Descriptor Variable Systems and Optimal State Regulation

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Abstract—Linear systems of the form $E\dot{x} = Ax + Bu$ with E singular are treated. It is desired to find a control which drives the system asymptotically to the origin, minimizing a quadratic cost functional. No restrictions are placed on initial conditions. The cost associated with the impulsive behavior of the system is examined as well as existence and uniqueness of the optimal control. Through a sequence of coordinate transformations it is proven that the optimal control can be found by solving a reduced order Riccati equation.

Introduction

THE problem of optimal state regulation of a linear state variable system $\dot{x} = Ax + Bu$ has been treated extensively in the literature, e.g., [1]–[4]. By far the most common cost functional considered in the problem with infinite terminal time has been

$$J(x,u) = \int_0^\infty \|x(t)\|^2 + \|u(t)\|^2 \, dt. \tag{1}$$

Often the conditions on J are relaxed letting $\|x(t)\|$ denote a seminorm. It is well known that when $\|x(t)\|$ is a norm, an optimal control exists if and only if (A, B) is a stabilizable pair. In this case the optimal control is unique and can be implemented by applying a linear feedback gain to the system. The appropriate feedback matrix can be calculated by solving a certain matrix Riccati equation whose order is equal to that of the system.

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We wish to show that analogous statements can be made about the descriptor variable system

$$E\dot{x} = Ax + Bu \tag{2}$$

where $x(t) \in R^n$ and E is singular. Such systems have been considered in [5], [6], [9]–[13], [15], [16]. In particular, we have derived general results concerning feedback in descriptor systems in [5]. This paper may be viewed as a continuation of [5], in that *optimal* feedback gains will be discussed for the system (2) with cost (1). It will be seen that the appropriate feedback matrix can be computed by solving a Riccati equation whose order is rank E.

Note that a state variable system may be viewed as a special case of (2) with $E = I$. The treatment of the descriptor problem is then a generalization of that for the state variable problem, both requiring the solution of a Riccati equation of order equal to rank E. In the descriptor case, E may have less than full rank so the corresponding Riccati equation may be considered to have reduced order.

The optimal regulator problem for discrete-time descriptor systems has already been studied in [7]. Unfortunately, the nature of the discrete-time problem is so different from that of the continuous-time one, that the solution of our problem cannot be deduced without an independent analysis.

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More similar to our results are those of [9]. However, there are several important ways in which the results of [9] differ from those we will obtain. First, the problem formulation of [9] excludes the possibility of "inconsistent" initial conditions as defined in [12]. It is shown in [6], [11], [12] that such an assumption avoids those cases where impulses occur in the natural response of (2). We intend to allow all initial conditions in our analysis. Second, the approach of [9] yields a control scheme which may involve successive integrals of the state variable. Our results will show that in order to implement the optimal control, only a *constant* feedback matrix is necessary (in the case of infinite terminal time). Finally, calculation of the optimal control scheme derived in [9] involves solving a Riccati equation whose order is greater than n—much greater in many cases. Our solution involves a Riccati equation whose order is less than n.

The approach that we will take is geometric in nature and involves a reformulation of the optimization problem as a minimum norm problem in Hilbert space. It is hoped that this will avoid many of the technical complications of the variational approach used in [9].

We choose to view the equation (2) as the limit of a singularly perturbed system. For example, consider a system described by

$$\begin{bmatrix} -\epsilon & 1 \\ 0 & -\epsilon \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3}$$

where $\epsilon > 0$ is a small number. We consider only $t \geqslant 0$. For an initial condition $x(0)$, the unforced solution of (3) is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t/\epsilon} & -\frac{t}{\epsilon^2} e^{-t/\epsilon} \\ 0 & e^{-t/\epsilon} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}. \tag{4}$$

It is easy to show that the upper right-hand term in (4) tends to a delta function as $\epsilon \to 0$. Hence, we say that the limiting system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5}$$

will behave impulsively unless some sort of compensation scheme is devised. Obviously, a state-variable model would not adequately characterize such a system.

The initial condition in (3) and (5) might result from a random disturbance entering the system. This viewpoint suggests that a feedback controller is called for, since the precise value of $x(0)$ is unpredictable and the control is likely to depend upon $x(0)$. In this example, the only "consistent" initial condition is $x(0) = 0$. Numerous physical examples of systems with impulsive behavior can be found in [11]–[13].

We may now proceed with the formal problem formulation. Let X and U be real Euclidean spaces with norms and inner products related by $\|x\|^2 = \langle x, x\rangle$. We will not distinguish notationally between norms or inner products on X and U. Let A and B be linear transformations from X into X and from U into X, respectively. Also let E be a singular transformation from X into X. We make the standard assumption throughout, that $\det(Es - A)$ is not the zero polynomial. Systems for which this assumption does not hold have not been treated extensively in the literature. It can be shown, however, that such systems may have nonunique solutions (see [14]).

A canonical decomposition of (2) was proposed in [14], [15] and adopted in [6], [12], [13], [16]. We interpreted it geometrically in [5] yielding subspaces

$$S \oplus F = X \tag{6}$$

and subsystems

$$\dot{x}_s = L_s x_s + B_s u \tag{7}$$

$$L_f \dot{x}_f = x_f + B_f u \tag{8}$$

where L_f is nilpotent. Each subsystem acts on its corresponding subspace. Since general linear feedback does not preserve the structure (6) (see [5]) and since the same control u is applied to both subsystems, it is not possible to solve the optimization problem by considering (7) and (8) independently.

We will use the theory of distributions as developed in [17] and extended to vector spaces in [18]. Briefly, the space K is the set of C^∞ mappings from R into R with bounded support. The sets of distributions K'_X and K'_U are the dual spaces of continuous linear transformations from K into X and U, respectively. A distribution f acting on $\phi \in K$ is denoted (f, ϕ). We will have need only for those distributions with support in $[0, \infty)$. We denote the spaces of such distributions by K_X^+ and K_U^+.

According to the interpretation of [6], the system (2) has a unique solution for any initial condition x_0 and any $u \in K_U^+$. x_0 is to be interpreted in the 0^- sense, i.e., $x_0 = x(0^-)$. Letting $e(A)$ be defined on $[0, \infty)$ by

$$e(A)(t) = e^{tA}$$

and decomposing $x_0 = x_{0s} + x_{0f}$ according to (6), the solutions of (7) and (8) are

$$x_s = e(L_s)x_{0s} + e(L_s) * B_s u \tag{9}$$

$$x_f = -\sum_{i=1}^{q-1} \delta^{i-1} L_f^i x_{0f} - \sum_{i=0}^{q-1} L_f^i B_f u^i \tag{10}$$

where δv is the delta function along $v \in F$ defined by $(\delta v, \phi) = \phi(0)v$, "$*$" denotes convolution, u^i, δ^i denote the ith derivatives in the distribution sense, and q is the index of nilpotency of L_f. The solution of (2) is

$$x = x_s + x_f. \tag{11}$$

Note that if u is C^∞ then $x_f(0^+) = -\Sigma_{i=0}^{q-1} L_f^i B_f u^i(0)$. Hence, x_0 and $x(0^+)$ may not be equal.

Let L_X^2 and L_U^2 be the square integrable mappings from $[0, \infty)$ into X and U. $\sqrt{J(x, u)}$ as defined in (1) is a standard norm on $L_X^2 \times L_U^2$ which we denote by $\|(x, u)\|_2$.

$L_X^2 \times L_U^2$ can be naturally imbedded in $K_X^+ \times K_U^+$. Clearly, even if x and u satisfy (2), they need not both be L^2 functions. Hence, we need to interpret $J(x, u)$ when $(x, u) \in (K_X^+ \times K_U^+) - (L_X^2 \times L_U^2)$.

EXTENSION OF J TO $K_X^+ \times K_U^+$

The first question that comes to mind with regard to extending the cost functional beyond the L^2 functions involves the value of $\int_0^\infty \|\delta(t)v\|^2\, dt$. As exhibited by (10), delta functions can occur in x even when $u = 0$, so we need to decide on the cost that we will impose on such a trajectory. Since (2) can be viewed as the limit of a singularly perturbed system ([6], [11]–[13]), it is natural to approximate δv by functions whose cost we already know. For example, we could choose

$$f_n(t) = \begin{cases} nv, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 0, & \dfrac{1}{n} < t. \end{cases}$$

Then $f_n \to \delta v$ in the topology of K_F^+ and

$$\int_0^\infty \|f_n(t)\|^2\, dt = n\|v\|^2 \to \infty$$

so we might claim that it is most reasonable to set

$$\int_0^\infty \|\delta(t)v\|^2\, dt = \infty.$$

However, we do not know yet whether choosing a different approximating sequence would yield a different limit. The following result shows us how to proceed.

Proposition 1: Let Z be a real Euclidean space, (z_n) a sequence in L_Z^2, and $z \in K_Z^+ - L_Z^2$. If $z_n \to z$ in the K_Z^+ topology, then $\|z_n\|_2 \to \infty$.

Proof: Consider K with the L^2 norm imposed on it. Since K is dense in L^2, the dual of K is L_Z^2 so z must be unbounded on K with respect to $\|\cdot\|_2$. Thus,

$$\sup_{\substack{\phi \in K \\ \|\phi\|_2 = 1}} \|(z, \phi)\| = \infty.$$

Let $M > 0$ be given. There exists $\psi \in K$ with $\|\psi\|_2 = 1$ such that

$$\|(z, \psi)\| > M.$$

Since $(z_n, \psi) \to (z, \psi)$, for sufficiently large n we must have

$$\begin{aligned} \|z_n\|_2 &= \sup_{\substack{\phi \in K \\ \|\phi\|_2 = 1}} \|(z_n, \phi)\| \\ &\geqslant \|(z_n, \psi)\| \\ &> M. \end{aligned}$$

Since M was arbitrary, $\|z_n\|_2 \to \infty$. □

According to Proposition 1, not only should we interpret $\|\delta v\|_2$ as infinite, but we should set $\|f\|_2 = \infty$ for *all* $f \in (K_X^+ \times K_U^+) - (L_X^2 \times L_U^2)$. In this way J can be extended to all possible distribution pairs

$$J(x, u) = \begin{cases} \|(x, u)\|_2, & (x, u) \in L_X^2 \times L_U^2 \\ \infty, & \text{otherwise.} \end{cases}$$

By an optimal solution we mean, then, a pair (x^*, u^*) satisfying (2) with $J(x^*, u^*) < \infty$ and $J(x^*, u^*) \leqslant J(x, u)$ for all other (x, u) satisfying (2).

EXISTENCE AND UNIQUENESS OF THE OPTIMAL CONTROL

We wish to find conditions under which a solution to the problem of minimizing J exists and when it is unique. We will do this through an application of the Hilbert space projection theorem. But first we must take a closer look at the geometry of the problem. Let

$$\Lambda = \left\{ (x, u) \in K_X^+ \times K_U^+ \,|\, x = e(L_s) * B_s u - \sum_{i=0}^{q-1} L_f^i B_f u^i \right\}.$$

It is easily verified that Λ is a subspace of $K_X^+ \times K_U^+$. Let

$$\mathfrak{N}(x_0) = \left(e(L_s)x_{0s} - \sum_{i=1}^{q-1} \delta^{i-1} L_f^i x_{0f}, 0 \right) \in K_X^+ \times K_U^+.$$

The first entry of $\mathfrak{N}(x_0)$ is the natural response of the system (2). In fact, it is clear from (9)–(11) that $x \in K_X^+$ is generated by applying $u \in K_U^+$ to the system (2) if and only if

$$(x, u) \in \mathfrak{N}(x_0) + \Lambda.$$

Hence, (2) and the initial condition x_0 determine the constraint set $\mathfrak{N}(x_0) + \Lambda$ over which $J(x, u)$ must be minimized.

For an optimal point (x^*, u^*) to exist there must be at least one point (x, u) such that $J(x, u) < \infty$ or, equivalently, $(x, u) \in L_X^2 \times L_U^2$. In order to establish conditions under which this is true we need two preliminary results. The first follows from uniqueness of solutions of (2).

Lemma 1: Consider two descriptor systems, both with initial condition x_0

i) $E\dot{x} = Ax + Bu$

ii) $E\dot{y} = (A + BK)y + Bv$

where K: $X \to U$ is a linear transformation. If $v \in K_U^+$ is chosen and applied to ii), generating y, and we apply $u = Ky + v$ to i), then y is the solution of i). □

The second is an extension of a structural result proved in [5]. It was shown in [5] that, when

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F$$

one can construct a linear transformation K: $X \to U$ such that

$$\operatorname{Ker} K \supset S \tag{12}$$

and

$$\det(Es - A - BK) \not\equiv 0 \tag{13}$$

and such that the closed-loop system

$$E\dot{y} = (A + BK)y + Bv, \qquad \text{i.c. } x_0 \tag{14}$$

exhibits no impulsive behavior. Furthermore, a geometric decomposition of (14) can be employed yielding subspaces

$$S_K \oplus F_K = X$$

where

$$\dim S_K = \operatorname{rank} E$$

and corresponding subsystems

$$\dot{y}_s = L_{sK} y_s + B_{sK} v, \qquad \text{i.c. } x_{0sK} \tag{15}$$

$$0 = y_f + B_{fK} v, \qquad \text{i.c. } x_{0fK} \tag{16}$$

where $x_0 = x_{0sK} + x_{0fK}$.

It was shown further in [5] that the closed-loop system (14) has, as its set of eigenvalues, those of the original system along with some others that were induced by applying feedback. If none of the induced eigenvalues is equal to any of the original ones, a decomposition of subsystem (15) may then be invoked to prove the following.

1) Each controllable mode (eigenvalue and corresponding eigenspace) of the original system (2) is also controllable as a mode of (14).

2) Each induced mode of (14) is controllable.

Hence, stabilizability of (L_s, B_s) implies stablizability of (L_{sK}, B_{sK}).

The required decomposition cannot, however, be employed if any of the induced eigenvalues are equal to any of the original. Even though this situation seems likely to be a pathological case, we choose to eliminate the problem by introducing the next result.

Proposition 2: Statements 1) and 2) above are true for any feedback matrix K satisfying (12) and (13).

Proof: It is clear from the development of [5] that we need only show that

$$\operatorname{Im}(\lambda E - A - BK) + \operatorname{Im} B = X$$

for each number λ satisfying

$$\operatorname{Im}(\lambda I - L_s) + \operatorname{Im} B_s = S.$$

From [5] we know that there exists an invertible $M: X \to X$ such that $ME|S = I$, $ME|F = L_f$, $MA|S = L_s$, $MA|F = I$, $P_{SF}MB = B_s$, and $P_{FS}MB = B_f$ where P_{SF} and P_{FS} are projection operators and the vertical bar denotes restriction. Let $K_f = K|F$. For any $x \in X$ decompose $Mx = x_1 + x_2$ with $x_1 \in S$, $x_2 \in F$ and choose $y_1 \in S$, $w \in U$ such that

$$(\lambda I - L_s)y_1 + B_s w = x_1.$$

Define

$$y_2 = (\lambda L_f - I)^{-1}(x_2 - B_f w) \in F$$
$$v = w + K_f y_2$$
$$y = y_1 + y_2.$$

Then, by a straightforward calculation,

$$\begin{aligned} M(\lambda E - A - BK)y + MBv &= (\lambda I - L_s)y_1 - B_s K_f y_2 \\ &\quad + (\lambda L_f - I - B_f K_f)y_2 + B_s v + B_f v \\ &= Mx. \end{aligned}$$

The desired result then follows from the invertibility of M. □

Corollary 2.1: For any K satisfying (12) and (13):

1) if (L_s, B_s) is controllable, then (L_{sK}, B_{sK}) is controllable,

2) if (L_s, B_s) is stabilizable, then (L_{sK}, B_{sK}) is stabilizable.

Corollary 2.2: If (L_s, B_s) is stabilizable and

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F$$

there exists a linear transformation $\bar{K}: X \to U$ satisfying (13) such that the closed-loop system contains no impulses and is asymptotically stable.

Proof: Let $K: X \to U$ satisfy (12) and (13) and eliminate impulses. Then the closed-loop system (14) has decomposition (15), (16) with (L_{sK}, B_{sK}) stabilizable. Choose $\tilde{K}: S_K \to U$ so that all eigenvalues of $L_{sK} + B_{sK}\tilde{K}$ have strictly negative real parts and define

$$\bar{K} = K + \tilde{K} P_{S_K F_K}$$

where $P_{S_K F_K}$ is a projection operator. Applying $u = \bar{K}x + v$ to (2) yields subsystems

$$\dot{y}_s = (L_{sK} + B_{sK}\tilde{K})y_s + B_{sK}v$$

$$0 = B_{fK}\tilde{K}y_s + y_f + B_{fK}v$$

which has the desired properties. Clearly,

$$\det(Es - A - BK) = \alpha \det(Is - L_{sK} - B_{sK}\tilde{K}) \not\equiv 0$$

for some constant $\alpha \neq 0$. □

We are now in a position to establish necessary and sufficient conditions under which, for any initial condition, a pair (x, u) satisfying (2) exists with $J(x, u) < \infty$. We would like to have conditions that work for all initial conditions simultaneously since in practice initial conditions are often unknown.

Theorem 1: $(\mathfrak{N}(x_0) + \Lambda) \cap (L_X^2 \times L_U^2) \neq \varnothing$ for all $x_0 \in X$ if and only if (L_s, B_s) is stabilizable and

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F.$$

Proof:

(Sufficiency): As shown in Corollary 2.2, the two conditions of Theorem 1 guarantee the existence of a linear transformation $\bar{K}$ such that, for any x_0, the closed-loop system

$$E\dot{y} = (A + B\bar{K})y$$

exhibits no impulsive behavior and has all its eigenvalues with strictly negative real part. Hence, its solution y is in L_X^2. Let $u = \bar{K}y$. Then, setting $v = 0$ in Lemma 1, the solution of

$$E\dot{x} = Ax + Bu$$

is $x = y$ so

$$(x, u) = (y, \bar{K}y) \in (\mathfrak{N}(x_0) + \Lambda) \cap (L_X^2 \times L_U^2).$$

(Necessity): Suppose that for every $x_0 \in X$ there exists

$$(x, u) \in (\mathfrak{N}(x_0) + \Lambda) \cap (L_X^2 \times L_U^2).$$

Then for each $x_{0s} \in S$ there is a $u \in L_U^2$ such that

$$e(L_s)x_{0s} + e(L_s) * B_s u \in L_S^2$$

which implies that (L_s, B_s) is stabilizable. Also, for each $x_{0f} \in F$ there must exist a $u \in L_U^2$ such that

$$\sum_{i=1}^{q-1} \delta^{i-1} L_f^i x_{0f} + \sum_{i=0}^{q-1} L_f^i B_f u^i \in L_F^2.$$

Multiplying by L_f^{q-2} yields

$$\delta L_f^{q-1} i x_{0f} + L_f^{q-1} B_f \dot{u} + L_f^{q-2} B_f u \in L_F^2.$$

Since $u \in L_U^2$, we must have

$$\delta L_f^{q-1} x_{0f} + L_f^{q-1} B_f \dot{u} \in L_F^2.$$

Suppose

$$L_f^{q-1} \operatorname{Im} B_f \neq \operatorname{Im} L_f^{q-1}$$

and choose x_{0f} such that

$$L_f^{q-1} x_{0f} \notin L_f^{q-1} \operatorname{Im} B_f.$$

Then $\delta L_f^{q-1} x_{0_f}$ and $L_f^{q-1} B_f \dot{u}$ are linearly independent members of K_F^+. Since

$$\delta L_f^{q-1} x_{0f} \notin L_F^2$$

the sum of the two cannot be in L_F^2. Hence,

$$L_f^{q-1} \operatorname{Im} B_f = \operatorname{Im} L_f^{q-1}.$$

An easy result concerning nilpotent operators gives

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F. \quad \square$$

If the conditions of Theorem 1 hold, then

$$\inf\{J(x, u) | (x, u) \in \mathfrak{N}(x_0) + \Lambda\} < \infty \tag{17}$$

and the optimization problem has at least an ϵ-optimal solution. Since $J = \infty$ for points outside $L_X^2 \times L_U^2$, we may restrict our attention to the smaller constraint set of Theorem 1. Since all points in this set are also in $L_X^2 \times L_U^2$ and since J is the square of an L^2 norm, the entire problem reduces to a Hilbert space optimization problem.

We would next like to know whether the infimum (17) is achieved by some (x^*, u^*) and, if so, whether the optimal pair is unique. The next result answers these questions.

Proposition 3: Under the conditions of Theorem 1, $(\mathfrak{N}(x_0) + \Lambda) \cap (L_X^2 \times L_U^2)$ is a closed linear variety with respect to the L^2 norm.

Proof: The set is obviously of a linear variety. We need only show that $\Lambda \cap (L_X^2 \times L_U^2)$ is closed. Suppose $u_n \to u$, $x_n \to x$ with respect to the L^2 norm and $(x_n, u_n) \in \Lambda$. Decompose x, x_n according to $S \oplus F = X$. Then $x_{ns} \to x_s$ and $x_{nf} \to x_f$. We now have

$$x_{ns} = e(L_s) * B_s u_n$$

$$x_{nf} = -\sum_{i=0}^{q-1} L_f^i B_f u_n^i.$$

L^2 convergence is stronger than convergence with respect to the K' topology so $u_n \to u$, $x_{ns} \to x_s$, and $x_{nf} \to x_f$ in K_U^+, K_S^+, and K_F^+. Since convolution and differentiation are continuous operations in K', it follows that

$$x_s = e(L_s) * B_s u$$

$$x_f = -\sum_{i=0}^{q-1} L_f^i B_f u^i.$$

Thus,

$$(x, u) \in \Lambda. \quad \square$$

Corollary: If (L_s, B_s) is stabilizable and

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F$$

then there exists a unique (x^*, u^*) such that

$$J(x^*, u^*) = \inf\{J(x, u) | (x, u) \in \mathfrak{N}(x_0) + \Lambda\}.$$

Proof: This follows immediately from Proposition 3, the fact that

$$J(x, u) = \|(x, u)\|_2^2,$$

and the projection theorem.

We have, therefore, that the conditions of Theorem 1 guarantee that the set of pairs (x, u) satisfying (2) with finite cost is nonempty and that a unique optimum pair (x^*, u^*) exists. If the conditions of Theorem 1 are not met, then all pairs (x, u) satisfying (2) have infinite cost and the optimization problem makes little sense.

Implementation of u^*

In this section we will show the following. 1) As in the state variable case, u^* can be implemented by a linear feedback law that is independent of x_0. 2) The feedback law can be found by solving a (rank E)th order Riccati equation and applying some coordinate transformations.

For the remainder of the paper we adopt the two conditions of Theorem 1. Thus, we are assured that a unique optimum point (x^*, u^*) exists.

We intend to introduce a sequence of transformations on $K_X^+ \times K_U^+$ whose inverses will take the original optimization problem into a form that we already know how to solve. The solution can then be mapped back through the transformations to give us the solution of the original problem.

To reformulate the problem in terms of the closed-loop system with no impulses, let

$$\mathcal{T}: K_X^+ \times K_U^+ \to K_X^+ \times K_U^+$$

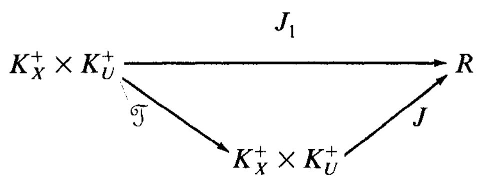
be defined by

$$\mathcal{T}(y, v) = (y, Ky + v)$$

where K: $X \to U$ satisfies (12) and (13) and eliminates impulses. It is routine to verify that $\mathcal{T}$ is an automorphism with an inverse given by

$$\mathcal{T}^{-1}(x, u) = (x, -Kx + u).$$

Let J_1 be defined by the commutative diagram



Since $\mathcal{T}$ is an automorphism, $\mathcal{T}(L_X^2 \times L_U^2) = L_X^2 \times L_U^2$, and $\sqrt{J}$ restricted to $L_X^2 \times L_U^2$ is a norm, it follows that $\sqrt{J_1}$ is a norm on $L_X^2 \times L_U^2$. Finally, let

$$\mathcal{N}_1(x_0) = (e(L_{sK})x_{0sK}, 0) \in K_X^+ \times K_U^+$$

$$\Lambda_1 = \{(y, v) \in K_X^+ \times K_U^+ | y = e(L_{sK}) * B_{sK}v - B_{fK}v\}.$$

Proposition 4: $\mathcal{T}(\mathcal{N}_1(x_0) + \Lambda_1) = \mathcal{N}(x_0) + \Lambda$.

Proof: Choose $(y, v) \in \mathcal{N}_1(x_0) + \Lambda_1$. Then y is the solution of (14). From Lemma 1, y is also the solution of (2) where $u = Ky + v$, so

$$\mathcal{T}(y, v) = (y, Ky + v) \in \mathcal{N}(x_0) + \Lambda.$$

Hence,

$$\mathcal{T}(\mathcal{N}_1(x_0) + \Lambda_1) \subset \mathcal{N}(x_0) + \Lambda.$$

To show the converse, choose u and let x be the solution of (2). If we choose $v = -Kx + u$, Lemma 1 gives

$$\mathcal{T}^{-1}(x, u) = (x, -Kx + u) \in \mathcal{N}_1(x_0) + \Lambda_1$$

so

$$(x, u) \in \mathcal{T}(\mathcal{N}_1(x_0) + \Lambda_1). \qquad \square$$

By applying the transformation $\mathcal{T}^{-1}$ to the original optimization problem we arrive at the equivalent one of minimizing J_1 over $\mathcal{N}_1(x_0) + \Lambda_1$. It is important to note that, from the definition of J_1, and since $\mathcal{T}$ is an automorphism, we are guaranteed that the transformed problem has a unique solution. Unfortunately, we still do not know how to calculate the optimum point. Hence, we must apply additional transformations.

We may now reduce the dimension of the problem by eliminating the subsystem corresponding to F_K. Consider another transformation

$$\mathcal{T}_1: K_{S_K}^+ \times K_U^+ \to K_X^+ \times K_U^+$$

defined by

$$\mathcal{T}_1(y_s, v) = (y_s - B_{fK}v, v).$$

It is easily shown that $\mathcal{T}_1$ is a monomorphism

$$\operatorname{Im}\mathcal{T}_1 = \{(y, v) | y + B_{fK}v \in K_{S_K}^+\}$$

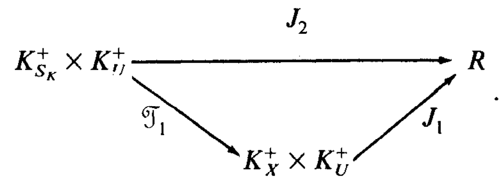
and

$$\mathcal{T}_1^{-1}(y, v) = (y + B_{fK}v, v)$$

whenever $(y, v) \in \operatorname{Im}\mathcal{T}_1$. From (15) and (16) it is clear that

$$\mathcal{N}_1(x_0) + \Lambda_1 \subset \operatorname{Im}\mathcal{T}_1.$$

Let J_2 be defined by



Finally, let

$$\mathcal{N}_2(x_0) = (e(L_{sK})x_{0sK}, 0) \in K_{S_K}^+ \times K_U^+$$

$$\Lambda_2 = \{(y_s, v) | y_s = e(L_{sK}) * B_{sK}v\}.$$

Proposition 5: $\mathcal{T}_1(\mathcal{N}_2(x_0) + \Lambda_2) = \mathcal{N}_1(x_0) + \Lambda_1$.

Proof: Suppose y_s is the solution of (15). Then, from the equivalence of (14) with (15) and (16), $y_s - B_{fK}v$ is the solution of (14) so

$$\mathcal{T}_1(y_s, v) = (y_s - B_{fK}v, v) \in \mathcal{N}_1(x_0) + \Lambda_1.$$

Conversely, let y be the solution of (14). Then $y = y_s - B_{fK}v$ where y_s is the solution of (15). Hence, $(y, v) \in \operatorname{Im}\mathcal{T}_1$ and

$$\mathcal{T}_1^{-1}(y, v) = (y_s, v) \in \mathcal{N}_2(x_0) + \Lambda_2$$

so

$$(y, v) \in \mathcal{T}_1(\mathcal{N}_2(x_0) + \Lambda_2). \qquad \square$$

Since $\mathcal{T}_1(L_{S_K}^2 \times L_U^2)$ is a subsystem of $L_X^2 \times L_U^2$, $\sqrt{J_2}$ is a norm on $L_X^2 \times L_U^2$. Again, we are guaranteed existence of a unique solution to the transformed problem, but a method for computing that solution is not yet clear. One more transformation is needed to simplify the performance index.

In constructing the final transformation we need to define

$$K_1 = K | S_K$$
$$K_2 = K | F_K$$
$$R = B_{fK}'B_{fK} + (I - K_2 B_{fK})'(I - K_2 B_{fK})$$

where a prime denotes the adjoint operator.

Proposition 6: R is positive definite.

Proof: R is positive semidefinite since it is the sum of positive semidefinite transformations. Suppose $\langle R\alpha, \alpha \rangle = 0$

for some $\alpha \in U$. Then

$$\langle B'_{fK} B_{fK} \alpha, \alpha \rangle = 0$$
$$\langle (I - K_2 B_{fK})'(I - K_2 B_{fK})\alpha, \alpha \rangle = 0$$

so

$$\langle B_{fK}\alpha, B_{fK}\alpha \rangle = 0$$

and

$$B_{fK}\alpha = 0.$$

Thus,

$$\langle \alpha, \alpha \rangle = \langle (I - K_2 B_{fK})\alpha, (I - K_2 B_{fK})\alpha \rangle = 0$$

and

$$\alpha = 0. \qquad \square$$

Next, let $P_{S_K}: X \to S_K$ be the orthogonal projection operator onto S_K along $S_K^\perp$ and define

$$P = P_{S_K} | F_K$$
$$N = PB_{fK} + K_1'(K_2 B_{fK} - I).$$

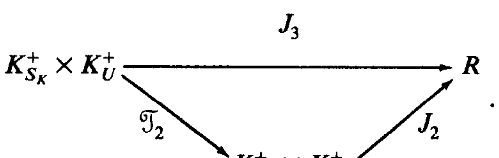
Let

$$\mathfrak{T}_2 : K^+_{S_K} \times K^+_X \to K^+_{S_K} \times K^+_X$$

be given by

$$\mathfrak{T}_2(z, w) = (z, R^{-1}N'z + w).$$

$\mathfrak{T}_2$ is an automorphism. Let J_3 be defined by



Finally, let

$$\mathfrak{N}_3(x_0) = \left(e\left(L_{sK} + B_{sK}R^{-1}N'\right)x_{0sK}, 0\right) \in K^+_{S_K} \times K^+_U$$
$$\Lambda_3 = \{(z, w) | z = e(L_{sK} + B_{sK}R^{-1}N') * B_{sK}w\}.$$

The constraint set $\mathfrak{N}_3(x_0) + \Lambda_3$ corresponds to the system

$$\dot{z} = (L_{sK} + B_{sK}R^{-1}N')z + B_{sK}w, \qquad \text{i.c. } x_{0sK}. \quad (18)$$

A proof similar to that of Proposition 4 shows that

$$\mathfrak{T}_2(\mathfrak{N}_3(x_0) + \Lambda_3) = \mathfrak{N}_2(x_0) + \Lambda_2.$$

We have shown that the problem of minimizing J_3 over the set $\mathfrak{N}_3(x_0) + \Lambda_3$ or, alternatively, with respect to the system (18) is equivalent to the original problem of minimizing J with respect to the descriptor system (2). We now claim that we already know how to solve the transformed problem involving J_3.

Let

$$Q = I + K_1'K_1 - NR^{-1}N'.$$

Proposition 7: Q is positive definite and

$$J_3(z, w) = \int_0^\infty \langle Qz(t), z(t) \rangle + \langle Rw(t), w(t) \rangle \, dt.$$

Proof: A routine calculation gives

$$\begin{aligned} J_3(z, w) &= J_2(z, R^{-1}N'z + w) \\ &= J_1\left(z - B_{fK}R^{-1}N'z - B_{fK}w, R^{-1}N'z + w\right) \\ &= J\left(z - B_{fK}R^{-1}N'z - B_{fK}w, K_1 z \right. \\ &\quad \left. + (I - K_2 B_{fK})(R^{-1}N'z + w)\right) \\ &= \int_0^\infty \|z(t) - B_{fK}R^{-1}N'z(t) - B_{fK}w(t)\|^2 \\ &\quad + \|K_1 z(t) + (I - K_2 B_{fK}) \\ &\quad \cdot (R^{-1}N'z(t) + w(t))\|^2 \, dt \\ &= \int_0^\infty \langle (I + K_1'K_1 - NR^{-1}N')z(t), z(t) \rangle \\ &\quad + \langle Rw(t), w(t) \rangle \, dt. \end{aligned}$$

Since $\sqrt{J_2}$ is a norm on $L^2_{S_K} \times L^2_U$ and $\mathfrak{T}_2$ is an automorphism, $\sqrt{J_3}$ is also a norm so $I + K_1'K_1 - NR^{-1}N'$ must be positive definite. $\square$

Let

$$G = L_{sK} + B_{sK}R^{-1}N'$$

and note that, since (L_{sK}, B_{sK}) is stabilizable, (G, B_{sK}) must also be stabilizable. Hence, the optimal z^*, w^* for the transformed problem is related by

$$w^* = -R^{-1}B'_{sK}\Sigma z^*$$

where Σ is the unique positive definite solution of the (rank E)th order matrix Riccati equation

$$\Sigma G + G'\Sigma - \Sigma B_{sK}R^{-1}B'_{sK}\Sigma + Q = 0.$$

It remains only to map (z^*, w^*) through the transformations $\mathfrak{T}$, $\mathfrak{T}_1$, and $\mathfrak{T}_2$ to obtain (x^*, u^*). We have

$$\begin{aligned} (x^*, u^*) &= \mathfrak{T}(\mathfrak{T}_1(\mathfrak{T}_2(z^*, w^*))) \\ &= \mathfrak{T}\left(\mathfrak{T}_1(z^*, R^{-1}N'z^* + w^*)\right) \\ &= \mathfrak{T}\left(z^* - B_{fK}R^{-1}N'z^* - B_{fK}w^*, R^{-1}N'z^* + w^*\right) \\ &= \left(z^* - B_{fK}R^{-1}N'z^* - B_{fK}w^*, K_1 z^* \right. \\ &\quad \left. + (I - K_2 B_{fK})(R^{-1}N'z^* + w^*)\right) \\ &= \left(z^* - B_{fK}R^{-1}(N' - B'_{sK}\Sigma)z^*, \right. \\ &\quad \left. \left(K_1 + (I - K_2 B_{fK})R^{-1}(N' - B'_{sK}\Sigma)\right)z^*\right). \end{aligned}$$

Thus,

$$x^* = z^* - B_{fK}R^{-1}(N' - B'_{sK}\Sigma)z^*. \quad (19)$$

Let $\Gamma: S_K \to X$ be given by

$$\Gamma\alpha = \alpha - B_{fK}R^{-1}(N' - B'_{sK}\Sigma)\alpha.$$

From (19), $x^*(t)$ must be in $\operatorname{Im}\Gamma$ for all t so

$$z^* = \Gamma^\dagger x^*$$

where $\Gamma^\dagger$ denotes the pseudoinverse. Hence,

$$\begin{aligned} u^* &= \left(K_1 + (I - K_2B_{fK})R^{-1}(N' - B'_{sK}\Sigma)\right)z^* \\ &= \left(K_1 + (I - K_2B_{fK})R^{-1}(N' - B'_{sK}\Sigma)\right)\Gamma^\dagger x^*. \end{aligned}$$

To show that u^* can be implemented simply by applying the feedback gain

$$\Omega = \left(K_1 + (I - K_2B_{fK})R^{-1}(N' - B'_{sK}\Sigma)\right)\Gamma^\dagger$$

to the system (2), define

$$\hat{A} = A + B\Omega$$

and note that x^* is the solution of

$$E\dot{y} = (\hat{A} - B\Omega)y + Bu^*, \qquad \text{i.c. } x_0.$$

Then, from Lemma 1, x^* is the solution of

$$E\dot{x} = \hat{A}x + Bu$$

where $u = -\Omega x^* + u^*$. But $u^* = \Omega x^*$, so $u = 0$ and x^* is the solution of

$$E\dot{x} = (A + B\Omega)x, \qquad \text{i.c. } x_0$$

which is the desired result. This argument also shows that there is precisely one pair $(x, u) \in \mathfrak{N}(x_0) + \Lambda$ satisfying $u = \Omega x$. Note that the feedback matrix Ω is independent of the initial condition.

Conclusions

We have shown that the problem of minimizing the cost functional J as defined in (1) with respect to the descriptor system (2) has a solution if and only if (L_s, B_s) is stabilizable and $\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F$. In this case the solution is unique and can be implemented by applying a linear feedback law independent of the initial condition.

It was seen that the only reasonable cost to assign to trajectories containing impulses is infinity. Hence, inherent in the optimization problem is the problem of eliminating impulses. The solution presented in this paper is thus an energy optimal solution to the general problem, treated in [5], of eliminating impulses with feedback.

In motivating the infinite cost of impulsive trajectories we hinted at the connection between the descriptor regulator problem and the analogous one for singularly perturbed systems. A descriptor system is obviously the limiting case of a singularly perturbed system, but the relationship between the optimal controls, trajectories, and costs for the two problems is not so obvious. This is a topic for further research.

Other topics for future work include the finite time problem and the generalized infinite time problem where the integrand in (1) contains a seminorm. It is our belief that both these problems can be solved by using the above approach in conjunction with slightly different norms on $L_X^2 \times L_U^2$.

Appendix

The results of [5] combine with those of this paper to give a theoretical basis for calculating the optimal feedback gain Ω. In this Appendix we wish to collect all the essential facts, translate them into matrix terminology, and organize them into one concise algorithm.

It is assumed that, faced with the abstract results we have already derived, the reader is capable of writing down the aforementioned algorithm almost by inspection. Hence, we will not justify each step of the algorithm, but will take the correctness of the procedure to be self-evident. Our intention is mainly to save the reader the organizational work involved.

The algorithm we will present represents the most straightforward interpretation of the available abstract results and is not necessarily the most efficient computational approach to the problem. The question of computational efficiency is a topic for further study.

In matrix form, the problem we have considered requires initial data consisting of a matrix triple (E, A, B) where E, A are $n \times n$, B is $n \times m$, and

$$\Delta(s) = \det(Es - A) \not\equiv 0.$$

A pair of positive definite matrices $Q_1(n \times n)$ and $R_1(m \times m)$ are also assumed given, determining the cost functional

$$J(x, u) = \int_0^\infty x(t)'Q_1x(t) + u(t)'R_1u(t)\,dt.$$

Here, the prime indicates the matrix transpose. It is also assumed that (E, A, B) satisfies the existence conditions of Theorem 1.

Steps 1)–6) calculate the canonical decomposition of (E, A, B) as described in [14], [15] and interpreted geometrically in [5].

1) Find the distinct roots $\lambda_1, \cdots, \lambda_p$ of $\Delta(s)$ and choose a single number $\lambda \neq \lambda_i$, $i = 1, \cdots, p$. Let $r = \deg \Delta$.

2) $(\lambda E - A)^{-1}E$ has, as its eigenvalues, 0 and $1/(\lambda - \lambda_i)$, $i = 1, \cdots, p$. Find a complete set of eigenvectors and generalized eigenvectors $\{e_1, \cdots, e_n\}$ of $(\lambda E - A)^{-1}E$ with $e_1, \cdots, e_r$ corresponding to $1/(\lambda - \lambda_i)$, $i = 1, \cdots, p$ and $e_{r+1}, \cdots, e_n$ corresponding to the eigenvalue 0. It is assumed that the vectors are ordered so that e_{r+1} is an eigenvector with $e_{r+2}, \cdots, e_{r+q_1}$ its corresponding chain of generalized eigenvectors. Similarly, e_{r+q_1+1} is an eigenvector with corresponding chain $e_{r+q_1+q_2}$, and so forth up to the last eigenvector $e_{r+q_1+\cdots+q_{d-1}+1}$ with chain $e_{r+q_1+\cdots+q_{d-1}+2}, \cdots, e_{r+q_1+\cdots+q_d} = e_n$.

3) Form the vectors

$$v_{11} = e_{r+1}$$

$$v_{1j} = \sum_{k=0}^{j-2} (-1)^{k+1} \binom{j-2}{k} \lambda^{j-k-2} e_{r+k+2}; \qquad j = 2, \cdots, q_1$$

$$v_{i1} = e_{r+q_1+\cdots+q_{i-1}+1}$$

$$v_{ij} = \sum_{k=0}^{j-2} (-1)^{k+1} \binom{j-2}{k} \lambda^{j-k-2} e_{r+q_1+\cdots+q_{i-1}+k+2};$$

$$j = 2, \cdots, q_i; \; i = 2, \cdots, d.$$

4) Apply the similarity transformation

$$T = \left[e_1 \cdots e_r \mid v_{11} \cdots v_{1q_1} \mid \cdots \mid v_{d1} \cdots v_{dq_d} \right]$$

to $(\lambda E - A)^{-1} E$ yielding

$$T^{-1} (\lambda E - A)^{-1} E T = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

where J_1 is nonsingular with eigenvalues $1/(\lambda - \lambda_i)$ and J_2 is upper triangular and nilpotent.

5) Set

$$M = \begin{bmatrix} J_1^{-1} & 0 \\ 0 & (\lambda J_2 - I)^{-1} \end{bmatrix} T^{-1} (\lambda E - A)^{-1}.$$

Then

$$MET = \begin{bmatrix} I_r & 0 \\ 0 & L_f \end{bmatrix}$$

$$MAT = \begin{bmatrix} L_s & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

where $L_f = (\lambda J_2 - I)^{-1} J_2$ and $L_s = \lambda I - J_1^{-1}$. The vectors v_{ij} were chosen in such a way that L_f is in Jordan form with d blocks of sizes $q_1, \cdots, q_d$. L_s has eigenvalues λ_i, $i = 1, \cdots, p$ and L_f is nilpotent.

6) Calculate $B_s(r \times m)$ and $B_f(n - r \times m)$ from

$$MB = \begin{bmatrix} B_s \\ B_f \end{bmatrix}.$$

At this point the conditions of Theorem 1 may be checked to see if the optimization problem has a solution.

Steps 7)–9) generate a feedback gain K_f that eliminates impulses in the system.

7) Construct the $d \times d$ diagonal matrix

$$D = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_d \end{bmatrix}$$

where

$$\alpha_i = \begin{cases} 1 & \text{if } q_i = 1 \\ 0 & \text{if } q_i > 1 \end{cases}$$

and, after labeling the rows of B_f

$$B_f = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-r} \end{bmatrix}$$

form the $d \times m$ matrix

$$\bar{B} = \begin{bmatrix} b_{q_1} \\ b_{q_1+q_2} \\ \vdots \\ b_{q_1+\cdots+q_d} \end{bmatrix}.$$

8) Find an $m \times d$ matrix H so that $D + \bar{B} H$ is nonsingular. One approach is to apply a pole placement algorithm to the pair $(D, \bar{B})$. The existence condition

$$\operatorname{Im} L_f + \operatorname{Ker} L_f + \operatorname{Im} B_f = F \tag{A.1}$$

also guarantees that the eigenvalue 0 is a controllable mode of $(D, \bar{B})$. A simple alternative approach is provided by the following algorithm.

a) Let ψ be the number of 1's in D. Obtain a sequence of row and column interchanges to put D in the form

$$\begin{bmatrix} I_\psi & 0 \\ 0 & 0 \end{bmatrix}.$$

b) Perform the same *row* operations on $\bar{B}$ yielding

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where B_1 is $\psi \times m$ and B_2 is $d - \psi \times m$.

c) The Assumption (A.1) also guarantees that B_2 has rank $d - \psi$. Thus, we can find an $m \times d - \psi$ matrix H_2 that makes $B_2 H_2$ nonsingular. For example, $H_2 = B_2^\dagger$ is one possibility.

d) Perform the reverse sequence of *column* operations on the $m \times d$ matrix $[0 \quad H_2]$ to obtain H. $D + \bar{B} H$ must then be nonsingular, since

$$\begin{bmatrix} I_\psi & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [0 \quad H_2] = \begin{bmatrix} I_\psi & B_1 H_2 \\ 0 & B_2 H_2 \end{bmatrix}$$

is nonsingular.

9) Form the $m \times n - r$ matrix K_f by letting its first column be equal to the first column of H, its $(1 + q_1)$th column be the second column of H, its $(1 + q_1 + \cdots + q_{i-1})$th column the ith, etc. This determines d columns of K_f. The remaining may be chosen arbitrarily.

10) Repeat the decomposition 1)–6) on the triple

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & L_f \end{bmatrix}, \begin{bmatrix} L_s & B_s K_f \\ 0 & I + B_f K_f \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix} \right)$$

yielding nonsingular matrices T_K, M_K such that

$$M_K\begin{bmatrix} I_r & 0 \\ 0 & L_f \end{bmatrix} T_K = \begin{bmatrix} I_{r_K} & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_K\begin{bmatrix} L_s & B_sK_f \\ 0 & I + B_fK_f \end{bmatrix} T_K = \begin{bmatrix} L_{sK} & 0 \\ 0 & I_{n-r_K} \end{bmatrix}$$

$$M_K\begin{bmatrix} B_s \\ B_f \end{bmatrix} = \begin{bmatrix} B_{sK} \\ B_{fK} \end{bmatrix}$$

where $r_K = \operatorname{rank} E$, L_{sK} is $r_K \times r_K$, B_{sK} is $r_K \times m$, and B_{fK} is $n - r_K \times m$.

Steps 11)–16) compute the optimal feedback matrix Ω, taking into account the various coordinate changes, so that Ω may be applied directly to the original system.

11) Calculate $K_1(m \times r_K)$ and $K_2(m \times n - r_K)$ from

$$\begin{bmatrix} 0 & K_f \end{bmatrix} T_K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.$$

12) To form inner products and adjoint matrices in the transformed coordinates, obtain positive definite matrices $Q_{11}(r_K \times r_K)$ and $Q_{22}(n - r_K \times n - r_K)$ and an $r_K \times n - r_K$ matrix Q_{12} from

$$T_K'T'Q_1TT_K = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}.$$

13) Calculate

$$R = B_{fK}'Q_{22}B_{fK} + (I - K_2B_{fK})'R_1(I - K_2B_{fK})$$
$$N = Q_{12}B_{fK} + K_1'R_1(K_2B_{fK} - I)$$
$$Q = Q_{11} + K_1'R_1K_1 - NR^{-1}N'$$
$$G = L_{sK} + B_{sK}R^{-1}N'.$$

Q and R are positive definite and (G, B_{sK}) is a stabilizable pair. Q, R, and N are not exactly the matrix representations of the transformations Q, R, and N introduced in the body of the paper, but are defined in such a way that simplifies the calculations for the matrix case.

14) Find the unique positive definite solution $\Sigma(r_K \times r_K)$ of

$$\Sigma G + G'\Sigma - \Sigma B_{sK}R^{-1}B_{sK}'\Sigma + Q = 0.$$

15) Set

$$\Gamma = \begin{bmatrix} I_{r_K} \\ -B_{fK}R^{-1}(N' - B_{sK}'\Sigma) \end{bmatrix}$$

and calculate $\Gamma^\dagger$.

16)

$$\Omega = \left(K_1 + (I - K_2B_{fK})R^{-1}(N' - B_{sK}'\Sigma)\right)\Gamma^\dagger T_K^{-1}T^{-1}.$$

Note that when $E = I$ the entire algorithm reduces to that for the state-variable case.

With regard to the example (5), we have

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Suppose

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_1 = 1.$$

Steps 1)–6) need not be carried out for this system since the subspace S is zero-dimensional. That is, $E = L_f$, $A = I$, and $B = B_f$. Steps 7)–9) generate (nonuniquely) the matrix

$$K_f = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Choosing $\lambda = 0$, 10) yields

$$M_k = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_k = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

so the decomposition of the closed-loop system has the form

$$\dot{\bar{x}}_1 = -\bar{x}_1 + v$$
$$0 = \bar{x}_2 + v$$

where $\bar{x} = T_k^{-1}x$. This system has no impulses, but is not optimal.

Steps 11)–16) generate

$$\Sigma = \sqrt{2}, \quad \Gamma = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2} \end{bmatrix}.$$

Applying Ω to (5) then yields the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which has the solution

$$x(t) = e^{-\sqrt{2}t}x_{02}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

where

$$x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

Hence,

$$u(t) = -e^{-\sqrt{2}t}x_{02}$$

and

$$J(x, u) = \sqrt{2}\,x_{02}^2.$$

Note that

$$x(0^+) = x_{02}\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \neq \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$$

if $x_{01} \neq -\sqrt{2}\,x_{02}$. Thus, there may be a jump discontinuity in the optimal system trajectory.

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