Technical Notes and Correspondence

Robust Stablization Relative to the Unweighted H^∞ Norm Is Generically Unattainable in the Presence of Singular Plant Perturbations

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Abstract—Recent work by Vidyasagar gives a sufficient condition under which a singularly perturbed system can be robustly stabilized relative to the unweighted H^∞ norm. In this note we show that the same condition is also necessary. In addition, we prove that this condition is generically untrue with respect to the class of all linear singularly perturbed systems, as well as the class of all singular perturbations of any fixed, stabilizable, and detectable system.

I. INTRODUCTION

Considerable research has been devoted recently to the idea of using the H^∞ norm as a measure of closed-loop system performance (e.g., see [1]). An important contribution to this body of work has been that of Vidyasagar [2], [3] where the "graph topology" is introduced. Roughly speaking, the graph topology is the weakest topology such that every system has a stabilizing compensator, which makes closed-loop system performance (measured by the H^∞ norm) insensitive to small plant perturbations. We are particularly interested in deviations from the plant model which give rise to an increase in model order. Such perturbations are often called singular and are typically a consequence of unmodeled parasitic phenomena (see [5]).

Consider the linear, time-invariant, two-time-scale, singularly perturbed system

$$\begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = [C_1 C_2]x + D_1 u \tag{1}$$

where the matrices A_{ij}, B_i, C_i, and D_1 are independent of ϵ, and A_{22} is Hurwitz. Contained in [2] and [3] are results related to the problem of robustly compensating (1). One such result is that the system (1) converges in the graph topology as $\epsilon \to 0^+$ whenever

$$C_2(sI - A)^{-1}B_2 \equiv 0 \tag{2}$$

(see [2, Theorem 6.1] and [3, Proposition 7.2.62]). If this condition is met, any stabilizing compensator (in the input-output sense) for (1) at $\epsilon = 0$ must also stabilize (1) for sufficiently small ϵ, and the corresponding closed-loop transfer matrix must converge in H^∞ as $\epsilon \to 0^+$. In Section II we prove that this sufficient condition is also necessary. In Section III we show that the class of all systems (1) which satisfy (2) is, in a natural sense, the complement of a generic set, and that singular perturbations of any fixed, stabilizable, detectable system (A, B, C, D) generically do not satisfy (2). Hence, we conclude that closed-loop performance of almost any feedback system is highly sensitive to singular perturbations, if the unweighted H^∞ norm is adopted as the performance measure.

II. A NECESSARY CONDITION FOR ROBUST STABILIZABILITY

In this section we prove that (2) is not only sufficient but also necessary for robust stabilizability.

Manuscript received April 10, 1986; revised August 4, 1986.
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IEEE Log Number 8611300.

Theorem 1: Let $P_\epsilon(s)$ be the transfer matrix of (1) for every $\epsilon \geq 0$, and let the triple (A, B, C) be stabilizable and detectable, where $A = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B = B_1 - A_{12}A_{22}^{-1}B_2$, and $C = C_1 - C_2A_{22}^{-1}A_{21}$. Then $P_\epsilon \to P_0$ in the graph topology as $\epsilon \to 0^+$ if and only if $C_2(sI - A_{22})^{-1}B_2 \equiv 0$.

Proof:

(Sufficient): See [2] or [3].

(Necessary): Suppose $P_\epsilon \to P_0$, and let F be any matrix that makes $A - FC$ Hurwitz. Using the notation of [2], define

$$\tilde{D}_0(s) = I - C(sI - A + FC)^{-1}F$$

$$\tilde{N}_0(s) = C(sI - A + FC)^{-1}(B - FD) + D$$

where $D = D_1 - C_2A_{22}^{-1}B_2$, and let

$$\tilde{D}_\epsilon(s) = I - C_0(sI - \tilde{A}_\epsilon)^{-1}F_0$$

$$\tilde{N}_\epsilon(s) = C_0(sI - \tilde{A}_\epsilon)^{-1}(B_\epsilon - F_0D_1) + D_1$$

where

$$\tilde{A}_\epsilon = \begin{bmatrix} A_{11} - FC_1 & A_{12} - FC_2 \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{bmatrix}, \quad B_\epsilon = \begin{bmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{bmatrix}$$

$$C_0 = [C_1 C_2], \quad F_0 = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

By [2, Lemma 6.1], $(\tilde{D}_\epsilon, \tilde{N}_\epsilon)$ is a left-coprime factorization of P_ϵ for every $\epsilon \geq 0$. According to [3, Lemma 7.2.20, part (ii), and Theorem 4.1.43], there exist unitary rational matrices $\{U_\epsilon | \epsilon > 0\}$ (not necessarily convergent) such that $U_\epsilon\tilde{N}_\epsilon \to \tilde{N}_0$ and $U_\epsilon\tilde{D}_\epsilon \to \tilde{D}_0$ with respect to the H^∞ norm. (In this context, a unitary matrix is one whose determinant has relative degree zero and has all poles and zeros in the open left half complex plane.) Thus,

$$U_\epsilon(i\alpha/\epsilon)\tilde{N}_\epsilon(i\alpha/\epsilon) - \tilde{N}_0(i\alpha/\epsilon) \to 0 \tag{3}$$

$$U_\epsilon(i\alpha/\epsilon)\tilde{D}_\epsilon(i\alpha/\epsilon) - \tilde{D}_0(i\alpha/\epsilon) \to 0 \tag{4}$$

as $\epsilon \to 0^+$ for every $\alpha \in \mathbb{R} - \{0\}$. We have

$$\tilde{D}_\epsilon(i\alpha/\epsilon) = I - \epsilon[C_1C_2] \cdot \begin{bmatrix} i\alpha I - \epsilon(A_{11} - FC_1) & -\epsilon(A_{12} - FC_2) \\ -A_{12} & i\alpha I - A_{22} \end{bmatrix}^{-1} \begin{bmatrix} F \\ 0 \end{bmatrix} \to I$$

and

$$\tilde{D}_0(i\alpha/\epsilon) = I - \epsilon C(i\alpha I - \epsilon(A - FC))^{-1}F \to I.$$

Hence, from (3), $U_\epsilon(i\alpha/\epsilon) \to$ I. Next, observe that

$$\tilde{N}_\epsilon(i\alpha/\epsilon) = [C_1C_2] \begin{bmatrix} i\alpha I - \epsilon(A_{11} - FC_1) & -\epsilon(A_{12} - FC_2) \\ -A_{21} & i\alpha I - A_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} \epsilon(B_1 - FD_1) \\ B_2 \end{bmatrix} + D_1 \to C_2(i\alpha I - A_{22})^{-1}B_2 + D_1$$

and

$$\tilde{N}_0(i\alpha/\epsilon) = \epsilon C(i\alpha I - \epsilon(A - FC))^{-1}(B - FD) + D \to D.$$

From (4) and the fact that $U_\epsilon(i\alpha/\epsilon) \to I$,

$$C_2(i\alpha I - A_{22})^{-1}B_2 + C_2 A_{22}^{-1} B_2 = 0 \tag{5}$$

for every $\alpha \in \mathbb{R} - \{0\}$.

We claim that the second term in (5) vanishes. Indeed,

$$C_2 A_{22}^{-1} B_2 = \lim_{\alpha \to \infty} \left(\frac{1}{\alpha} C_2 \left(iI - \frac{1}{\alpha} A_{22} \right)^{-1} B_2 + C_2 A_{22}^{-1} B_2 \right)$$

$$= \lim_{\alpha \to \infty} \left(C_2(i\alpha I - A_{22})^{-1} B_2 + C_2 A_{22}^{-1} B_2 \right)$$

so from (5) we have $C_2 A_{22}^{-1} B_2 = 0$. It therefore follows that $C_2(sI - A_{22})^{-1} B_2 = 0$ for infinitely many s and, hence, for all $s \in \mathbb{C}$. □

At this stage, a few comments are in order. Although detectability of (A, B, C) was used in the proof of Theorem 1 in assuming the existence of a stabilizing matrix F, stabilizability of (A, B, C) was not used. In fact, examination of the sufficiency proof of Theorem 1 from [2] or [3] reveals that all arguments could equally well be carried out by assuming only stabilizability and relying on right-coprime factorizations. The same holds true for our necessity proof. This weakening of assumptions is probably of limited interest, however, since any system which is not both stabilizable and detectable cannot be *internally* stabilized by any compensator.

As pointed out in [2] and [3], the majority of results surrounding the graph topology do not rely on any properties indigenous to the H^∞ norm except that the H^∞ norm should make the ring of proper, stable rational functions a topological ring with i) the set of units open and ii) inversion a continuous operation on the set of units. Thus, the graph topology is actually a family of topologies, each corresponding to a different notion of robustness. The singular perturbation result we have addressed in Theorem 1, however, does utilize specific properties of the H^∞ norm beyond i) and ii) (see [3, p. 257]). Hence, it is not known whether either the necessary or sufficient part of Theorem 1 generalizes to other robustness measures.

III. Genericity of Undesirable Perturbations

In this section we explore the implications of Theorem 1 to the problem of robust compensation under parasitic uncertainty. In particular, we seek to understand just how large a class of plant perturbations is taken into account by condition (2). An easy result may be obtained by viewing the 4-tuple

$$\xi = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, [C_1 C_2], D_1 \right)$$

as a point in $\mathbb{R}^{n(n+m+p)+pm}$. Since (2) determines finitely many nonzero polynomials on $\mathbb{R}^{n(n+m+p)+pm}$, the set of all ξ satisfying (2) is a proper algebraic variety. Consider the class of all singularly perturbed systems $\Sigma = \{\xi | A_{22} \text{ is Hurwitz}\}$. It is apparent that the class of singularly perturbed systems which do not converge in the graph topology is open and dense in the open set Σ. The class of convergent singularly perturbed systems is nowhere dense and has vanishing Lebesgue measure (see [6]).

A more interesting—and perhaps more convincing—result involves singular perturbations of a single given system. Suppose we wish to design a compensator based on a fixed plant model (A, B, C, D) which is stabilizable and detectable. Any good compensator should be robust with respect to a reasonably defined class of parasitics. For our analysis, we take this to mean all systems of the form (1) which reduce to the given model when $\epsilon = 0$. That is, we consider all singularly perturbed systems (1) with

$$A_{11} - A_{12} A_{22}^{-1} A_{21} = A, \quad B_1 - A_{12} A_{22}^{-1} B_2 = B$$

$$C_1 - C_2 A_{22}^{-1} A_{21} = C, \quad D_1 - C_2 A_{22}^{-1} B_2 = D \tag{6}$$

(and, of course, A_{22} Hurwitz). Equations (6) determine a proper variety in $\mathbb{R}^{n(n+m+p)+pm}$ which we denote by V. Let W be the variety determined by (2), and let $V \cap \Sigma$ inherit relative topology from $\mathbb{R}^{n(n+m+p)+pm}$.

Theorem 2: $V \cap \Sigma - W$ is open and dense in $V \cap \Sigma$.

Proof: Let $\xi \in V \cap \Sigma$ with $\xi \notin W$. Since W is a variety, there exists a neighborhood U of ξ such that $U \cap W = \phi$. $U \cap V \cap \Sigma$ is a relative neighborhood of ξ which does not intersect W, and $V \cap \Sigma - W$ is open.

To show density, let ξ be any point in $V \cap \Sigma \cap W$ and choose a sequence $(A_{22k}, B_{2k}, C_{2k}) \to (A_{22}, B_2, C_2)$ such that (2) is violated for every k. Such a sequence must exist, since (2) determines a proper variety in $\mathbb{R}^{q(q+m+p)}$. (A_{22} is $q \times q$.) Let A_{12} and A_{21} remain constant, and define

$$A_{11k} = A + A_{12} A_{22k}^{-1} A_{21}, \quad B_{1k} = B + A_{12} A_{22k}^{-1} B_{2k}$$

$$C_{1k} = C + C_{2k} A_{22k}^{-1} A_{21}, \quad D_{1k} = D + C_{2k} A_{22k}^{-1} B_{2k}.$$

This determines a sequence $\xi_k \to \xi$ with $\xi_k \notin W$ and $\xi_k \in V \cap \Sigma$ for sufficiently large k. □

The results of [2] and [3] indicate that perturbations of a plant model which are not small in the graph topology lead either to instability in the closed-loop system, for arbitrarily small ϵ, or to discontinuity in H^∞ performance at $\epsilon = 0$, regardless of what compensator is chosen. Hence, Theorem 2 shows that, in a generic sense, closed loop performance of any feedback system is highly sensitive to singular perturbations, if the H^∞ norm is adopted as the performance measure. It is important to note, however, that our results apply only to the ***unweighted*** H^∞ norm. Examination of the proof of Theorem 1 reveals that difficulties can arise only in the high-frequency behavior of the closed-loop system. Our results suggest that, in order to guarantee compensator robustness, one would have to formulate the design problem with relatively little emphasis placed on high-frequency performance. This could only be implemented by choosing a weighting function which rolls off in an appropriate manner at high frequency. On the other hand, high-frequency rolloff is only a necessary condition on the weighting function; more work is required to determine the class of weighting functions which actually have the properties required to guarantee robustness.

Another important observation follows from the results of [4]. Suppose a stabilizing compensator is chosen having a strictly proper transfer function matrix. Then [4] indicates that sufficiently small singular perturbations of the plant can never destabilize the closed-loop configuration. Since (1) does not converge in the graph topology, we must conclude from [2] that the closed-loop system exhibits a discontinuity in H^∞ performance at $\epsilon = 0$. Hence, strictly proper compensators lead to discontinuous H^∞ performance as a function of ϵ, while nonstrictly proper compensators lead to either instability or discontinuous H^∞ performance.

It is our contention that any good compensator design must be robust with respect to a large class of parasitic uncertainties. The precise class which needs to be considered in practice is still a highly controversial issue. We feel that the results of this paper focus in on what might be considered a "minimal" class. Systems of the form (1) have been extensively studied over the past 25 years by countless researchers (see [5]). This fact is partly due to the tractability of problems associated with (1), but primarily relates to the fact that (1) has been shown to accurately describe a wide variety of physical processes. Surely, any perturbational analysis of a given plant model should include all the perturbations described in Theorem 2. In fact, one would probably be more correct in considering a much broader class of perturbations, such as those characterized by multiple time scales, perturbations in the matrices A_{ij}, B_i, $\cdots$, small nonlinearities, etc. It seems unlikely, however, that such a generalized approach would do anything but strengthen our genericity result.

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