one degree of singularity is present in the plant (4) (rank $E = n - 1$). In physical terms this can be interpreted as meaning that a static compensator can handle only a first-order unmodeled dynamic element. In our opinion, this indicates that some basic assumptions which are as yet not well understood are conventionally placed on system models in engineering practice.

For a mathematical explanation of how nonrobust compensators may fail to stabilize a system, consider the matrix condition in part 2) of Theorem 4.4. This condition determines an open affine half-space in the set $\mathbb{R}^{mp}$ of compensation gains K. Examination of the proof of Theorem 4.4 reveals that, for systems which are fast cyclic, impulse controllable, and impulse observable, a static compensator results in positive feedback either for all admissible perturbations simultaneously or for none at all. The half-space of robust feedback gains is simply the set of all K with the appropriate sign to guarantee negative feedback for all perturbations of the system (4). The system (6) illustrates this point. The robust gains are simply those satisfying $K < 1$. On the other hand, part 1) of Theorem 4.4 maintains that unless the plant is fast cyclic, impulse controllable, and impulse observable, the class of admissible perturbations is so broad that any compensator results in positive feedback with respect to some perturbation; hence, no compensator is robust. This is illustrated by (2) and (3).

Another important point to note at this stage is that, although all definitions and technical arguments until now have been couched in terms of sequences, each statement applies equally well to nets in the various topological spaces. This observation is important, since the space of distributions $\mathfrak{D}_+$ does not satisfy the first axiom of countability (see [20]).

To conclude this section we compare our results to those of [5]. Specifically, [5, Theorem 1] shows that, for any system

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{42}$$

and any compensation matrix K, there exists a singular perturbation of (42) of the form (1) which destabilizes the closed-loop system. (The result of [5] is somewhat more general in that it applies to all dynamic compensators which are proper but not strictly proper.) According to Theorem 4.4, if we take such a perturbation and set $\epsilon = 0$, we obtain a nominal system

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = [C_1 \ C_2] x \tag{43}$$

which must either fail to be fast cyclic, impulse controllable, or impulse observable. For example, setting $\epsilon = 0$ in (3) yields a system of the form (43) which can be shown to be not fast cyclic. While the result of [5] illustrates that a specialized class of parasitics can lead to closed-loop destabilization, our results characterize the same phenomenon but in the context of a broader class of perturbations and a larger family of nominal systems. For example, our Theorem 4.4 applies to systems of the form (43) with A_{22} singular (as long as $|sE - A| \not\equiv 0$ is satisfied), while [5] considers only the case of A_{22} nonsingular. Our result also shows when destabilization can occur as a result of perturbations to a given order; the perturbed order required to destabilize the closed-loop system in [5] is not specified.

V. Genericity

We now consider the class of systems (4) for which there exists a robust compensator K. The sets of impulse controllable and impulse observable systems were shown in [13] to be dense in the system space $\mathcal{L}$. The next result characterizes those systems which are also fast cyclic.

Proposition:

1) ξ is fast cyclic iff $\xi \in \mathcal{L}^n \cup \mathcal{L}^{n-1}$.
2) $(\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_{ic} \cap \mathcal{L}_{io}$ is open in $\mathcal{L}$.
3) $\mathcal{L}^{n-1} \cap \mathcal{L}_{ic} \cap \mathcal{L}_{io}$ is dense in $\mathcal{L}^s$.

Proof:

1) Let (E, A, B, C) be any representative of ξ. If $r = \text{ord } \xi$, the Weierstrass decomposition (9) shows that rank $E = r +$ rank A_f. But ξ is fast cyclic if and only if either $r = n$ or rank $A_f = n - r - 1$. Hence, ξ is fast cyclic iff rank $E = n$ or rank $E = n - 1$.

2) Let

$$\xi = [E, A, B, C] \in \Omega = (\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_{ic} \cap \mathcal{L}_{io}$$

and apply the decomposition (9). Then A_f is cyclic, and Ker $A_f \subset$ Im A_f. Since ξ is impulse controllable and impulse observable.

$$\text{Im } A_f + \text{Im } B_f = \text{Im } A_f + \text{Ker } A_f + \text{Im } B_f = \mathbb{R}^{n-r}$$

$$\text{Ker } A_f \cap \text{Ker } C_f = \text{Ker } A_f \cap \text{Im } A_f \cap \text{Ker } C_f = 0$$

so $\xi \in \mathcal{L}_{fc} \cap \mathcal{L}_{fo}$ (see [14]). It follows that

$$\Omega = (\mathcal{L}^n \cap \mathcal{L}^{n-1}) \cap \mathcal{L}_{fc} \cup \mathcal{L}_{fo}.$$

We know from [13] that $\mathcal{L}_{fc}$ and $\mathcal{L}_{fo}$ are both open, so Ω is the finite intersection of open sets.

3) It was shown in [13] that $\mathcal{L}_{co} \cap \mathcal{L}^{n-1}$ is dense in $\mathcal{L}^s$. Our result follows immediately, since $\mathcal{L}_{ic} \cap \mathcal{L}_{io} \supset \mathcal{L}_{co}$. □

Note that part 3) is stated in terms of the singular subspace $\mathcal{L}^s$. Since every point in the regular subspace $\mathcal{L}^n$ is necessarily fast cyclic, impulse controllable, and impulse observable (see [14]) and since $\mathcal{L}^n$ is dense in $\mathcal{L}$, density of $(\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_{ic} \cap \mathcal{L}_{io}$ in $\mathcal{L}$ is trivial. Part 3) is a much stronger result.

VI. Discussion and Conclusions

In this section we discuss some of the implications of our theory and use these to suggest further research. Theorem 4.5 shows that a generic class of systems can be robustly compensated using static compensators K. This does not mean, however, that the complement of the open and dense subset $(\mathcal{L}^n \cup \mathcal{L}^{n-1}) \cap \mathcal{L}_{ic} \cap \mathcal{L}_{io}$ does not contain interesting systems. On the contrary, it is easy to show that all systems of the form (4) with $r < n - 1$ and A_{22} nonsingular lie outside the generic class described by Theorem 4.5. Another interesting observation is that even a system which does lie in the generic set can be trivially augmented so that it sits outside the generic set in a higher dimensional system space. For example, the dimension of (4) may be increased simply by defining a new (scalar-valued) state variable $z = 0$ and noting that

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = [C \ 0] \begin{bmatrix} x \\ z \end{bmatrix}. \tag{44}$$

System (44) is a member of $\mathcal{L}(n + 1, m, p)$. It is easy to show that (44) is not fast cyclic and, hence, cannot be robustly compensated. The latter point can be countered by arguing that only variables of interest should be included in a well-devised state-space model; therefore, the variable z would never be present.

There are at least a couple of avenues of research which might eventually resolve these issues. Dynamic compensation is still relatively unexplored in the context of singular perturbations. One promising result is [5, Theorem 2] which suggests that, when parasitics are present, strictly proper compensators are more robust than nonstrictly proper ones. Since [5] treats only the single time-scale case, more work needs to be done to see whether this result stands up to a larger class of perturbations. As pointed out in Section I, the issue of which class of perturbations is meaningful in a given system analysis is of fundamental importance. Our main results can in fact be proven under a somewhat more general definition of system perturbation than the one provided here (strong convergence). However, preliminary work suggests that even such a generalization might be too restrictive to allow a coherent robustness theory to be developed. We intend to explore these issues more fully in the future.

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