Controllability, Observability, and Duality in Singular Systems

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Abstract—**The concepts of controllability and observability for systems of the form $E\dot{x} = Ax + Bu$, $y = Cx$, E singular are considered. A theory is presented which unifies the three main approaches to this topic already existing in the literature. The development includes a generalization of the duality theorem from state-space theory.**

I. Introduction

RECENT papers by numerous researchers [1]–[4], [7], [8], [11], [12], [14] have sought to generalize many of the elementary concepts of linear system theory to the realm of singular linear time-invariant systems:

$$\theta : \begin{array}{c} E\dot{x} = Ax + Bu \\ y = Cx \end{array}.$$

Here E, $A:\mathbb{R}^n \to \mathbb{R}^n$, $B:\mathbb{R}^m \to \mathbb{R}^n$, $C:\mathbb{R}^n \to \mathbb{R}^k$ are linear maps with E singular and

$$|Es - A| \not\equiv 0. \tag{2}$$

We are mainly interested in the concepts of controllability and observability and in extending their definitions from the case of state-space systems ($E = I$) to include the system θ.

Several attempts have already been made in this direction. Taking the Laplace transform of θ, Rosenbrock [8] and Verghese *et al.* [12] have developed fairly extensive theories of input and output "decoupling zeros" which are, in a sense, frequency domain analogs to controllability and observability. In the time-domain, the work of Yip *et al.* [14] has proceeded along more traditional lines, treating the differential equation θ directly. Although other work in this area has certainly appeared (e.g., [2], [7], [11]), we view the results contained in [8], [12], and [14] as being representative of the main points of view.

It is our intent to develop a theory which unifies the most fundamental elements of the existing time-domain and frequency-domain approaches and which also rectifies what we see as their primary shortcomings. In particular, we are interested in exploring four central issues: 1) A basic difference between the theories of [8] and [12] lies in their definitions of controllability and observability at infinity. It is natural to ask whether either approach is, in some sense, more "correct." 2) The developments of [8] and [12] give what are essentially frequency-domain theories. Although [12] does discuss some time-domain implications, we feel that there is a need for a more explicit and mathematically precise time-domain formulation. 3) As we will see in Sections III and IV the definition of controllability presented in [14] is consistent with that of [8]; however, the notion of observability in [14] does not take into account infinite frequency behavior as described in either [8] or [12]. 4) In [14], as the authors themselves point out, controllability and observability are not algebraically dual concepts. Also, "inconsistent" initial conditions (as defined in [1]) are not accounted for. We will see that these two difficulties are closely related.

As mentioned in 3) above, the definition of controllability at infinity used by Rosenbrock [8] is consistent with the corresponding concept (C-controllability) in Yip [14]. We accept this approach since it is clearly motivated by the idea of reachable states. With this as our starting point, we will deal with 1) and 2) by extending the ideas of [14] to give a complete time-domain characterization of controllability as described in [8] and [12]. Such a theory should then clarify the conflict between the two definitions of controllability at infinity. It will be seen that Verghese's controllability at infinity, although not directly related to reachable states, is equivalent to one's ability to generate a *maximal* class of impulses using piecewise smooth, nonimpulsive controls. (Verghese should be given credit for essentially this idea couched in frequency domain terminology. In [12] he says that "controllable impulsive modes are those that can be excited from zero initial conditions using nonimpulsive inputs.") This motivates the term "impulse controllability." The importance of impulse controllability will be shown by demonstrating its relation to certain feedback compensation problems.

Issues 3) and 4) will be dealt with by deviating from the approach of [14] and defining observability in a way that is compatible with [8] and [12], accounts for inconsistent initial conditions, and allows full algebraic duality. We will then proceed to extend the standard state-space duality theorem to singular systems.

II. Preliminaries

We begin by noting that condition (1) is necessary and sufficient for existence and uniqueness of solutions in θ [5, p. 452]. It will be convenient to decompose θ into subsystems. Let

$$r = \deg\ |Es - A|.$$

Then (see [5, p. 28], [3]) there exist a linear nonsingular $M:\mathbb{R}^n \to \mathbb{R}^n$ and subspaces $S \oplus F = \mathbb{R}^n$ with $\dim S = r$ that decompose the equivalent system

$$ME\dot{x} = MAx + MBu$$

$$y = Cx$$

into

$$\theta_s : \begin{array}{c} \dot{x}_s = A_s x_s + B_s u \\ y_s = C_s x_s \end{array}$$

$$\theta_f : \begin{array}{c} A_f \dot{x}_f = x_f + B_f u \\ y_f = C_f x_f \end{array}$$

with

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$$y = y_s + y_f.$$ [1]

A_f is a nilpotent operator with index q. Let the initial condition of θ_s be

$$x_s(0) = x_{0s}.$$

We will need the following spaces of functions. Let C^i be the i times continuously differentiable maps, and C_p^i the i times piecewise continuously differentiable maps on $\mathbb{R}$ with range depending on context. Also, let C_p^{i+} be the same as C_p^i, but with domain $[0, \infty)$. Let $\mathfrak{D}$ be the C^∞ functions $\phi: \mathbb{R} \to \mathbb{R}$ with bounded support, and $\mathfrak{D}'$ the space of distributions on $\mathbb{R}$ (as in [10]). We define $\mathfrak{D}_p'$ to be the *piecewise continuous distributions:* $\mathfrak{D}_p'$ consists of those distributions f for which there exist points $\cdots, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \cdots$ in $\mathbb{R}$ (finitely many in any bounded interval) and a piecewise continuous function g such that $f = g$ on (τ_{i-1}, τ_i) for every i. Finally, let $\mathfrak{D}^+$ and $\mathfrak{D}^\tau$ be the spaces of distributions with support in $[0, \infty)$ and with point support at τ, respectively. Elements of $\mathfrak{D}'$, $\mathfrak{D}_p'$, $\mathfrak{D}^+$, and $\mathfrak{D}^\tau$ all have range spaces depending on context.

In order to speak in precise terms about the impulsive part of x and y we will need the concept of *restrictions* of elements in $\mathfrak{D}_p'$. For any $\tau \in \mathbb{R}$ and any $f \in \mathfrak{D}_p'$ there exist $\epsilon > 0$ and a piecewise continuous g such that $f = g$ on $(\tau - \epsilon, \tau)$. Define

$$f|[\tau, \infty) \in \mathfrak{D}'$$

by

$$\langle f|[\tau, \infty), \phi\rangle = \begin{cases} 0 & \text{if supp } \phi \subset (-\infty, \tau] \\ \langle f, \phi\rangle - \int_{\tau-\epsilon}^{\tau} g(t)\phi(t)\, dt & \text{if supp } \phi \subset [\tau-\epsilon, \infty). \end{cases}$$

This determines $f|[\tau, \infty)$ uniquely since

$$\mathfrak{D}_{(-\infty,\tau]} + \mathfrak{D}_{[\tau-\epsilon,\infty)} = \mathfrak{D}$$

where $\mathfrak{D}_I$ denotes the subspace of $\mathfrak{D}$ consisting of those ϕ with support in I. Defining $f|(-\infty, \tau]$ similarly, we set

$$f|[\tau_1, \tau_2] = f|[\tau_1, \infty) + f|(-\infty, \tau_2] - f, \qquad \tau_2 \geq \tau_1$$

$$f[\tau] = f|[\tau, \tau]$$

$$f_+ = f|[0, \infty).$$

Then, for any $f \in \mathfrak{D}_p'$, we have $f_+ \in \mathfrak{D}^+$ and $f[\tau] \in \mathfrak{D}^\tau$. $f[\tau]$ may be thought of as the impulsive part of f at τ. Note that f_+ includes any impulses in f at the origin.

Left- and right-hand limits in $\mathfrak{D}_p'$ may be defined by $f(\tau^-) = g(\tau^-)$ and $f(\tau^+) = g(\tau^+)$ where g is piecewise continuous and $f = g$ on $(\tau - \epsilon, \tau)$ and $(\tau, \tau + \epsilon)$. Then

$$\Delta_\tau f = f(\tau^+) - f(\tau^-)$$

is the jump in f at τ.

Let $\delta_\tau \in \mathfrak{D}^\tau$ be the unit impulse at τ. We then have the following obvious result.

Proposition 1: For any $f \in \mathfrak{D}_p'$

1) $(\dot{f}|[\tau, \infty)) = \dot{f}|[\tau, \infty) + \delta_\tau f(\tau^-)$

2) $(\dot{f}[\tau]) = \dot{f}[\tau] - \delta_\tau \cdot \Delta_\tau f$.

We may write the unique solution of θ_f (see [5, p. 48]) as

$$x_f = -\sum_{i=0}^{q-1} A_f^i B_f u^i \tag{2}$$

where u^i denotes the ith derivative in the distribution sense. In the terminology of [1], an initial condition x_0 is said to be "inconsistent" if $x_0 \notin S$. This concept originates from the fact that, as (2) indicates, the response of θ_f is determined completely by u alone.

In discussing controllability and observability, it will be necessary to work with x_+ and $x[\tau]$. These distributions can best be described by a corresponding pair of differential equations.

Proposition 2: Let $u \in C_p^{q-1}$ be given and $x \in \mathfrak{D}_p'$ be a solution of θ.

1) If z is the unique distribution in $\mathfrak{D}^+$ satisfying $E\dot{z} = Az + Bu_+ + \delta_0 Ex(0^-)$, then $z = x_+$.

2) If z is the unique distribution in $\mathfrak{D}^\tau$ satisfying $E\dot{z} = Az - \delta_\tau E\Delta_\tau x$, then $z = x[\tau]$.

Proof: The statements follow immediately after applying Proposition 1 to the system θ. □

Hence, we have the equations

$$\theta^+: \begin{array}{l} E(\dot{x}_+) = Ax_+ + Bu_+ + \delta Ex(0^-) \\ y_+ = Cx_+ \end{array}$$

$$\theta^\tau: \begin{array}{l} E(\dot{x}[\tau]) = Ax[\tau] - \delta_\tau E(\Delta_\tau x) \\ y[\tau] = Cx[\tau] \end{array}$$

characterizing the system response for $t \geq 0$ due to the initial condition $x(0^-)$, and the impulsive behavior of the system at $t = \tau$. Note that this formalism is in keeping with the idea that impulses occur in the natural response ($u_+ = 0$) whenever $x_f(0^-) \notin \operatorname{Ker} E$ (see [12]). θ^+ and θ^τ may also be decomposed in the same way as θ, yielding θ_s^+, θ_f^+, θ_s^τ, and θ_f^τ.

A crucial observation that will be used to simplify matters later on concerns

$$\theta_f^\tau: \begin{array}{l} A_f(\dot{x}_f[\tau]) = x_f[\tau] - \delta_\tau A_f(\Delta_\tau x_f) \\ y_f[\tau] = C_f x_f[\tau]. \end{array}$$

Since $u \in C_p^{q-1}$, $\Delta_\tau x_s = 0$ so $\Delta_\tau x = \Delta_\tau x_f$. Thus, the expression analogous to (2) for θ_f^τ is

$$\begin{aligned} x_f[\tau] &= -\sum_{i=0}^{q-1} A_f^i(-\delta_\tau^i A_f(\Delta_\tau x_f)) \\ &= \sum_{i=1}^{q-1} \delta_\tau^{i-1} A_f^i(\Delta_\tau x). \end{aligned} \tag{3}$$

This means that the impulsive behavior of θ at τ depends *only on the jump in x at* τ. All forcing functions u that produce the same $\Delta_\tau x$ result in the same $x[\tau]$.

III. Controllability and Impulse Controllability

In this section, we start by briefly summarizing the theory of controllability for singular systems as developed by Yip *et al.* in [14].

Definition: θ is *controllable* if for every $\tau > 0$, $x_{0s} \in S$, and $w \in \mathbb{R}^n$ there exists $u \in C^{q-1}$ such that $x(\tau) = w$.

Define the subspaces

$$\mathfrak{R}_s = \sum_{i=0}^{r-1} \operatorname{Im}(A_s^i B_s), \qquad \mathfrak{R}_f = \sum_{i=0}^{q-1} \operatorname{Im}(A_f^i B_f)$$

$$\mathfrak{R} = \mathfrak{R}_s \oplus \mathfrak{R}_f$$

$\mathfrak{R}$ is the *controllable subspace.*

Theorem 1 [14]:

1) Let $\tau > 0$, $x_{0s} \in S$, and $w \in \mathbb{R}^n$. There exists $u \in C^{q-1}$ such that $x(\tau) = w$ iff $w \in \mathfrak{R}$.

2) θ_s is controllable iff

$$\operatorname{Im}(\lambda E - A) + \operatorname{Im} B = \mathbb{R}^n$$

for every $\lambda \in \mathbb{C}$.

[1] An appropriate choice of basis in $\mathbb{R}^n$ would yield a block-diagonal matrix representation for the decomposed system. Many of the authors listed in the references take this approach; however, we prefer coordinate-free interpretations whenever possible.

3) The following statements are equivalent.
a) θ_f is controllable.
b) $\mathcal{R}_f = F$.
c) $\text{Im } A_f + \text{Im } B_f = F$.
d) $\text{Im } E + \text{Im } B = \mathbb{R}^n$.
4) The following are equivalent.
a) θ is controllable.
b) θ_s and θ_f are both controllable.
c) $\mathcal{R} = \mathbb{R}^n$.

We now extend the theory in a way that elucidates the differences between [8] and [12]. Since jumps and impulses in the natural response are so clearly a feature unique to singular systems, we would like to augment the preceding results with statements concerning θ^+ and θ^τ. First we treat jumps.

Theorem 2: There exists $u \in C_p^{q-1}$ such that $\Delta_\tau x = w$ iff $w \in \mathcal{R}_f$.

Proof:
Necessary:

$$\Delta_\tau x = -\sum_{i=0}^{q-1} A_f^i B_f(\Delta_\tau u^i) \in \sum_{i=0}^{q-1} \text{Im } (A_f^i B_f).$$

Sufficient: Choose $w_0, \cdots, w_{q-1}$ such that

$$-\sum_{i=0}^{q-1} A_f^i B_f w_i = w$$

and let

$$u(t) = \begin{cases} w_0 + (t-\tau)w_1 + \frac{1}{2}(t-\tau)^2 w_2 + \cdots \\ \quad + \frac{1}{(q-1)!}(t-\tau)^{q-1} w_{q-1}, & t \geq \tau. \\ 0 \end{cases}$$

□

Moving on to impulses, we need some preliminary definitions. As demonstrated in (3), we can define a map $I_\tau: F \to \mathfrak{D}^\tau$ which takes the jump in x into its corresponding impulsive part. That is,

$$I_\tau(w) = \sum_{i=1}^{q-1} \delta_\tau^{i-1} A_f^i w. \quad (4)$$

Definition: θ is *impulse controllable* if for every $\tau \in \mathbb{R}$, $w \in F$ there exists $u \in C_p^{q-1}$ such that $x[\tau] = I_\tau(w)$.

Impulse controllability guarantees our ability to generate a maximal set of impulses, at each instant, in the following sense: Suppose E and A are given but B and u are allowed to vary over all values. Setting $B = I$ gives $\mathcal{R}_f = F$, and Theorem 2 then indicates that any $\Delta_\tau x \in F$ can occur. Hence, from (4), the largest set of impulsive distributions that can possibly be generated at τ, for fixed E and A, is just $I_\tau(F) \subset \mathfrak{D}^\tau$. On the other hand, if B is also given, Theorem 2 and (4) show that the set of possible impulses at τ is $I_\tau(\mathcal{R}_f)$. Thus, an equivalent definition of impulse controllability is that θ have

$$I_\tau(\mathcal{R}_f) = I_\tau(F).$$

The ability to generate the maximal class of impulses is important, for example, in problems where random disturbances are present. To see this, consider the system

$$E\dot{x} = Ax + Bu + v$$

where v is a disturbance input. Clearly, any jump $\Delta_\tau x \in F$ can be induced by v alone, resulting in perhaps any distribution from $I_\tau(F)$. Impulse controllability is equivalent, therefore, to our ability to *cancel* all such impulses by choosing u. It will be seen that such a feat can be achieved with linear feedback.

We are now ready to characterize impulse controllability algebraically. Let

$$\mathcal{I}_r = \sum_{i=1}^{q-1} \text{Im } (A_f^i B_f).$$

Theorem 3: For any $w \in F$, there exists $u \in C_p^{q-1}$ such that $x[\tau] = I_\tau(w)$ iff $I_\tau(w) \in \mathcal{I}_r$.[2]

Proof: From (3) and (4), $x[\tau] = I_\tau(w)$ iff $A_f(\Delta_\tau x) = A_f w$ so, from Theorem 2, existence of an appropriate u is equivalent to $w \in \mathcal{R}_f + \text{Ker } A_f$. Under this condition,

$$\begin{aligned} I_\tau(w) &\in \sum_{i=1}^{q-1} A_f^i(\mathcal{R}_f + \text{Ker } A_f) \\ &= \sum_{i=1}^{q-1} A_f^i \mathcal{R}_f \\ &= \sum_{i=1}^{q-1}\sum_{j=0}^{q-1} \text{Im } (A_f^{i+j} B_f) \\ &= \sum_{i=1}^{q-1} \text{Im } (A_f^i B_f). \end{aligned}$$

On the other hand, if $I_\tau(w) \in \mathcal{I}_r$, then $A_f^i w \in \mathcal{I}_r = A_f \mathcal{R}_f$ for $i = 1, \cdots, q-1$. In particular,

$$A_f w \in A_f \mathcal{R}_f$$

or

$$w \in \mathcal{R}_f + \text{Ker } A_f. \quad \square$$

Using Theorem 3, we may describe in more detail the set $I_\tau(\mathcal{R}_f)$ of impulses which u can generate. In particular, the following result applies to the maximal set $I_\tau(F)$ (by letting $B = I$).

Proposition 3: $I_\tau(\mathcal{R}_f)$ is a subspace of $\mathfrak{D}^\tau$, isomorphic to $\mathcal{I}_r$.

Proof: Since I_τ is linear, $I_\tau(\mathcal{R}_f)$ is a subspace. Let $\phi \in \mathfrak{D}$ satisfy $\phi(\tau) \neq 0$, $\phi^i(\tau) = 0$, $i = 1, \cdots, q-2$. Theorem 3 shows that the map $f: I_\tau(w) \mapsto \langle I_\tau(w), \phi \rangle$, where w ranges over $\mathcal{R}_f$, takes $I_\tau(\mathcal{R}_f)$ into $\mathcal{I}_r$. Since f is linear, we need only show that f is bijective. If $\langle I_\tau(w), \phi \rangle = 0$ then, from (4), $\phi(\tau)A_f w = 0$ and $I_\tau(w) = 0$. Thus f is one-to-one. If $v \in \mathcal{I}_r (= A_f \mathcal{R}_f)$, choose $w \in \mathcal{R}_f$ so that $\phi(\tau)A_f w = v$. Then $f: I_\tau(w) \mapsto v$, and f is onto. □

Theorem 4: The following statements are equivalent.
1) θ is impulse controllable.
2) θ_f is impulse controllable.
3) $\mathcal{R}_f + \text{Ker } A_f = F$.
4) $\mathcal{I}_r = \text{Im } A_f$.
5) $\text{Im } A_f + \text{Im } B_f + \text{Ker } A_f = F$.
6) There exists a linear $K: \mathbb{R}^n \to \mathbb{R}^m$ such that $\deg |Es - (A + BK)| = \text{rank } E$.

Proof: The equivalence of 1) and 2) follows immediately from the definition of impulse controllability and the observation that $x[\tau] = x_f[\tau]$ must always hold. Equivalence of 1) and 3) follows from the same definition and the first sentence in the proof of Theorem 3. Since

$$\mathcal{I}_r = A_f \mathcal{R}_f = A_f(\mathcal{R}_f + \text{Ker } A_f).$$

statements 3) and 4) are equivalent. That 3) and 4) imply 5) follows from

[2] For a distribution f, we say $f \in V \subset \mathbb{R}^n$ if $\langle f, \phi \rangle \in V$ for all $\phi \in \mathfrak{D}$. Intuitively, $I_\tau(w)$ "points along" $\mathcal{I}_r$.

$$\text{Im } A_f + \text{Im } B_f + \text{Ker } A_f = \mathcal{I}_r + \text{Im } B_f + \text{Ker } A_f$$
$$= \mathcal{R}_f + \text{Ker } A_f$$
$$= F.$$

Conversely, 5) implies 3) since

$$\text{Im } A_f^{i+1} + \text{Im } A_f^i B_f = \text{Im } A_f^i, \qquad i = 1, \cdots, q-1$$

implies

$$F = \text{Ker } A_f + \text{Im } B_f + \text{Im } A_f B_f + \cdots + \text{Im } A_f^{q-1} B_f + \text{Im } A_f^q$$
$$= \text{Ker } A_f + \mathcal{R}_f.$$

Finally, the equivalence of 5) and 6) was established in [3]. □

With Theorem 3 in mind, we call $\mathcal{I}_r$ the *impulse controllable subspace*. Theorems 2 and 3 complement Theorem 1 in describing the type of behavior that can be induced in x through choice of u; only jumps in $\mathcal{R}_f$ and impulses (and their derivatives) along $\mathcal{I}_r$ can be generated.

Statement 6) in Theorem 4 is added for completeness. In [3] and [4] we solved problems related to the elimination of impulses in θ. We obtained results involving the application of a linear feedback law $u = Kx$ which reorganizes the structure of θ so that $A_f = 0$ and, consequently, so that no impulses can exist. Theorem 4 states that, if impulses can be eliminated at all, they can be eliminated with linear feedback.

We are now in a position to interpret some of the definitions in [8] and [12] in the light of Theorems 1–4.

Theorem 5:

1) θ is controllable at infinity in the sense of Rosenbrock [8] iff θ_f is controllable.

2) θ is controllable in the sense of Rosenbrock iff θ is controllable.[3]

3) θ is controllable at infinity in the sense of Verghese [12] iff θ is impulse controllable.

4) θ is controllable in the sense of Verghese iff θ_s is controllable and θ is impulse controllable.

Proof: 1) The condition in [8] for controllability at infinity is that the matrix $[E - sA \; B]$ have full rank for $s = 0$. This is equivalent to Theorem 1, part 3d).

2) From [9], θ has no finite input decoupling zeros iff $[sE - A \; B]$ has full rank for all $s \in \mathbb{C}$; this is equivalent to Theorem 1, part 2). In combination with 1), we have controllability of both subsystems so the result follows from Theorem 1, part 4b).

3) As [12] indicates, uncontrollability at infinity is equivalent to the existence of a vector $v \neq 0$ such that (in matrix terminology)

$$v'[sE - AB] = w'[E \; 0]$$

for some w and all $s \in \mathbb{C}$. Passing to the Weinerstrass cononical form of $sE - A$ via an equivalence transformation (see [5]), it is clear that uncontrollability is equivalent to the existence of $\bar{v} = [\bar{v}_1/\bar{v}_2] \neq 0$ such that

$$[\bar{v}_1' \bar{v}_2']' \begin{bmatrix} sI - A_s & 0 & B_s \\ 0 & sA_f - I & B_f \end{bmatrix} = [\bar{w}_1' \bar{w}_2'] \begin{bmatrix} I & 0 & 0 \\ 0 & A_f & 0 \end{bmatrix}$$

for some $\bar{w}$. Since $\bar{v}'_1(sI - A_s) = \bar{w}_1'$ must hold for all s, $\bar{v}_1 = 0$. Also, $\bar{v}_2'(sA_f - I) = \bar{w}_2' A_f$ implies that $\bar{v}_2' A_f = 0$ or $\bar{v}_2 \in \text{Im } A_f^\perp$. Hence, $-\bar{v}_2' = \bar{w}_2' A_f$ and $\bar{v}_2' B_f = 0$ or $\bar{v}_2 \in \text{Ker } A_f^\perp$ and $\bar{v}_2 \in \text{Im } B_f^\perp$. Controllability at infinity is therefore equivalent to

$$\text{Im } A_f^\perp \cap \text{Im } B_f^\perp \cap \text{Ker } A_f^\perp = 0$$

or

$$\text{Im } A_f + \text{Im } B_f + \text{Ker } A_f = F.$$

[3] For the sake of uniformity, we are taking slight liberties with Rosenbrock's terminology. More correctly, 1) and 2) should begin with "θ has no input decoupling zeros" instead of "θ is controllable."

4) In [12] a system is said to be controllable if it satisfies the matrix condition in the proof of part 3) as well as the conditions for having no finite input decoupling zeros presented in [9]. □

IV. Observability and Impulse Observability

Our task in this section is to define observability for singular systems in a way that reduces to the state-space definition when $E = I$ and allows for a set of results analogous to Theorems 1–5. In particular, the observability equivalent of Theorem 5 must hold if we are to have a time-domain characterization of results in [8] and [12]. Clearly, ours is not the only way to generalize state-space observability (e.g., [14]); however, our insistence on duality narrows the field to only one definition as far as we can see. The definition is justified by the body of results which succeed it.

Definition: θ is *observable* if knowledge of $u_+ \in C_p^{q-1+}$, $y_+ \in \mathfrak{D}^+$, and $y(0^-)$ is sufficient to determine $x(0^-)$. θ is *impulse observable* if, for every $\tau \in \mathbb{R}$, knowledge of $y[\tau]$ is sufficient to determine $x[\tau]$.

We also define

$$\mathfrak{N}_s = \bigcap_{i=0}^{r-1} \text{Ker } (C_s A_s^i), \qquad \mathfrak{N}_f = \bigcap_{i=0}^{q-1} \text{Ker } (C_f A_f^i)$$
$$\mathfrak{N} = \mathfrak{N}_s \oplus \mathfrak{N}_f$$
$$\mathcal{I}_n = \bigcap_{i=1}^{q-1} \text{Ker } (C_f A_f^i).$$

In a manner analogous to Wonham [13] we call $\mathfrak{N}$ the *unobservable subspace* and $\mathcal{I}_n$ the *impulse unobservable subspace*. The following results justify this terminology; they are organized in such a way as to point out the symmetry between controllability and observability for singular systems. Theorem $k + 5$ is analogous for $k = 1, \cdots, 5$.

Theorem 6:

1) Let $u_+ = 0$ in θ^+. Then $y_+ = 0$, $y(0^-) = 0$ iff $x(0^-) \in \mathfrak{N}$.

2) θ_s is observable iff $\text{Ker } (\lambda E - A) \cap \text{Ker } C = 0$ for every $\lambda \in \mathbb{C}$.

3) The following statements are equivalent.

a) θ_f is observable.

b) $\mathfrak{N}_f = 0$.

c) $\text{Ker } A_f \cap \text{Ker } C_f = 0$.

d) $\text{Ker } E \cap \text{Ker } C = 0$.

4) The following are equivalent.

a) θ is observable.

b) θ_s and θ_f are both observable.

c) $\mathfrak{N} = 0$.

Proof:

1) We need only consider y_{f+} since $y_{s+} = 0$ is equivalent to $x_s(0^-) \in \mathfrak{N}_s$. Since $u_+ = 0$,

$$y_{f+} = -\sum_{i=1}^{q-1} \delta^{i-1} C_f A_f^i x_f(0^-)$$

so $y_{f+} = 0$, $y_f(0^-) = 0$ is equivalent to

$$x_f(0^-) \in \text{Ker } C_f A_f^i, \qquad i = 0, \cdots, q-1.$$

2) From the decomposition θ_s, θ_f we see that

$$ME|S = 1, \; ME|F = A_f, \; MA|S = A_s, \; MA|F = I,$$
$$C_s = C|S, \; C_f = C|F. \qquad (5)$$

Since $\lambda A_f - I$ is nonsingular,

$$\text{Ker } (\lambda E - A) \cap \text{Ker } C = \text{Ker } (\lambda ME - MA) \cap \text{Ker } C$$
$$= \text{Ker } (\lambda I - A_s) \cap \text{Ker } C_s.$$

3) Equivalence of a) and b) follows from part 1) and the definition of observability. Since $\mathfrak{N}_f$ is defined in a manner identical to $\mathfrak{N}_s$, we know from state-space theory that a) is equivalent to

$$\text{Ker } (\lambda I - A_f) \cap \text{Ker } C_f = 0$$

for every $\lambda \in \mathbb{C}$. To obtain c), set $\lambda = 0$. Finally, to demonstrate equivalence of c) and d), apply the transformation M

$$\text{Ker } E \cap \text{Ker } C = \text{Ker } ME \cap \text{Ker } C$$
$$= \text{Ker } A_f \cap \text{Ker } C_f.$$

4) The equivalence of a), b), and c) follows from the definitions of observability and $\mathfrak{N}$, and from 1). □

Theorem 7:

$$y[\tau] = 0, \ \Delta_\tau y = 0 \text{ iff } \Delta_\tau x \in \mathfrak{N}_f.$$

Proof: Since $\Delta_\tau y = C_f(\Delta_\tau x)$ and $y[\tau] = \Sigma_{i=1}^{q-1} \delta_\tau^{i-1} C_f A_f^i(\Delta_\tau x)$, the condition $y[\tau] = 0$, $\Delta_\tau y = 0$ is equivalent to

$$\Delta_\tau x \in \text{Ker } (C_f A_f^i) \qquad i = 0, \cdots, q-1. \qquad \square$$

Corollary: Knowledge of $y[\tau]$ and $\Delta_\tau y$ are sufficient to determine $\Delta_\tau x$ (and therefore $x[\tau]$) iff $\mathfrak{N}_f = 0$.

Theorem 8:

$$y[\tau] = 0 \text{ iff } \Delta_\tau x \in \mathfrak{I}_n.$$

Proof: $y[\tau] = 0$ is equivalent to

$$C_f A_f^i(\Delta_\tau x) = 0, \qquad i = 1, \cdots, q-1$$

or

$$\Delta_\tau x \in \mathfrak{I}_n. \qquad \square$$

Theorem 9: The following are equivalent.

1) θ is impulse observable.
2) θ_f is impulse observable.
3) $\mathfrak{N}_f \cap \text{Im } A_f = 0$.
4) $\mathfrak{I}_n = \text{Ker } A_f$.
5) $\text{Ker } A_f \cap \text{Ker } C_f \cap \text{Im } A_f = 0$.
6) There exists a linear $K: \mathbb{R}^k \to \mathbb{R}^n$ such that $\deg |Es - (A + KC)| = \text{rank } E$.

Proof: Equivalence of 1) and 2) follows from the definition of impulse observability and the observation that $y_f[\tau] = y[\tau]$ and $x_f[\tau] = x[\tau]$. To prove the equivalence of 1) and 4), note that from (3) $x[\tau] = 0$ iff $\Delta_\tau x \in \text{Ker } A_f$. Combining this with Theorem 8 and the definition of impulse observability gives the desired result. To prove the equivalence of 3), 4), and 5) we need only choose inner products on F and $\mathbb{R}^k$, and apply Theorem 4 to the adjoint transformations A_f' and C_f'. This gives that the three conditions

$$\sum_{i=0}^{q-1} (\text{Ker } (C_f A_f^i)^\perp) + \text{Im } A_f^\perp = F$$

$$\text{Ker } A_f^\perp = \sum_{i=1}^{q-1} (\text{Ker } (C_f A_f^i)^\perp)$$

$$\text{Ker } A_f^\perp + \text{Ker } C_f^\perp + \text{Im } A_f^\perp = F$$

are equivalent to one another. Take the orthogonal complement of both sides in each expression. We postpone treatment of 6) until the next section. □

Theorem 10:

1) θ is observable at infinity in the sense of Rosenbrock [8] iff θ_f is observable.

2) θ is observable in the sense of Rosenbrock iff θ is observable.[4]

[4] As in Theorem 5, 1) and 2) might begin more correctly with "θ has no output decoupling zeros."

3) θ is observable at infinity in the sense of Verghese [12] iff θ is impulse observable.

4) θ is observable in the sense of Verghese iff θ_s is observable and θ is impulse observable.

Proof:

1) The condition for observability at infinity in [8] is that $[E' - sA'C']'$ have full rank at $s = 0$. This is equivalent to Theorem 6, part 3d).

2) From [9], θ has no finite output decoupling zeros iff $[sE' - A'C']'$ has full rank for all $s \in \mathbb{C}$. The result follows from Theorem 6, parts 2) and 4b).

3) As shown in [12], unobservability at infinity is equivalent to the existence of a vector $v \neq 0$ such that

$$\begin{bmatrix} sE - A \\ C \end{bmatrix} v = \begin{bmatrix} E \\ 0 \end{bmatrix} w$$

for some w and all $s \in \mathbb{C}$. The remainder of the proof is analogous to that of Theorem 5, part 2).

4) Analogous to Theorem 5, part 4). □

Note that, if inconsistent initial conditions are not allowed, our definition of observability reduces to that of [14]. It is only our taking inconsistent initial conditions into account that has allowed the formulation of Theorems 6–10. It can easily be shown that θ is "observable" in the sense of [14] if and only if θ_s is observable in our sense; θ_f is always "observable" in the sense of [14]. This spoils the symmetry with Theorem 1.

Our final observation before turning to the duality theorem is that the subspaces $\mathfrak{R}_f$ and $\mathfrak{N}_f$ are defined in terms of A_f, B_f, and C_f in exactly the same way as are $\mathfrak{R}_s$ and $\mathfrak{N}_s$ in terms of A_s, B_s, and C_s. Therefore, the standard fourfold Kalman decomposition [6] of θ_s can be applied equally well to θ_f, yielding an eightfold decomposition of the entire system. We thus have a geometric characterization of the Kalman decomposition for singular systems as originally described in [12] for the frequency domain. For the sake of brevity, we omit the details.

V. Duality

Clearly, there is a strong sense of symmetry between Theorems 1–5 and Theorems 6–10. We now extend this idea and show that the subspaces $\mathfrak{R}_s$, $\mathfrak{N}_f$, $\mathfrak{I}_r$, etc., and the notions of observability, impulse controllability, etc., have been defined in such a way that controllability and observability are dual concepts in an algebraic sense. Corresponding to θ we define the *dual system*

$$\bar{\theta}: \begin{array}{l} E'\dot{x} = A'x + C'u \\ y = B'x \end{array}$$

where a prime denotes the adjoint transformation with respect to some given inner product. In state-space theory ($E = I$) we know that $\bar{\theta}$ has controllable and unobservable subspaces $\bar{\mathfrak{R}} = \mathfrak{N}^\perp$, $\bar{\mathfrak{N}} = \mathfrak{R}^\perp$. (An overbar will hereafter denote quantities related to $\bar{\theta}$.) To generalize this to singular systems we need some preliminary results.

Theorem 11:

$$\bar{S} = (AF)^\perp \quad \bar{F} = (ES)^\perp.$$

Proof: Define

$$T_\lambda = (\lambda E - A)^{-1} \tag{6}$$

where λ is chosen to make $\lambda E - A$ nonsingular. From [3] we know that $T_\lambda ES = S$ and $T_\lambda AF = F$ so

$$T_\lambda^{-1} S = ES, \qquad T_\lambda^{-1} = AF. \tag{7}$$

Also,

$$\bar{S} = \text{Ker} \prod_{i=1}^{p} \left(T_\lambda' E' - \frac{1}{\lambda^* - \lambda_i^*} I \right)^{n_i}$$

where p is the number of distinct roots of $|Es - A|$ and n_i is the multiplicity of the ith root. An asterisk denotes the complex conjugate. Thus

$$\bar{S} = \operatorname{Im} \prod_{i=1}^{p} \left(ET_\lambda - \frac{1}{\lambda - \lambda_i} I \right)^{n_i \perp}$$

$$= \left(T_\lambda^{-1} \operatorname{Im} \prod_{i=1}^{p} \left(T_\lambda E - \frac{1}{\lambda - \lambda_i} I \right)^{n_i} \right)^{\perp}$$

$$= (T_\lambda^{-1} F)^{\perp} = (AF)^{\perp}$$

$$\bar{F} = \operatorname{Ker} (T_\lambda' E')^{n-r} = \operatorname{Im} (ET_\lambda)^{n-r\perp}$$

$$= (T_\lambda^{-1} \operatorname{Im} (T_\lambda E)^{n-r})^{\perp} = (T_\lambda^{-1} S)^{\perp}$$

$$= (ES)^{\perp}. \qquad \square$$

If U and V are subspaces of $\mathbb{R}^n$ and $U \oplus V = \mathbb{R}^n$, let $\mathcal{P}(U, V): \mathbb{R}^n \to \mathbb{R}^n$ be the transformation that projects vectors on U along V. Also, let

$$P = \mathcal{P}(S, F)$$

$$Q = \mathcal{P}(F, S).$$

Lemma:

1) $\bar{P} = (T_\lambda^{-1} P T_\lambda)'$ and $\bar{Q} = (T_\lambda^{-1} Q T_\lambda)'$ where T_λ is defined in (6).

2) $\bar{M} = M'$.

Proof:

1) From Theorem 11 and (7)

$$\bar{P} = \mathcal{P}(\bar{S}, \bar{F}) = \mathcal{P}((AF)^{\perp}, (ES)^{\perp})$$

$$= \mathcal{P}(ES, AF)' = \mathcal{P}(T_\lambda^{-1} S, T_\lambda^{-1} F)'$$

$$= (T_\lambda^{-1} \mathcal{P}(S, F) T_\lambda)'$$

$\bar{Q}$ is handled similarly.

2) From (5), $MEP + MAQ = I$ so

$$M^{-1} = EP + AQ.$$

From part 1)

$$\bar{M}^{-1} = E'\bar{P} + A'\bar{Q}$$

$$= (T_\lambda^{-1}(PT_\lambda E + QT_\lambda A))'.$$

But P, Q, $T_\lambda E$ and $T_\lambda A$ all commute so

$$\bar{M}^{-1} = (EP + AQ)'. \qquad \square$$

Theorem 12 (Duality Theorem):

1) $\bar{\mathcal{R}}_s = (E\mathfrak{N}_s)^{\perp} \cap \bar{S}$

2) $\bar{\mathcal{R}}_f = (A\mathfrak{N}_f)^{\perp} \cap \bar{F}$

3) $\bar{\mathcal{R}} = (E\mathfrak{N}_s \oplus A\mathfrak{N}_f)^{\perp}$

4) $\bar{\mathfrak{N}}_s = (E\mathcal{R}_s)^{\perp} \cap \bar{S}$

5) $\bar{\mathfrak{N}}_f = (A\mathcal{R}_f)^{\perp} \cap \bar{F}$

6) $\bar{\mathfrak{N}} = (E\mathcal{R}_s \oplus A\mathcal{R}_f)^{\perp}$

7) $\bar{\mathfrak{I}}_r = (A\mathfrak{I}_n)^{\perp} \cap \bar{F}$

8) $\bar{\mathfrak{I}}_n = (A\mathfrak{I}_r)^{\perp} \cap \bar{F}$.

Proof:

1) From (5), $M^{-1}|S = E|S$ and $M^{-1}|F = A|F$. Let P_r be the same as P but with range restricted to S. Then (see [3]) $B_s = P_r MB$. From the lemma

$$\bar{\mathcal{R}}_s = \sum_{i=0}^{r-1} \operatorname{Im} (\bar{A}_s^i \bar{B}_s)$$

$$= \sum_{i=0}^{r-1} \operatorname{Im} ((M'A' | \bar{S})^i \bar{P}_r M'C')$$

$$= \sum_{i=0}^{r-1} \operatorname{Im} ((M'A')^i \bar{P} M'C')$$

$$= \sum_{i=0}^{r-1} \operatorname{Ker} (CMT_\lambda^{-1} P T_\lambda (AM)^i)^{\perp}.$$

Since MT_λ^{-1}, MA, and P commute,

$$\bar{\mathcal{R}}_s = \left(\bigcap_{i=0}^{r-1} \operatorname{Ker} (CP(MA)^i M) \right)^{\perp}$$

$$= \left(M^{-1} \left(\bigcap_{i=0}^{r-1} \operatorname{Ker} (C_s A_s^i) \oplus F \right) \right)^{\perp}$$

$$= (E\mathfrak{N}_s \oplus AF)^{\perp}$$

$$= (E\mathfrak{N}_s)^{\perp} \cap (AF)^{\perp}.$$

The result follows from Theorem 11. Part 2) can be proven similarly.

3) $\bar{\mathcal{R}} = ((E\mathfrak{N}_s)^{\perp} \cap \bar{S}) \oplus ((A\mathfrak{N}_f)^{\perp} \cap \bar{F})$

$$\bar{\mathcal{R}}^{\perp} = (E\mathfrak{N}_s \oplus AF) \cap (A\mathfrak{N}_f \oplus ES)$$

$$= M^{-1}((\mathfrak{N}_s \oplus F) \cap (\mathfrak{N}_f \oplus S))$$

$$= M^{-1}(\mathfrak{N}_s \oplus \mathfrak{N}_f)$$

$$= E\mathfrak{N}_s \oplus A\mathfrak{N}_f$$

4) $\bar{\mathfrak{N}}_s = \bigcap_{i=0}^{r-1} \operatorname{Ker} (\bar{C}_s \bar{A}_s^i)$

$$= \bigcap_{i=0}^{r-1} \operatorname{Ker} ((B' | \bar{S})(M'A' | \bar{S})^i)$$

$$= \bigcap_{i=0}^{r-1} \operatorname{Ker} (B'(M'A')^i \bar{P}) \cap \bar{S}$$

$$= \left(\sum_{i=0}^{r-1} \operatorname{Im} (T_\lambda^{-1} P T_\lambda (AM)^i B) \right)^{\perp} \cap \bar{S}$$

$$= \left(M^{-1} \sum_{i=0}^{r-1} \operatorname{Im} (P(MA)^i MB) \right)^{\perp} \cap \bar{S}$$

$$= \left(M^{-1} \sum_{i=0}^{r-1} \operatorname{Im} (A_s^i P_r MB) \right)^{\perp} \cap \bar{S}$$

$$= (E\mathcal{R}_s)^{\perp} \cap \bar{S}.$$

The proofs of 5) and 6) are similar to those for 4) and 3).

7) As in the proof of 1),

$$\bar{\mathfrak{I}}_r = \sum_{i=1}^{q-1} \operatorname{Im} (\bar{A}_f^i \bar{B}_f)$$

$$= \left(M^{-1} \left(\bigcap_{i=0}^{q-1} \operatorname{Ker} (C_f A_f^i) \oplus S \right) \right)^{\perp}$$

$$= (A\mathfrak{I}_n \oplus ES)^{\perp}$$

$$= (A\mathfrak{I}_n)^{\perp} \cap \bar{F}.$$

The proof of 8) is similar to 7).

Corollary:

1) θ is controllable iff $\bar{\theta}$ is observable.

2) θ is impulse controllable iff $\bar{\theta}$ is impulse observable.

Proof:

1) $\bar{\mathcal{R}} = \mathbb{R}^n$ iff $E\mathfrak{N}_s = A\mathfrak{N}_f = 0$. From (5), $E|S$ and $A|F$ are injections and $ES \cap AF = 0$ so $\bar{\mathcal{R}} = \mathbb{R}^n$ iff $\mathfrak{N}_s = \mathfrak{N}_f = 0$.

2) From the definition of $\mathcal{I}_r$ and Theorem 4, part 4), θ is impulse controllable iff dim $\mathcal{I}_r = \rho - r$ where ρ = rank E. Similarly, from Theorem 9, part 4), θ is impulse observable iff dim $\bar{\mathcal{I}}_n = n - \rho$. Theorem 12, part 8) shows

$$\bar{\mathcal{I}}_n = (A\mathcal{I}_r \oplus ES)^{\perp}$$

so

$$\dim \bar{\mathcal{I}}_n = n - (\dim A\mathcal{I}_r + \dim ES)$$
$$= n - r - \dim \mathcal{I}_r. \qquad \square$$

Note that Theorems 11 and 12 reduce to the standard state-space results when $E = I$.

The duality theorem may be used to prove Theorem 9, part 6). We know from Theorem 4, part 6), that a linear $K:\mathbb{R}^k \to \mathbb{R}^n$ exists such that

$$\deg |E's - (A' + C'K')| = \text{rank } E$$

iff $\bar{\theta}$ is impulse controllable. It follows that

$$\deg |Es - (A + KC)| = \text{rank } E$$

iff $\bar{\bar{\theta}} = \theta$ is impulse observable.

VI. Conclusions

In order to unify various theories related to singular systems, we have defined observability to allow for the possibility of inconsistent initial conditions. Also, a new pair of time-domain concepts unique to singular systems, impulse controllability, and impulse observability, have been described. All definitions were motivated not only by dynamic system considerations, but also to make the theories of controllability and observability algebraic duals of one another. We believe our theory to be the simplest and most intuitively appealing possibility.

We have also seen that the results described in Theorem 4 relate directly to previous work in [3] and [4] dealing with impulse cancellation. Exactly how the theory presented here relates to more advanced topics in linear system theory, as applied to singular systems, is a topic for further research.

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