Optimal Design of Overmoded Waveguide Tapers Using the Pontryagin Minimum Principle With Application to TE_{0n} Circular Waveguide Taper Designs

J. Shafii, J. D. Cobb, and R. J. Vernon
Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Engineering Drive, Madison, WI 53706

Twentieth International Conference on Infrared and Millimeter Waves

Orlando, Florida

December 11-14, 1995

Optimal Design of Overmoded Waveguide Tapers Using the Pontryagin Minimum Principle With Application to TE_{0n} Circular Waveguide Taper Designs

J. Shafii, J. D. Cobb and R. J. Vernon
Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Engineering Drive, Madison, WI 53706

Abstract

Oversized waveguide tapers are frequently used to gradually change the waveguide cross section in transmission systems for microwave and millimeter-wave sources. In this paper we introduce the use of a new methodology, namely the Pontryagin Minimum Principle, for optimal design of radius profiles of such overmoded tapers. We discuss the existence of optimal solutions. We present an optimal taper design for the case of a TE_{02} mode in a circular waveguide at 60 GHz.

The Pontryagin Minimum Principle

The Minimum Principle [1], [2] is a generalization of the calculus of variations, and is used in optimal control theory for optimal-system designs. In this context, the coupled mode equations describing the interaction of modes in a varying-radius waveguide taper [3] can be viewed as state equations of our device. In our search for the optimal solution, we can use the necessary conditions required by the Minimum Principle to isolate the admissible solutions that are candidates for optimality. The Minimum Principle is somewhat similar to the condition that the derivative vanishes at the local minima of an ordinary function. The Minimum Principle reduces the optimal design problem to the solutions of a system of differential equations with two-point boundary conditions.

Optimal Taper Design Problem

Our goal is to design a tapered overmoded circular waveguide with a given initial radius a_1, final radius a_2, and length l excited by a single mode. We assume that the waveguide is straight at both the input and output, i.e., $a'(z{=}0) = a'(z{=}l) = 0$, where $a(z)$ is the radius profile of the taper and $a'(z)$ is its first derivative respect to z. Here, z is the coordinate axis along the waveguide. We neglect the backward coupled modes, and consider the optimal design for the case where modes involved are all above cutoff throughout the taper. We denote by $\phi(z)$ the total power in the spurious modes. Our optimal problem is then to obtain the radius profile $a(z)$, from the class of functions with piecewise continuous second derivatives, that minimizes the total power in the spurious modes at the output, i.e., the radius profile which minimizes the cost function, J, given by

$$J = \phi(l) \ . \qquad (1)$$

Discussion of the Optimal Solution

We have shown that, for the above stated problem there does not exist a minimizing solution that satisfies all the necessary conditions that are required by the Minimum Principle. The problem above is in the category of singular optimization problems. This is because the necessary condition that the Hamiltonian function is minimized along the optimal trajectory (optimal $a(z)$) does not provide us with a well defined expression for the optimal control function, u(z). (We have chosen $u(z) = a''(z)$, the second derivative of the radius function which is related to the axial curvature of the waveguide radius, to be our control function.) On more intuitive ground, the above optimal design problem is a singular optimization problem because no upper bound has been placed on the size of the guide radius. Hence, the waveguide radius could become infinitely large.

However, we can modify the design problem such that it is a regular optimization problem. To this end, we choose the cost function as

$$J = \phi(l) + \varepsilon \int_0^l \left(a''(z) \right)^2 dz \qquad (2)$$

where ε is a fixed positive small parameter. So our problem now is to obtain a profile $a(z)$ that, for a given ε, minimizes the above cost function. The integral term above can be interpreted to be related to the total integrated square of the curvature of radius profile. This integral term places a soft constraint on a''. The solutions of the resulting system of differential equations with two-point boundary conditions correspond to the minima of J.

An Optimal Design for a TE_{02} Circular Waveguide Taper at 60 GHz

Here, we present an optimal design for a downtaper excited by the TE_{02} mode at 60 GHz with the following parameters: a_1 = 3.175 cm, a_2 = 1.389 cm, and l = 40 cm. In Fig. 1, the optimal radius profiles corresponding to different values of ε have been plotted. The cost function J given by (2) and the output power in the spurious TE_{01} mode defined by $\phi(l)$ are plotted in Fig. 2. The power contents of the TE_{01} and TE_{02} modes along the optimal tapers are plotted in Fig. 3. Figure 1 illustrates that the optimal radius profiles converge uniformly to a radius function as ε approaches zero. In fact, there is a negligible difference in optimal radius profiles for ε smaller than 7.0×10^{-7}. Hence, although the singular problem with the cost function (1) does not have a minimizing solution, the regular optimal problem has a well-behaved solution even in the limit as ε approaches zero. As seen from Fig. 2, the power in the TE_{01} spurious mode at the output is very small for small values of ε. For instance, for $\varepsilon = 1.0\times10^{-10}$, $\phi(l)$ is about 1.1×10^{-10} for unit TE_{02} input power. However, we can argue by contradiction that $\phi(l)$ will not be zero. If $\phi(l) = 0$, and since $\phi(l)$ is bounded below by zero, it will indicate that we have achieved the optimal solution for the singular problem. However, as stated before, the singular problem does not have a minimizing solution. Therefore, we believe that as ε approaches zero, $\phi(l)$ approaches its infimum (greatest lower bound), a small positive number, which from Fig. 2 can be predicted to be in the order of 1.0×10^{-10}.

There may be more than one solution u(z) that satisfies all the necessary conditions of the Minimum Principle. These solutions locally minimize the cost function J. The global optimal solution is the u(z), from these local solutions, that gives the smallest value for J. In our computer simulation for the optimal TE_{02} taper design, we have found one solution for $a(z)$ for each ε.

In this study we have included only one spurious mode. Including more than one spurious mode is a straightforward procedure, and does not change the above conclusions.

The coupled mode equations (state equations) are valid only for a slowly varying waveguide. As seen in Fig. 2, the optimal radius profiles corresponding to different ε change gradually along the z axis . Here, the maximum slope and the maximum axial curvature of the optimal tapers are all within the range of validity of the coupled mode equations that the optimal design procedure was based on.

The Minimum Principle can also be used to minimize the length as well as the output power level of the spurious modes.

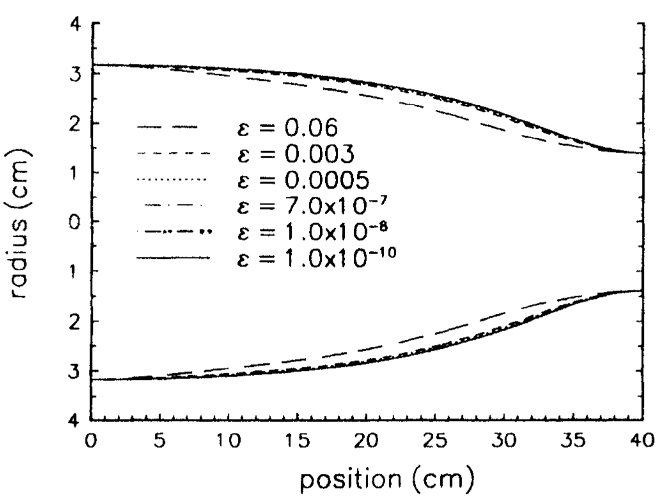

Fig. 1. The optimal radius profiles for a downtaper excited by a TE_{02} mode at 60 GHz with a_1=3.175 cm, a_2=1.389 cm, and l=40 cm. For each ε the corresponding profile minimizes the cost function J given by (2).

Acknowledgment

This work was supported by the U.S. Department of Energy under contract DE-FG02-85ER52122.

References

[1] L. S. Pontryagin, *et al.*, *The Mathematical Theory of Optimal Processes*, John Wiley and Sons, New York, NY, 1963.

[2] M. Athans and P. L. Falb, *Optimal Control*, McGraw-Hill Book Company, New York, NY, 1966.

[3] F. Sporleder and H-G Unger, *Waveguide Tapers, Transitions and Couplers*, Peter Peregrinus Ltd., London, U.K., 1979, ch. 6.

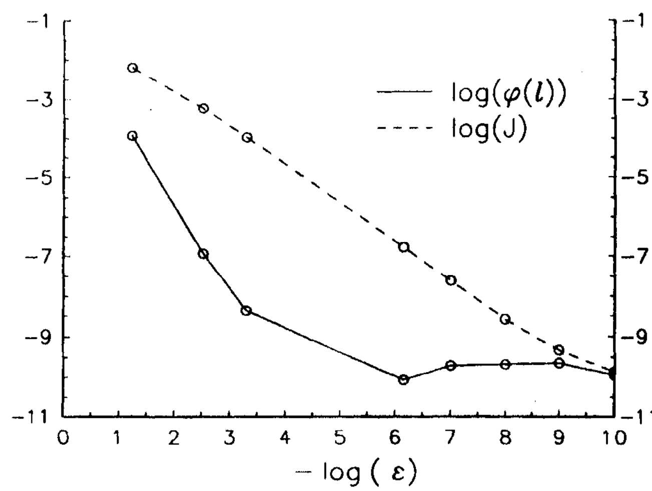

Fig. 2. Plot of the cost function J given by (2) and $\phi(l)$, the output power in the spurious TE_{01} mode, versus ε. Both axes are in logarithmic scale.

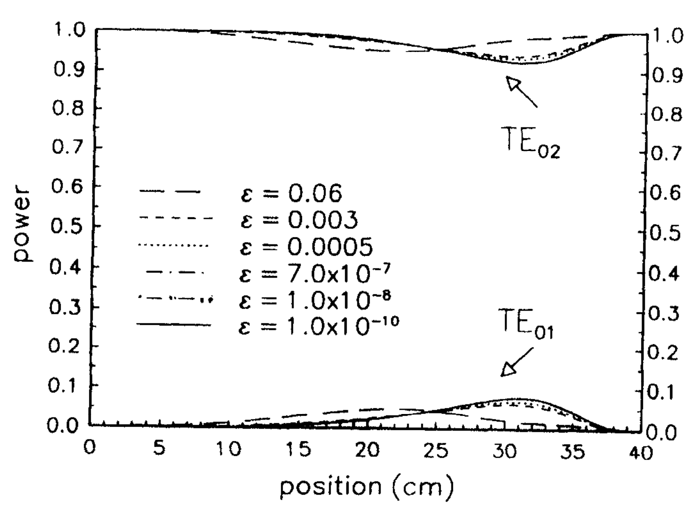

Fig. 3. Plot of the power contents of the incident TE_{02} and the spurious TE_{01} modes along the optimal tapers.