# On the Topology of Spaces of Controllable and Observable Systems

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*Abstract*—**Topological properties of classes of state-space systems $(A, B, C) \in R^{n(n+m+p)}$ are considered, using the natural Euclidean topology. In particular, the connected components of the spaces of controllable, observable, and controllable and observable systems are characterized. Similar results are then easily established for corresponding spaces of rational transfer matrices.**

## INTRODUCTION

In this note we examine certain topological properties of important subsets of the space $R^{n(n+m+p)}$ of linear systems $(A, B, C)$, using the natural Euclidean topology. In particular, we are interested in the sets $\Sigma_c$, $\Sigma_o$, and $\Sigma_{co}$ of controllable, observable, and controllable and observable systems, respectively. It was first shown in [1] that $\Sigma_c$, $\Sigma_o$, and $\Sigma_{co}$ are open and dense in $R^{n(n+m+p)}$ (see also [2]). Another natural question to ask is that of how many connected components comprise each set.

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The issue of connectedness was shown in [3] to be fundamental to identification theory, although the results of [3] apply more directly to spaces of transfer functions, not $(A, B, C)$ triples. Specifically, it is shown in [3] that the space rat $(n)$ of strictly proper scalar transfer functions of degree $n$ has $n + 1$ connected components. A strictly proper transfer function $\mathcal{K}$ of degree $n$ is one which can be expressed as

$$\mathcal{K}(S) = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

where numerator and denominator are coprime. The $n + 1$ components of rat $(n)$ are determined by the $n + 1$ possible values of the Cauchy index $\mathcal{I}(\mathcal{K})$, defined as the number of jumps of $\mathcal{K}(\sigma)$ from $-\infty$ to $+\infty$ minus the number of jumps from $+\infty$ to $-\infty$ as $\sigma \in R$ varies from $-\infty$ to $+\infty$ (see [4]). By considering the topology of systems $(A, B, C)$, we will be able to easily extend the results of [3] to the multivariable case.

We refer to the multivariable generalization of rat $(n)$ as rat $(n, m, p)$, i.e., the space of all $p \times m$ strictly proper rational matrices $\mathcal{K}$ with degree $n$. Here, the degree of $\mathcal{K}$ is defined to be the degree of the least common denominator of all minors of $\mathcal{K}$; the least common denominator is called the characteristic polynomial of $\mathcal{K}$ (see [5]). The obvious topology to use on rat $(n, m, p)$ is the one obtained by viewing

$$\mathcal{K}(S) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}(b_{n-1}s^{n-1} + \cdots + b_0) \tag{1}$$

as an element of $R^{n(mp+1)}$, where $b_0, \cdots, b_{n-1}$ now are $p \times m$ matrices and $s^n + a_{n-1}s^{n-1} + \cdots + a_0$ is the characteristic polynomial of $\mathcal{K}$; in this sense, each transfer matrix $\mathcal{K}$ has a unique representation of the form (1). Convergence in rat $(n, m, p)$ thus corresponds to convergence of every $a_i$ and $b_i$ sequence. Note that the map $(A, B, C) \to \mathcal{K}$ is continuous on $\Sigma_{co}$ since

$$\mathcal{K}(S) = \frac{1}{\det (sI - A)} C(\text{adj } (sI - A))B.$$

Corresponding to each triple $(A, B, C) \in R^{n(n+m+p)}$ are its controllability matrix $U$ and observability matrix $V$. We will see that $\mathcal{I}(\mathcal{K})$ along with sgn det $U$ and sgn det $V$, if $U$ or $V$ is square, completely determines the connected components of $\Sigma_c$, $\Sigma_o$, and $\Sigma_{co}$.

Finally, let $GL(n, R)$ be the general linear group of all nonsingular $n \times n$ real matrices. As a starting point for our work, we cite [6] where it is shown that $GL(n, R)$ consists of two components, determined by sgn det $(\cdot)$.

## MAIN RESULTS

We can immediately state and prove our main results. These will be followed by an easy corollary which describes the topology of rat $(n, m, p)$ when either $m > 1$ or $p > 1$.

*Theorem:* 1) If $m = p = 1$, $\Sigma_c$ (respectively, $\Sigma_o$) has two connected components, determined by sgn det $U$ (respectively, sgn det $V$).

2) If $m = p = 1$, $\Sigma_{co}$ has $2(n + 1)$ connected components, determined by the pair (sgn det $U$, $\mathcal{I}(\mathcal{K})$) or, equivalently, by (sgn det $V$, $\mathcal{I}(\mathcal{K})$).

3) If $m = 1$ and $p > 1$, $\Sigma_c$ and $\Sigma_{co}$ each have two connected components, determined by sgn det $U$; $\Sigma_o$ is connected.

4) If $m > 1$ and $p = 1$, $\Sigma_o$ and $\Sigma_{co}$ each have two connected components, determined by sgn det $V$; $\Sigma_c$ is connected.

5) If $m > 1$ and $p > 1$, $\Sigma_c$, $\Sigma_o$, and $\Sigma_{co}$ are each connected.

*Proof:* 1) Since $(A, B, C) \to \det U$ is continuous with range $R - \{0\}$, sgn det $U$ determines two disjoint open subsets $W_1 \cup W_2 = \Sigma_c$. It is easily verified that the map $(A, B, C) \to (S^{-1}AS, S^{-1}B, CS)$ is a homeomorphism from $W_1$ onto $W_2$, where

$$S=\begin{vmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1\end{vmatrix}.$$

Let

$$\bar{A}=\begin{vmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{vmatrix}, \quad \bar{B}=\begin{vmatrix}0\\ \vdots \\ 0 \\ 1\end{vmatrix}. \tag{2}$$

We need only show that the subset containing $(\bar{A}, \bar{B}, 0)$, say $W_1$, is connected. This we will do by constructing a finite sequence of continuous paths joining any given $(A_1, B_1, C_1) \in W_1$ to $(\bar{A}, \bar{B}, 0)$.

Choose a nonsingular $T_1$ that takes $(A_1, B_1, C_1)$ via similarity transformation into controllable canonic form (see [5]), and define

$$A_2=T_1^{-1}A_1T_1=\begin{vmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0 & \cdots & & \cdots & -a_{n-1}\end{vmatrix}$$

$$B_2=T_1^{-1}B_1=\begin{vmatrix}0\\ \vdots \\ 0 \\ 1\end{vmatrix}, \quad C_2=C_1T_1=[c_0 \cdots c_{n-1}]. \tag{3}$$

We have $U_1 = T_1^{-1}U_2$ where $U_1$ and $U_2$ are the corresponding controllability matrices. Since $(A_1, B_1, C_1)$ and $(\bar{A}, \bar{B}, 0)$ are both in $W_1$, it follows from (2) and (3) that det $U_1$ and det $U_2$ have the same sign. But $U_2 = T_1^{-1}U_1$, so det $T_1 > 0$. From [6, p. 131], we know that there exists a continuous curve $T$: $[0, 1] \rightarrow GL(n, R)$ joining $I$ and $T_1$. Thus, $\alpha \rightarrow (T(\alpha)^{-1}AT(\alpha), T(\alpha)^{-1}B, CT(\alpha))$ describes a continuous curve in $\Sigma_c$, starting at $(A_1, B_1, C_1)$ and terminating at $(A_2, B_2, C_2)$.

Next, define $(F, G, H)$: $[0, 1] \rightarrow \Sigma_c$ by

$$F(\alpha)=\begin{vmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (\alpha-1)a_0 & \cdots & & \cdots & (\alpha-1)a_{n-1}\end{vmatrix}$$

$$G(\alpha)=B_2, \quad H(\alpha)=(1-\alpha)C_2.$$

Then $(F, G, H)$ continuously connects $(A_2, B_2, C_2)$ with $(\bar{A}, \bar{B}, 0)$.

Observability can be handled by analogous reasoning using observable canonic form.

2) Reasoning as in 1), it is easy to see that $W_1 \cap \Sigma_{co}$ and $W_2 \cap \Sigma_{co}$ are homeomorphic. We need to show that $W_1 \cap \Sigma_{co}$ has $n + 1$ components, determined by $\mathcal{I}(\mathcal{H})$. Again arguing as in 1), $(A_1, B_1, C_1)$ can be continuously joined to some $(A_2, B_2, C_2)$ in controllable canonic form. The set of all controllable canonic from systems is a linear variety $\Sigma_{can} \subset R^{n(n+2)}$. Furthermore, $\Sigma_{can} \cap \Sigma_{co}$ is homeomorphic to rat $(n)$ since the coefficients of $\mathcal{H}_2(s) = C_2(sI - A_2)^{-1}B_2$ are just the variable entries of $A_2$ and $C_2$. We know from [3] that rat $(n)$ has $n + 1$ components, determined by $\mathcal{I}(\mathcal{H})$, so it remains to show that two points $(F_1, G_1, H_1)$ and $(F_2, G_2, H_2)$ in $\Sigma_{can} \cap \Sigma_{co}$ cannot be joined by a continuous path in $\Sigma_{co}$ if $\mathcal{I}(\mathcal{H}_1) \neq \mathcal{I}(\mathcal{H}_2)$.

Suppose the converse is true. Then there exists a continuous $(F, G, H)$: $[0, 1] \rightarrow \Sigma_{co}$ connecting the two points. We know that $(A, B, C) \rightarrow \mathcal{H}$ is continuous so $\alpha \rightarrow H(\alpha)(sI - G(\alpha))^{-1}F(\alpha)$ is also continuous and connects $\mathcal{H}_1$ and $\mathcal{H}_2$. But this is impossible, since $\mathcal{H}_1$ and $\mathcal{H}_2$ lie in different components of rat $(n)$.

By using observable canonic form, the analogous proof can be constructed using $V$ instead of $U$. Hence, sgn det $V$ may be substituted for sgn det $U$.

3) With regard to $\Sigma_c$, all arguments are identical to those in 1), except that

$$C_2=[c_0 \cdots c_{n-1}]$$

with each $c_i \in R^p$. To prove the result for $\Sigma_{co}$, we first observe that, as in 1), $W_1 \cap \Sigma_{co}$ and $W_2 \cap \Sigma_{co}$ are disjoint homeomorphic open sets whose union equals $\Sigma_{co}$. To show that $W_1 \cap \Sigma_{co}$ is connected, we recall that $R^{n(m+1+p)}$ is locally connected so any given $(A_1, B_1, C_1) \in W_1 \cap \Sigma_{co}$ has a connected neighborhood $W_3 \subset W_1 \cap \Sigma_{co}$. Since the set of all $(A, B, C)$ which are observable through each output is dense in $R^{n(n+1+p)}$, there exists a continuous path connecting $(A_1, B_1, C_1)$ with some system $(A_2, B_2, C_2) \in W_1 \cap \Sigma_{co}$ which is observable through each output.

Choose a matrix $T_1$ with det $T_1 > 0$ which takes $(A_2, B_2, C_2)$ into a controllable canonic form system $(A_3, B_3, C_3)$, and construct a corresponding continuous path. The parametrization $(F_1, G_1, H_1)$: $[0, 1] \rightarrow \Sigma_{co}$ defined by

$$F_1(\alpha)=A_3, \quad G_1(\alpha)=B_3$$

$$H_1(\alpha)=\begin{vmatrix} \alpha+(1-\alpha) & c_{01}(1-\alpha)c_{11} & \cdots & (1-\alpha)c_{n-1,1} \\ (1-\alpha)c_{02} & & & \\ \vdots & & & \vdots \\ (1-\alpha)c_{0,n-1} & & \cdots & (1-\alpha)c_{n-1,n-1} \\ c_{0p} & & \cdots & c_{n-1,p}\end{vmatrix}$$

takes $(A_3, B_3, C_3)$ into $(A_3, B_3, C_4)$ where

$$C_4=\begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 0 \\ c_{0p} & \cdots c_{n-1,p}\end{vmatrix}.$$

Finally, we may construct a continuous path $(F_2, G_2, H_2)$: $[0, 1] \rightarrow \Sigma_{co}$ according to

$$F_2(\alpha)=\begin{vmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (\alpha-1)a_0 & \cdots & & \cdots & (\alpha-1)a_{n-1}\end{vmatrix}$$

$$G_2(\alpha)=B_3, \quad H_2(\alpha)=\begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ (1-\alpha)c_{0p} & \cdots & & (1-\alpha)c_{n-1,p}\end{vmatrix}$$

which joins $(A_3, B_3, C_4)$ with $(\bar{A}, \bar{B}, \bar{C})$ where

$$\bar{C}=\begin{vmatrix} 1 & 0 \cdots 0 \\ 0 & 0 \cdots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \cdots 0\end{vmatrix}.$$

Since $(A_1, B_1, C_1)$ was arbitrary, $W_1 \cap \Sigma_{co}$ is connected.

To see connectedness of $\Sigma_o$, observe that $\Sigma_o$ is open and $\Sigma_{co}$ is dense in $\Sigma_o$, so there exists a connected neighborhood of any given $(A_1, B_1, C_1) \in \Sigma_o$ containing a point in $\Sigma_{co}$. Thus, we need only show that $W_1 \cap \Sigma_{co}$ and $W_2 \cap \Sigma_{co}$ can be joined by a continuous path through $\Sigma_o$. Consider the parametrization $(F, G, H)$: $[0, 1] \to \Sigma_o$ defined by

$$F(\alpha)=\begin{vmatrix} 0 & 1-2\alpha & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{vmatrix}$$

$$G(\alpha)=\begin{vmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{vmatrix}, \quad H(\alpha)=\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

It is straightforward to check that $(F(\alpha), G(\alpha), H(\alpha)) \in \Sigma_o$ and that det $U(0)$ and det $V(1)$ have opposite sign where $U(0)$ and $U(1)$ are the controllability matrices for $\alpha = 0$ and $\alpha = 1$, respectively.

4) All arguments are dual to those in 3).

5) Since $R^{n(n+m+p)}$ is locally connected and $\Sigma_{co}$ is dense in both $\Sigma_c$ and $\Sigma_o$, we only need to demonstrate connectedness of $\Sigma_{co}$. Also, the class of $(A, B, C)$ with $A$ cyclic is dense in $\Sigma_{co}$, so we may begin by choosing $(A_1, B_1, C_1) \in \Sigma_{co}$ controllable and observable through each input and output and with $A_1$ cyclic. Transforming to canonical form yields $(A_2, B_2, C_2)$ where

$$B_2=\begin{vmatrix} 0 & b_{12} & \cdots & b_{1m} \\ \vdots & & & \\ 0 & \vdots & & \vdots \\ 1 & b_{n2} & \cdots & b_{nm} \end{vmatrix}.$$

Arguing as in 3) demonstrates that any system can be continuously joined to one of the form $(\bar{A}, \bar{B}, \bar{C})$ where now

$$\bar{B}=\begin{vmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{vmatrix}.$$

*Corollary:* If $m > 1$ or $p > 1$, rat $(n, m, p)$ is connected.

*Proof:* The map $(A, B, C) \to \mathcal{H}$ defined from $\Sigma_{co}$ onto rat $(n, m, p)$ is continuous, so, if both $m > 1$ and $p > 1$, the result is obvious. The cases where either $m = 1$ or $p = 1$ are dual, so we need only treat $m = 1$, $p > 1$. Here the two components of $\Sigma_{co}$ are determined by sgn det $U$. Suppose $\mathcal{H}(s) = C(sI - A)^{-1}B$ is given by $(A, B, C)$ and satisfies det $U < 0$. Letting

$$S=\begin{vmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{vmatrix}$$

we may define $(\bar{A}, \bar{B}, \bar{C}) = (S^{-1}AS, S^{-1}B, CS)$. Then $\mathcal{H}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$ but $\bar{U} = S^{-1}U$ so det $\bar{U} > 0$. Hence, any $\mathcal{H}$ can be realized by some $(A, B, C)$ with det $U > 0$. Therefore, rat $(n, m, p)$ is the continuous image of a connected set and is itself connected.

## Concluding Remarks

Although fundamental to various branches of control theory, an understanding of further topological properties of the sets of controllable and observable systems was originally necessitated by our investigation into the nature of singular system representations (see [7]). We intend to show in a future publication that the class of controllable and observable systems can become a connected set when singular systems are brought into the picture, even if $m = p = 1$. It is our contention that for this reason, as well as many others, the space of regular and singular systems should be viewed as a natural "completion" of the set of state-space systems.

## References

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