Global Analyticity of a Geometric Decomposition for Linear Singularly Perturbed Systems*

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Abstract. A geometric decomposition is developed for linear, time-invariant singularly perturbed systems of a general form. The decomposition is shown to be determined by a mapping d between two real analytic manifolds, the range of d being a manifold of canonical forms. Our main result establishes analyticity of d over its entire domain.

1. Introduction

We are interested in structural properties of autonomous singularly perturbed systems with external control:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, u) \\ \epsilon\dot{x}_2 &= f_2(x_1, x_2, u) \end{aligned} \tag{1}$$

Here, f_1 and f_2 are vector-valued. The literature dealing with such systems is vast (e.g. [1]-[4]) even when attention is restricted to the case of linear f_1 and f_2. In this case, (1) may be rewritten

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u \\ \epsilon\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2u \end{aligned} \tag{2}$$

where the A's and B's are matrices.

Recently, a great deal of interest has been generated, especially in the field of control theory, by the "singular" system formed by setting $\epsilon = 0$ in (1) or (2) ([5] - [9]). In the linear case, much of this work has been based on the Weierstrass decomposition for regular pencils as presented in Gantmacher [10]. The decomposition involves a nonsingular coordinate change applied

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to **(2)** with $\epsilon = 0$, yielding two subsystems

$$\dot{x}_s = A_s x_s + B_s u \tag{3a}$$

$$A_f \dot{x}_f = x_f + B_f u \tag{3b}$$

where A_f is nilpotent. The subscripts s and f refer to the singular perturbation terms "slow" and "fast." As demonstrated in **[10]**, subsystem **(3b)** contains all the "singular" behavior inherent in **(3)**.

A natural question to ask is whether the Weiersrass decomposition can be extended to **(2)** or even **(1)**. In other words, **can an invertible coordinate change depending smoothly on ϵ be found such that, when applied to a singularly perturbed system, a complete decomposition results with one subsystem containing all regular behavior and the other containing all singular behavior and with all matrices depending smoothly on ϵ?** A complete answer to this is not known even in a local sense for the general nonlinear case **(1)**. However, an affirmative answer has been obtained for the linear system **(2)**. The problem was first posed in **[11]** where we showed that a parameterized system of the form

$$E(\epsilon)\dot{x} = A(\epsilon) + B(\epsilon)u \tag{4}$$

where the matrices E, A, and B are continuous in ϵ at a given point ϵ_0, can be decomposed as in **(3)** such that A_s, A_f, B_s, and B_f all depend on ϵ and are continuous at ϵ_0. Later, in **[12]** it was further shown that analyticity in **(4)** with respect to ϵ implies analyticity in **(3)**. One of the goals of this paper is to show that any degree of smoothness carries over from **(4)** into **(3)**. (This will be seen to follow almost immediately from analyticity.)

The main issue we will address is that of the coordinate-free or geometric setting for the decompositon of **(4)**. Our results in **[11]** were presented in geometric terms while those of **[12]** were not. It is our aim to develop a coordinate-free decomposition of **(4)** analogous to that of **[12]**, thus extending the results of **[11]**. An advantage of the geometric setting is that we will be able to prove uniqueness of the decompositon—a result which is untrue in the coordinate-dependent context (see **[10]**).

2. Preliminaries

All Euclidean spaces considered are assumed to be real inner product spaces with norms defined by $\|x\|^2 = \langle x, x \rangle$. The class of linear systems under investigation consists of all those of the form

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5}$$

where E, $A : R^n \to R^n$, $B : \mathbf{R}^m \to \mathbf{R}^n$, $C : \mathbf{R}^n \to \mathbf{R}^p$ are linear with

$$\det(sE - A) \not\equiv 0 \tag{6}$$

Condition **(6)** is necessary and sufficient for existence and uniqueness of solutions in **(5)** and determines the complement of an algebraic variety. Thus we are actually dealing with the set of all (E, A, B, C) in an open, dense subset of $R^{n(2n+m+p)}$.

Let $0 \le r \le n$ and $\Gamma \subset C$ be a rectifiable Jordan curve encircling the origin. To construct a smooth decomposition of **(5)** we need to further restrict attention to

$$\Sigma = \{(E, A, B, C) \in \mathbf{R}^{n(2n+m+p)} \mid \det(sE - A) \text{ has exactly } r \text{ roots (counting multiplicities) encircled by } \Gamma\}$$

Since Γ encloses an open set and the roots of $\det(sE - A)$ are continuous functions of E and A (see **[11]**), Σ is open. Hence Σ is a real analytic manifold of dimension $n(2n + m + p)$. Our aim is to extend the decomposition **(3)** smoothly over Σ by letting Γ determine an eigenspace decomposition of R^n. But first we need some additional constructions.

3. A manifold of canonical forms

In this section we develop an analytic manifold, each point corresponding to a decomposed form **(3)**. The differentiable structure of the manifold will determine smooth variation of decomposing subspaces $S \oplus F = \mathbf{R}^n$ and the operators A_s, A_f, B_s, ...

As a first step, let $0 \le k \le n$ and

$$\Delta_k = \{(W, X, Y, Z) \mid W \in \mathbf{R}^{nk}, X \in \mathbf{R}^{k^2}, Y \in \mathbf{R}^{km}, z \in \mathbf{R}^{pk}, W \text{ has full rank}\}$$

Any nonsingular $T \in \mathbf{R}^{k^2}$ determines a free group action on Δ_k given by

$$(W, X, Y, Z) \to (WT,\ T^{-1}XT,\ T^{-1}Y,\ ZT)$$

This in turn yields a quotient set $\bar{\Delta}_k$.

Proposition 1. *$\bar{\Delta}_k$ is a real analytic manifold of dimension $k(n + m + p)$.*

Proof. Our approach is to generalize the standard Grassman manifold development (see **[14]**). Choose $(W, X, Y, Z) \in \Delta_k$. W has k linearly independent rows $e_1, \dots, e_k$ so we may define

$$T = \begin{bmatrix} e_1 \\ \cdot \\ \cdot \\ \cdot \\ e_k \end{bmatrix}^{-1}$$

T then takes (W, X, Y, Z) into a representative of the equivalence class $[W, X, Y, Z]$. A chart may be defined accordingly onto an open subset of $\mathbf{R}^{k(n+m+p)}$. For example, if $e_1, \ldots, e_k$ are the first k rows of W, we have the representative

$$(WT, T^{-1}XT, T^{-1}Y, ZT) = \left(\begin{bmatrix} I \\ \hat{W} \end{bmatrix}, \hat{X}, \hat{Y}, \hat{Z} \right)$$

where $\hat{W}, \hat{X}, \hat{Y}$, and $\hat{Z}$ may take on any values. This defines a chart $\phi_1 : U_1 \to \mathbf{R}^{k(n+m+p)}$ according to

$$[W, X, Y, Z] \to (\hat{W}, \hat{X}, \hat{Y}, \hat{Z})$$

where U_1 consists of all points in Δ_k where $e_1, \ldots, e_k$ are linearly independent. By selecting different sets of k rows, domains U_i are generated which cover $\bar{\Delta}_k$. Any two charts ϕ_i and ϕ_j determine a change of coordinates $\phi_j^{-1} \circ \phi_i$ on $U_i \cap V_j$ which is a rational function and hence analytic.

To complete the construction, consider the set of all 8-tuples $\eta = (S, A_s, B_s, C_s, F, A_f, B_f, C_f)$ where $S, F \subset \mathbf{R}^n$ are subspaces with dimensions r and $n-r$, respectively, and $A_s : S \to S$, $B_s : \mathbf{R}^m \to S$, $C_s : S \to \mathbf{R}^p$, $A_f : F \to F$, $B_f : \mathbf{R}^m \to F$, and $C_f : F \to \mathbf{R}^p$ are linear. Let

$$\mathcal{C} = \{\eta \mid S \cap F = 0\},$$

Proposition 2. *C is an open, dense submanifold of* $\bar{\Delta}_r \times \bar{\Delta}_{n-r}$ *with dimension* $(n+m+p)$.

Proof. Each equivalence class $[W, X, Y, Z] \in \bar{\Delta}_k$ determines uniquely a k-dimensional subspace $\operatorname{Im} W \subset \mathbf{R}^n$. All K-dimensional subspaces are obtained in this way. For each representative (W, X, Y, Z) the columns of W are basis vectors for $\operatorname{Im} W$ and X, Y, and Z are matrix representations of some linear operators A, B, and C with respect to those vectors and some fixed bases in $\mathbf{R}^m$ and $\mathbf{R}^p$. We need to show that any other representative $(\hat{W}, \hat{X}, \hat{Y}, \hat{Z})$ of the same equivalence class gives rise to the same operators; then $\mathcal{C}$ can be naturally embedded into $\bar{\Delta}_r \times \bar{\Delta}_{n-r}$. To do this, note that any two representatives determine a nonsingular T with $\hat{W} = WT$. Then, with respect to the columns of $\hat{W}$, A has matrix representation $T^{-1}XT = \hat{X}$, B has matrix $T^{-1}Y = \hat{Y}$, and C has $ZT = \hat{Z}$.

From the construction of $\bar{\Delta}_k$ it is clear that $\bar{\Delta}_r \times \bar{\Delta}_{n-r}$ inherits quotient set topology from $\Delta_r \times \Delta_{n-r}$ in which

$$\Lambda = \{(\eta,\tilde{\eta}) \in \Delta_r \times \Delta_{n-r} \mid \det[W\ \tilde{W}] \neq 0\}$$

is open. Hence, Λ consists precisely of those points where $\operatorname{Im} W \cap \operatorname{Im} \tilde{W} = 0$. Thus $\mathcal{C}$ is open. Furthermore, $\mathcal{C}$ has the same dimension as $\Delta_r \times \Delta_{n-r}$ which equals

$$r(n+m+p) + (n-r)(n+m+p) = n(n+m+p)$$

Finally, density of $\mathcal{C}$ follows immediately from density of Λ in $\Delta_r \times \Delta_{n-r}$.

As pointed out in the proof of Proposition 2, each $(W,X,Y,Z) \in \Delta_k$ uniquely determines a subspace and three operators acting on that subspace. Furthermore, any two points in Δ_k are equivalent if and only if they represent the same subspaces and operator triples. Thus, for example, $\mathcal{C}$ distinguishes state-space systems $(\mathbf{R}^n, A_i, B_i, C_i, 0, 0, 0, 0)$; $i = 1, 2$ as long as $(A_1, B_1, C_1) \neq (A_2, B_2, C_2)$.

$\mathcal{C}$ may be interpreted as a manifold of canonical forms. Each point $\eta = (S, A_s, \ldots, C_f)$ determines a subspace decomposition $S \oplus F = \mathbf{R}^n$ and a corresponding system (3). Thus each $\eta \in \mathcal{C}$ represents points in Σ in a natural way. In section 4 we will show that this correspondance is analytic; but first we offer a result which gives an alternative description of the topology of $\mathcal{C}$.

Proposition 3. *$\mathcal{C}$ is metrizable.*

Proof. We base our construction on the "opening" metric on the Grassman manifold $G_k(\mathbf{R}^n)$ (see [13]): For k-dimensional subspaces $R,S \subset \mathbf{R}^n$ define

$$\varrho_0^k(R,S) = \| P_R - P_S \|$$
$$= \max \left\{ \sup_{\substack{x\in R\\ \|x\|=1}} \inf_{y\in S} \|x-y\|,\ \sup_{\substack{y\in R\\ \|y\|=1}} \inf_{x\in S} \|x-y\| \right\}$$

where P_R and P_S are orthogonal projection operators. For any subspaces $K \subset \mathbf{R}^j$, $L \subset \mathbf{R}^k$ and linear $F: K \to L$ we also need to consider the minimum norm extension

$$\mu(F) : \mathbf{R}^j \to \mathbf{R}^k$$

defined by

$$\mu(F)x = \begin{cases} Fx \text{ if } x \in K \\ 0 \text{ if } x \in K^{\perp} \end{cases}$$

It is easy to verify that

$$\varrho((S_1, A_{s1}, B_{s1}, \ldots, C_{f1}), (S_2, \ldots, C_{f2}))$$

$$= \varrho_0^r(S_1, S_2) = \varrho_0^{n-r}(F_1, F_2) = \|\mu(A_{s1}) - \mu(A_{s2})\|$$

$$+ \|\mu(A_{f1}) - \mu(A_{f2})\| + \cdots + \|\mu(C_{f1}) = \mu(C_{f2})\|$$

defines a metric on $\mathcal{C}$.

It remains to show that ϱ induces manifold topology on $\mathcal{C}$. Consider any sequence $\eta_j \to \eta = (S, A_s, \ldots, C_f)$ in manifold topology. Then corresponding to the η_j is a convergent sequence of representatives $S(W_j, X_j, Y_j, Z_j)$. Let

$$[e_{1j} \cdots e_{rj}] = W_j$$

For each $e \in \mathbf{R}^n$

$$P_{S_j}e = \gamma_{1j}e_{1j} + \cdots + \gamma_{rj}e_{rj}$$

where the γ_{ij} are defined by

$$\begin{bmatrix} <e_{1j}, e_{1j}> & \cdots & <e_{1j}, e_{rj}> \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ <e_{rj}, e_{1j}> & \cdots & <e_{rj}, e_{rj}> \end{bmatrix} \begin{bmatrix} \gamma_{1j} \\ \\ \\ \\ \gamma_{rj} \end{bmatrix} = \begin{bmatrix} <e, e_{1j}> \\ \cdot \\ \cdot \\ \cdot \\ <e, e_{rj}> \end{bmatrix}$$

The Gram matrix converges and is nonsingular; hence, $\gamma_{1j}, \ldots, \gamma_{rj}$ converge and $P_{S_j} \to P_S$. Similarly, $P_{F_j} \to P_F$.

To prove convergence of $\mu(A_{sj})$ note that $x_j \to x$ and let

$$\begin{bmatrix} \alpha_{1j}^1 & \cdots & \alpha_{1j}^r \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \alpha_{rj}^1 & \cdots & \alpha_{rj}^r \end{bmatrix} = X_j$$

If $(v_1, \ldots, v_{n-r})$ is a given basis for $S^\perp$ and

$$v_{ij} = P_{S_j^\perp} v_i = (I - P_{S_j})v_i$$

then, $v_{ij} \to v_i$ and, for sufficiently large j, $(v_{1j}, \ldots, v_{n-r,j})$ is a basis for $S_j^\perp$. Thus, for any $e \in \mathbf{R}^n$

$$e = \beta_{1j}e_{1j} + \cdots + \beta_{rj}e_{rj} + \beta_{r+1,j}v_{1j} + \cdots + \beta_{nj}v_{n-r,j}$$

with $\beta_{ij} \to \beta_i$ for every i. Hence

$$\begin{aligned}\mu(A_{s_j})e &= \sum_{i=1}^{r} \beta_{ij} A_{sj} e_{ij} \\ &= \sum_{i=1}^{r} \sum_{k=1}^{r} \beta_{ij} \alpha_{kj}^{i} e_{kj} \\ &\to \sum_{i=1}^{r} \sum_{k=1}^{r} \beta_i \alpha_k^i e_k \\ &= \mu(A_s)e\end{aligned}$$

Treating B_{si}, C_{si}, ..., C_{fi} similarly yields $\varrho(\eta_j, \eta) \to 0$.

Conversely, suppose $\varrho(\eta_j, \eta) \to 0$ and let $(e_1, \ldots, e_r)$ be a basis for S. Then $P_{S_j} \to P_S$ implies that S_j has a basis given by

$$e_{ij} = P_{S_j} e_i \to e_i$$

Let $W_j = [e_{1j} \ldots e_{rj}]$. We also have $\mu(A_{sj}) \to \mu(A_s)$ so if we let

$$X_j = \begin{bmatrix} \alpha_{1j}^1 & \cdots & \alpha_{1j}^r \\ \vdots & & \vdots \\ \alpha_{rj}^1 & \cdots & \alpha_{rj}^r \end{bmatrix}$$

and observe that

$$\begin{aligned}\alpha_{1j}^i e_{1j} + \cdots + \alpha_{rj}^i e_{rj} &= \mu(A_{si}) e_{ij} \\ &\to \mu(A_s) e_i\end{aligned}$$

it follows that $X_j \to X$. Convergent Y_j, Z_j,... can be defined similarly. Thus η_j has a convergent representative sequence in $\Delta_r \times \Delta_{n-r}$. Since manifold topology is just quotient set topology, $\eta_j \to \eta$.

As a final comment concerning the topology of $\mathcal{C}$, we note that small variations on $\mathcal{C}$ correspond to small variations in the matrix representations of A_s, A_f, ... with respect to an appropriate basis. Indeed, let Π be a manifold and select a parametrization $\Pi \to \mathcal{C}$, both with some given degree of smoothness. Then, from the construction in the proofs of Propositions 1 and 2, on a sufficiently small neighborhood in $\mathcal{C}$ we can choose bases of S and F for each point in such a way that small variations in Π correspond to small variations in those basis vectors. Furthermore, the resulting matrix representations of A_s, A_f, ... also undergo only small changes.

We have, therefore, three different characterizations of the topology of $\mathcal{C}$: We may alternatively view it as manifold topology, a metric topology, or a quotient set topology on the space of bases and matrix representations. These same points of view carry over to the decomposed system **(3)**.

4. Main decomposition results

We are now ready to establish the coordinate-free analytic decomposition theorem for a parametrized system. Actually, we leave until the next section the task of showing that all degrees of smoothness carry over from a system into its decomposed form; for now we will be content with identifying the parameter space Π with the system space Σ and taking the parametrization $\psi: \Pi \to \Sigma$ to be the identity map. The main theorem describes an analytic map $d: \Sigma \to \mathcal{C}$ which takes each system into a form **(3)**. Analyticity of d guarantees that the parameterized decomposition $d \circ \psi$ is also analytic.

We need one preliminary result.

Lemma. *If* $det\,(sE-A) = \phi \prod_{i=1}^{k} (s-\lambda_i)$ *and* $\lambda \in \mathbf{C}$ *is such that* $\lambda E - A$ *is non-singular, then*

$$det(sI - (\lambda E-A)^{-1}E) = s^{n-k} \prod_{i=1}^{k} (s - \frac{1}{\lambda-\lambda_i}).$$

Proof. For $s=0$ and E singular the result is obvious. For $s=0$ and E non-singular, $k=n$ and

$$\begin{aligned} \det(-(\lambda E-A\ )^{-1}E) &= (-1)^n \det(\lambda I-E^{-1}A)^{-1} \\ &= (-1)^n \prod_{i=1}^{n} \frac{1}{\lambda-\lambda_i} \end{aligned}$$

since $\det(sE-A) = \det E \det(sI-E^{-1}A)$.

Finally, for $s \neq 0$

$$\begin{aligned} \det(sI-(\lambda E-A)^{-1}E) &= s^n \frac{det((\lambda E-A) - \frac{1}{s}E)}{\det(\lambda E-A)} \\ &= s^n \frac{\det((\lambda - \frac{1}{s})E-A)}{\det(\lambda E-A)} \end{aligned}$$

$$= s^{n} \frac{\phi \prod_{i=1}^{k} (\lambda - \frac{1}{s} - \lambda_i)}{\phi \prod_{i=1}^{k} (\lambda - \lambda_i)}$$

$$= s^{n-k} \prod_{i=1}^{k} \frac{(\lambda - \lambda_i)s - 1}{\lambda - \lambda_i}$$

Decomposition Theorem.

(1) *There exist unique maps* d: $\Sigma \to \mathcal{C}$ *and* M: $\Sigma \to \mathbf{R}^{n^2}$ *such that for each* $\xi = (E,A,B,C) \in \Sigma$.

(a) *S and F are both* $M(\xi)E-$ *and* $M(\xi)A-$*invariant*

(b) $M(\xi)E \mid S = I$

(c) $M(\xi)A \mid F = I$

(d) $M(\xi)A \mid S = A_s$

(e) $M(\xi)E \mid F = A_f$

(f) *the eignevalues of* A_s *are encircled by* Γ

(g) $P_{SF}M(\xi)B = B_s$

(h) $P_{FS}M(\xi)B = B_f$

(i) $C \mid S = C_s$

(j) $C \mid F = C_f$

where $d(\xi) = (S, A_s, B_s, C_s, F, A_f, B_f, C_f)$.

(2) *As defined in* (1), *d and M are analytic and for each* $\xi \in \Sigma$

(a) $M(\xi)$ *is nonsingular*

(b) $\det M(\xi) \det(sE-A) = \det(sI-A_s)\det(A_f s-I)$

(c) *if* $\det(sE-A)$ *has degree r,* A_f *is nilpotent.*

Proof. (1) Suppose that $\det(sE-A)$ has k roots $\lambda_1, \ldots, \lambda_k$ where $k \geq r$ and let the roots be indexed so that $\lambda_1, \ldots, \lambda_r$ are encircled by Γ. Define

$$S = \operatorname{Ker} \prod_{i=1}^{r} ((\lambda E-A)^{-1}E - \frac{1}{\lambda-\lambda_i} I)$$

$$F = \operatorname{Ker}((\lambda E-A)^{-1}E)^{n-k} \prod_{i=r+1}^{k} ((\lambda E-A)^{-1}E - \frac{1}{\lambda-\lambda_i} I)$$

where $\lambda \in \Gamma$.

Then $S \oplus F = \mathbf{R}^n$ and both subspaces are $(\lambda E - A)^{-1}E-$ invariant.
In order to construct $M(\xi)$, let

$$J_1 = (\lambda E - A)^{-1}E \mid S$$

$$J_2 = (\lambda E - A)^{-1}E \mid F$$

Then

$$\det(sI - J_1) = \prod_{i=1}^{r} (s - \frac{1}{\lambda - \lambda_i})$$

$$\det(sI - J_2) = s^{n-k} \prod_{i=r+1}^{k} (s - \frac{1}{\lambda - \lambda_i})$$

Hence J_1 is invertible and $\lambda J_2 - I$ has $n-k$ eigenvalues at negative unity and k eigenvalues at

$$\lambda\left(\frac{1}{\lambda - \lambda_i}\right) - 1 = \frac{\lambda_i}{\lambda - \lambda_i}$$

Since λ_i lies outside Γ for $i = r+1, \ldots, k$ and Γ encircles the origin, $\lambda J_2 - I$ is also invertible. Define a linear transformation $\bar{M}$ on $\mathbf{R}^n$ by

$$\bar{M}x = \begin{cases} J_1^{-1}x & \text{if } x \in S \\ (\lambda J_2 - I)^{-1}x & \text{if } x \in F \end{cases}$$

and let $M(\xi) = \bar{M}(\lambda E - A)^{-1}$. From the construction and the fact that

$$(\lambda E - A)^{-1}A = \lambda(\lambda E - A)^{-1}E - I$$

we have $M(\xi)E-$ and $M(\xi)A-$invariance of S and F with

$$M(\xi)E \mid S = \bar{M}(\lambda E - A)^{-1}E \mid S = I$$

anu

$$M(\xi)A \mid F = \bar{M}(\lambda E - A)^{-1}A \mid F = I.$$

If we simply define A_s and A_f as in (*d*) and (*e*), (*f*) follows, since $A_s = J_1^{-1}(\lambda J_1 - I)$ has eigenvalues

$$(\lambda - \lambda_i)(\frac{1}{\lambda - \lambda_i} - 1) = \lambda_i; \quad i = 1, \ldots, r$$

We may also define B_s, B_f, C_s, and C_f as in (g) – (j).

To prove uniqueness, suppose $M(\xi)$, S, and F are given satisfying (a)–(f). Then S and F are $(\lambda E - A)^{-1}E = (\lambda M(\xi)E - M(\xi)A)^{-1}M(\xi)E$ – invariant. Further, suppose

$$S \neq \tilde{S} = \operatorname{Ker} \prod_{i=1}^{r} ((\lambda E - A)^{-1}E - \frac{1}{\lambda - \lambda_i} I)$$

S and $\tilde{S}$ must have the same dimension so there exists $x \in S$ whose minimal polynomial with respect to $(\lambda E - A)^{-1}$ does not divide $\Pi_{i=1}^{r}\ (s - \frac{1}{\lambda - \lambda_i})$ (see [10]). But, from (b) and (d),

$$\det(sI - (\lambda E - A)^{-1}E \mid \tilde{S}) = \det(sI - (\lambda I - A_s)^{-1}) = \prod_{i=1}^{r} (s - \frac{1}{\lambda - \lambda_i})$$

which yields a contradiction. A similar contradiction can be derived by assuming

$$F \neq \operatorname{Ker}((\lambda E - A)^{-1}E)^{n-k} \prod_{i=r+1}^{k} ((\lambda E - A)^{-1}E - \frac{1}{\lambda - \lambda_i} I)$$

To see uniqueness of $M(\xi)$, observe that

$$(\lambda E - A)^{-1}ES = S$$

and

$$(\lambda E - A)^{-1}AF = F$$

so

$$ES \oplus AF = \mathbf{R}^n$$

and (b) and (c) determine $M(\xi)$. Uniqueness of A_s, ..., C_f then follows directly from (d), (e), and (g)–(j).

(2) Nonsingularity of $M(\xi)$ follows immediately from its definition. Part (b) is obvious from (1b)–(1e). If $k = r$ then J_2 is nilpotent and so is

$$A_f = (\lambda J_2 - I)^{-1} J_2$$

It remains to show analyticity of d and M. Since $\lambda_1, \ldots, \lambda_r$ are encircled by Γ and $\lambda_{r+1}, \ldots, \lambda_k$ are not,

$$\begin{aligned} S &= \operatorname{Im} \oint_\Gamma (sI - A_s)^{-1} ds \oplus \operatorname{Im} \oint_\Gamma (sA_f - I)^{-1} ds \\ &= \operatorname{Im} \oint_\Gamma (sME - MA)^{-1} ds \\ &= \operatorname{Im} \oint_\Gamma (sE - A)^{-1} ds \end{aligned}$$

Let

$$W = \oint_\Gamma (sE - A)^{-1} ds$$

The columns of W are analytic functions on all of Γ. If we let

$$\tilde{W} = \oint_{1/\Gamma} (sA - E)^{-1} ds$$

where $1/\Gamma$ consists of all points reciprocal to those in Γ, we have $F = \operatorname{Im} \tilde{W}$ and the columns of $\tilde{W}$ are analytic on Σ. Let

$$N = [W \tilde{W}],$$

From (1b) and (1c), the first r columns of $M^{-1}N$ are just those of EN and the last $n - r$ are those of AN. Hence M^{-1} and, therefore, M are analytic on Σ. Defining X to be the matrix representation of A_s with respect to the columns of W, we have

$$N^{-1}MAN = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}$$

so X is analytic on Σ. $Y, Z, \ldots$ can be handled similarly. Since $(W, X, Y, \ldots)$ is analytic on Σ, so is its induced map d into the quotient set $\mathcal{C}$.

5. Parametrizations

We can now show that our results yield invariance with respect to any degree of smoothness. If Γ and r are chosen, Σ is well defined and we may select a parameter manifold Π and a map $\psi: \Pi \to \Sigma$, both with some given degree of smoothness. ψ determines a parameterized system and $d \circ \psi$ its decomposed form. Since d is analytic, $d \circ \psi$ must have the same degree of smoothness as ψ. For example, let $\omega_0 \in \Pi$ be such that $\det(sE(\omega_0) - A(\omega_0))$ has exactly r roots, all encircled by Γ, where

$$\psi(\omega_0) = (E(\omega_0), A(\omega_0), B(\omega_0), C(\omega_0))$$

Similarly, for each $\omega \in \Pi$ we have a system

$$\psi(\omega): \quad \begin{aligned} E(\omega)\dot{x} &= A(\omega)x + B(\omega)u \\ y &= C(\omega)x \end{aligned}$$

where $\psi(\omega)$ has order at least r. The decomposition theorem gives the unique canonical form

$$d(\psi(\omega)): \quad \begin{aligned} \dot{x}_s &= A_s(\omega)x_s + B_s(\omega)u \\ A_f(\omega)\dot{x}_f &= x_f + B_f(\omega)u \\ y &= C_s(\omega)x_s + C_f(\omega)x_f \end{aligned}$$

corresponding to the subspace decomposition

$$S(\omega) \oplus F(\omega) = \boldsymbol{R}^n$$

Note that according to part (2c) of the theorem, $A_f(\omega_0)$ is nilpotent.

All maps associated with $d(\psi(\omega))$ must have the same degree of smoothness as do E, A, B, and C. In matrix terms, if any bases are chosen for $S(\omega)$ and $F(\omega)$, depending smoothly on ω, the matrix representations of $A_s(\omega)$, $A_f(\omega)$, ... must also vary smoothly with ω.

6. Conclusions

We have shown that existing smooth decomposition theorems for linear singularly preturbed systems can be generalized to account for all degrees of smoothness. This has been accomplished in a coordinate-free context. One advantage of the approach is that we have proven uniqueness of the canonical form – a result hitherto unestablished.

In a future publication we intend to show that by identifying "equivalent" system in Σ, a quotient manifold results which is diffeomorphic to the image of d. Hence, the space of canonical forms may be arrived at by two equivalent methods.

It is hoped that our work in the area will eventually lead to more insight into nonlinear singular perturbation problems.

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