Technical Notes and Correspondence

A Further Interpretation of Inconsistent Initial Conditions in Descriptor-Variable Systems

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Abstract —**Initial conditions for systems of the form $E\dot{x} = Ax + Bu$ are discussed. A new interpretation of inconsistent initial conditions is proposed and shown to apply to a larger class of problems than that which has been so far described in the literature.**

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I. Introduction

During the development of the theory of linear systems of the form

$$E\dot{x} = Ax + Bu \tag{1}$$

where $E, A \in \mathbf{R}^{n \times n}$, E is singular, and $B \in \mathbf{R}^{n \times m}$, two main points of view have appeared concerning initial conditions: that of allowing only "consistent" initial conditions [1], [2] and that of attempting to make physical sense out of arbitrary initial conditions [3], [4]. Those who have adopted the former attitude tend to argue that the set of values $x(0)$, as x ranges over all possible C^1 solutions of (1), forms a proper subset of $\mathbf{R}^n$. Therefore, only these values of $x(0)$ make sense in general. Those adopting the latter viewpoint argue in favor of "inconsistent" initial conditions, usually by citing examples of electric circuits which are formed at a certain instant of time with the opening and closing of switches, thus

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allowing any 0^- initial value to be present. Hence, it has been proposed [3] that arbitrary initial conditions have physical meaning for systems which are, in a sense, created or structurally changed at $t=0$.

In this note, we propose an alternative meaning for inconsistent initial values, which makes physical sense with regard to all systems described by (1). *The existence of abrupt structural changes is not a prerequisite.*

II. Background

The standard decomposition of (1) as described in [5] was interpreted geometrically in [6] as follows. Provided that $\det(Es-A)\not\equiv 0$, there exist subspaces

$$S\oplus F=\mathbb{R}^n$$

and subsystems

$$\dot{x}_s=A_s x_s+B_s u \tag{2}$$

$$A_f\dot{x}_f=x_f+B_f u \tag{3}$$

acting on S and F, respectively, with A_f nilpotent. Let $q=$ index of A_f ($A_f^m=0$, $m\geqslant q$; $A_f^m\neq 0$, $0\leqslant m<q$). Then (3) has solution

$$x_f=-\sum_{i=0}^{q-1}A_f^iB_fu^i \tag{4}$$

where u^i denotes the ith derivative. Hence, $x(0)$ is consistent in the sense of [1] if and only if

$$x(0)\in-\sum_{i=0}^{q-1}A_f^iB_fu^i(0)+S.$$

The "natural response" of (3) is invariably taken to be

$$\tilde{x}_f=-\sum_{i=1}^{q-1}\delta^{i-1}A_f^ix_f(0^-). \tag{5}$$

This formula has been arrived at through several different arguments including Laplace transformation [3] and singular perturbations [4]. We will soon demonstrate a further justification for its use.

To motivate the discussion of descriptor systems let us first consider a state-variable system [e.g., (2)] under the influence of a piecewise $q-1$ times continuously differentiable function $u\in C_p^{q-1}$ with bounded support: $\operatorname{supp} u\subset[a,b]$. u may be thought of as either a control input or a disturbance. For simplicity we also assume that $x(a)=0$.

Then, for $t\geqslant b$ we have

$$\begin{aligned}x_s(t)&=\int_a^b e^{(t-\tau)A_s}B_su(\tau)\,d\tau\\&=e^{(t-b)A_s}x_s(b).\end{aligned} \tag{6}$$

One might think of (6) as portraying a sort of duality between forced and natural responses in state-variable systems. Once the input u has disappeared, the system response x_s may be viewed as the forced response due to u or, alternatively, as the natural response due to the "initial condition" $x_s(b)$. Here, the initial state $x_s(b)$ is imposed on the system by the transient input u. This, of course, is a standard interpretation of initial values in state-space systems. In the presence of random disturbances, the value of this viewpoint is that one can deal with the effect of u by considering some initial value $x_s(b)$ without having to address the often difficult task of modeling the disturbance itself.

III. Natural and Forced Response Duality in Descriptor Systems

It is our intention to verify an equation analogous to (6) for descriptor systems. Actually, we need only consider subsystem (3) since (2) has already been discussed. It is clear from (4) and the piecewise smoothness of u that we are necessarily dealing with distributions. Hence, in order to isolate the part of x_f for $t\geqslant b$, we cannot simply restrict x_f to $[b,\infty)$. This is meaningless, for example, if δ-functions are present at $t=b$. We may, however, define a suitable generalization.

Let a distribution f be given and assume that for some $\epsilon>0$, $f=g$ on $(b-\epsilon,b)$ where g is a locally integrable function (see [7]). Then we may formally define the "restriction" $f\|[b,\infty)$ according to

$$\langle f\|[b,\infty),\phi\rangle=\begin{cases}\langle f,\phi\rangle-\int_{b-\epsilon}^b g(t)\phi(t)\,dt & \text{if } \operatorname{supp}\phi\subset[b-\epsilon,\infty)\\ 0, & \text{if } \operatorname{supp}\phi\subset(-\infty,b]\end{cases} \tag{7}$$

where $\langle h,\phi\rangle$ denotes operation by a distribution h acting as a linear functional on the test function ϕ. The definition (7) determines $f\|[b,\infty)$ uniquely since

$$\mathcal{D}_{(-\infty,b]}+\mathcal{D}_{[b-\epsilon,\infty)}=\mathcal{D}_{\mathbf{R}}$$

where $\mathcal{D}_I$ denotes the subspace of test functions with support in an interval I. As an example, observe that

$$(1+\delta)\|[0,\infty)=\theta+\delta$$

where $\theta=$ unit step.

Now, suppose that the transient input u is applied to (3). During the interval $[a,b]$, u drives the system to the state

$$x_f(b^-)=-\sum_{i=0}^{q-1}A_f^iB_fu^i(b^-). \tag{8}$$

Note that $x_f(b^-)$ may be inconsistent with respect to $u\|[b,\infty)=0$.

In order to apply the interpretation of Section II, we must verify that the forced response due to u and the natural response (as defined by [5]) due to $x_f(b^-)$ are equal.

Theorem:

$$\begin{aligned}x_f\|[b,\infty)&=-\sum_{i=0}^{q-1}A_f^iB_fu^i\|[b,\infty)\\&=-\sum_{i=1}^{q-1}\delta^{i-1}A_f^ix_f(b^-).\end{aligned}$$

Proof: It is easily shown that

$$u\|[b,\infty)=0$$

and

$$u^i\|[b,\infty)=-\sum_{j=0}^{i-1}\delta^{i-j-1}u^j(b^-),\qquad i>0.$$

From (4) and (7),

$$\begin{aligned}x_f\|[b,\infty)&=-\sum_{i=0}^{q-1}A_f^iB_fu^i\|[b,\infty)\\&=\sum_{i=1}^{q-1}\sum_{j=0}^{i-1}\delta^{i-j-1}A_f^iB_fu^j(b^-).\end{aligned}$$

On the other hand, from (8),

$$\begin{aligned}-\sum_{i=1}^{q-1}\delta^{i-1}A_f^ix_f(b^-)&=\sum_{i=1}^{q-1}\sum_{j=0}^{q-1}\delta^{i-1}A_f^{i+j}B_fu^j(b^-)\\&=\sum_{i=1}^{q-1}\sum_{j=0}^{q-i-1}\delta^{i-1}A_f^{i+j}B_fu^j(b^-)\\&=\sum_{k=1}^{q-1}\sum_{j=0}^{k-1}\delta^{k-j-1}A_f^kB_fu^j(b^-)\end{aligned}$$

where $k=i+j$. □

The preceding theorem establishes an equation of duality, analogous to (6), for descriptor systems. Since each $u^i(b^-)$ may take on any value, all initial conditions $x_f(b^-)$ are possible. Hence, any descriptor system may have inconsistent initial conditions in this sense. As with state-variable systems, such a viewpoint is useful in studying the effects of disturbances without having to actually model the disturbances. Examples of such an application include [6] and [8], where feedback compensators were derived to eliminate impulsive behavior due to inconsistent initial conditions in (1), and to regulate the system. This approach to dealing with disturbances in systems is, of course, precisely the same as that which was used by numerous researchers in developing the standard deterministic theory of state regulation.

As a final observation concerning the effect of the input u, note that if u is an arbitrary C_p^{q-1} function with possibly unbounded support, linearity of (4) in u indicates that a superposition principle may be invoked to separate the effects of $u|(-\infty, b)$ and $u|[b,\infty)$. The response due to $u|[b,\infty)$ is the forced response given by (4), and that due to $u|(-\infty, b)$ may be interpreted according to the theorem, as a natural response. Once again, the value of $x(b^-)$ imposed on the system by $u|(-\infty, b)$ may be inconsistent with respect to $u|[b,\infty)$.

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