Proceedings of the 27th Conference
on Decision and Control
Austin, Texas • December 1988

THE MINIMAL DIMENSION OF STABLE FACES REQUIRED TO GUARANTEE STABILITY OF A MATRIX POLYTOPE: D-STABILITY[1]

J. Daniel Cobb

Department of Electrical and Computer Engineering
University of Wisconsin-Madison
1415 Johnson Drive
Madison, WI 53706-1691

Abstract

We consider the problem of determining whether a polytope $\mathcal{P}$ of n×n matrices is D-stable -- i.e. whether each point in $\mathcal{P}$ has all its eigenvalues in a given nonempty, open, convex, conjugate-symmetric subset D of the complex plane. Our approach is to check D-stability of certain faces of $\mathcal{P}$. In particular, for each D and n we determine the smallest integer *m* such that D-stability of every *m*-dimensional face guarantees D-stability of $\mathcal{P}$.

1. Introduction

Let $D\subset\mathbb{C}$ be nonempty, open, convex, and conjugate-symmetric (symmetric about the real axis), and define an n×n real matrix M to be D-stable if each eigenvalue λ of M satisfies $\lambda\subset D$; otherwise, M is D-unstable. We consider the problem of determining whether certain subsets of $\mathbb{R}^{n\times n}$ consist entirely of D-stable matrices. To facilitate discussion we begin with some definitions.

A (convex) polytope $\mathcal{P}$ in a vector space V is the convex hull conv(Ω) of any nonempty finite subset $\Omega\subset V$. The dimension of $\mathcal{P}$ is the dimension of the affine hull aff($\mathcal{P}$) of $\mathcal{P}$. A face of $\mathcal{P}$ is any set of the form $\Pi\cap\mathcal{P}$, where Π is a supporting hyperplane of $\mathcal{P}$. Finally, a k-dimensional half-plane in V is any nonempty set of the form $\mathcal{H}=R\cap S$, where R is a closed half-space, S is a k-dimensional affine subspace, and $S\not\subset R$. (Note that this implies that aff($\mathcal{H}$) is simply S.)

In the robust control literature, considerable interest has been generated by the problem of determining whether a family of linear systems can be shown to consist entirely of D-stable systems by checking D-stability of certain representative members of that family. In many cases, such problems can be reduced to that of determining whether a polytope or other subset of $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ consists entirely of D-stable points [1],[2]. (D-stability of a vector $x\in\mathbb{R}^n$ means simply that the polynomial $s^n+x_ns^n+\ldots+x_1$ has all its roots in D.) We are primarily interested in the technique of checking D-stability of lower dimensional faces of a polytope in order to guarantee D-stability of the entire set.

Most "facial" results pertain to continuous-time (CT) stability -- i.e. where D is the open left half complex plane. The seminal result [3] for polynomial polytopes motivates the approach. In [3] it is shown that a polynomial polytope of a particular simple structure (an "interval polynomial") is CT stable whenever four specially constructed vertices are CT stable. A more recent result [1] demonstrates that, for an arbitrary polynomial polytope, checking all edges is sufficient to guarantee CT stability. With respect to polytopes in $\mathbb{R}^{n\times n}$, is has been shown [4] that 1) an arbitrary polytope is CT stable if all (2n-4)-dimensional faces are CT stable and 2) there exist CT unstable polytopes such that all (2n-5)-dimensional faces are CT stable; hence, the value 2n-4 is minimal. In this paper we extend the results of [4] to D-stability where D may be any nonempty, open, convex, conjugate-symmetric subset of $\mathbb{C}$.

We note that for the cases n=0 and n=1 our problem has a trivial solution: D-stability of vertices guarantees D-stability of the polytope. To handle $n\geq2$ we need to partition the family of stability sets D according to the following two assumptions.

Assumption A: D is of the form $D=\{s\in\mathbb{C}\,|\,a<\text{Re } s<b\}$, where $-\infty\leq a<b\leq\infty$.

Assumption B: D is a nonempty, open, convex, conjugate-symmetric set not satisfying Assumption A.

In addition, we define $m_A(n)=\begin{cases}1, & n=2\\ 2n-4, & n>2\end{cases}$ and $m_B(n)=2n-2$. We intend to show that m_A and m_B are the values of *m* that we seek for cases A and B.

2. Sufficiency of m_A and m_B

Throughout our analysis, we will make extensive use of the fact that any affine, one-to-one map $f:\mathbb{R}^k\longrightarrow\mathbb{R}^{n^2}$ determines an affine isomorphism between $\mathbb{R}^k$ and $f(\mathbb{R}^k)$. Among other things, this implies that, for any polytope $\mathcal{P}\subset\mathbb{R}^k$, $f(\mathcal{P})$ is also a polytope of the same dimension as $\mathcal{P}$; furthermore, f sets up a one-to-one correspondence between the q-dimensional faces of $\mathcal{P}$ and the q-dimensional faces of $f(\mathcal{P})$. In addition, f maps each k-dimensional half-plane in $\mathbb{R}^k$ into another k-dimensional half-plane (e.g. see [5]). Finally, we note that every polytope is compact and that any set of the form $\{x\in\mathbb{R}^k\,|\,\|x\|_\infty\leq\gamma\}$, where $\gamma>0$, is a polytope whose q-dimensional faces are generated by fixing k-q entries of x at either $\pm\gamma$ and letting the remaining q entries vary independently over $[-\gamma,\gamma]$.

With these observations in mind, we prove a result characterizing the affine structure of the set of D-unstable points in $\mathbb{R}^{n\times n}$.

Lemma 2.1 1) If D satisfies Assumption A, then for each D-unstable $M\in\mathbb{R}^{n\times n}$ there exists an (n^2-m_A)-dimensional half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that a) $M\in\mathcal{H}$ and b) $N\in\mathcal{H}$ implies N is D-unstable.

2) If D satisfies Assumption B, then for each D-unstable $M\in\mathbb{R}^{n\times n}$ there exists an (n^2-m_B)-dimensional half-plane $\mathcal{H}\subset\mathbb{R}^{n\times n}$ such that a) $M\in\mathcal{H}$ and b) $N\in\mathcal{H}$ implies N is D-unstable.

[1] This work was supported by NSF Grant ECS-8612948.

Theorem 2.2 1) Under Assumption A, D-stability of every matrix in every m_A-dimensional face of $\mathcal{P}$ guarantees D-stability of every matrix in $\mathcal{P}$.

2) Under Assumption B, D-stability of every matrix in every m_B-dimensional face of $\mathcal{P}$ guarantees D-stability of every matrix in $\mathcal{P}$.

Sketch of Proof 1) If $\mathcal{P}$ contains an unstable A, from Lemma 2.1, part 1), there exists an (n^2-m_A)-dimensional half-plane $\mathcal{H}$ consisting entirely of D-unstable points and containing A. From dimensionality arguments, such a plane must intersect an m_A-dimensional face of $\mathcal{P}$. (See [6] for details.)

2) Same as part 1). □

3. Minimality of m_A and m_B

Our next task is to show that m_A and m_B are the smallest integers such that D-stability of all m_A-dimensional or m_B-dimensional faces of $\mathcal{P}$ guarantees D-stability of $\mathcal{P}$ under Assumptions A or B, respectively. In order to prove this, we need a lemma which may be interpreted as a multivariable extension of L'Hospital's rule. For any k×k matrices Q and R, we use the notation Q>0 and R<0 to signify that Q is positive definite and R is negative definite, respectively.

Theorem 3.1 1) Suppose D satisfies Assumption A. For each n there exists an m_A-dimensional polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ containing a D-unstable point and such that all (m_A-1)-dimensional faces of $\mathcal{P}$ are D-stable.

2) Suppose D satisfies Assumption B. For each n there exists an m_B-dimensional polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ containing a D-unstable point and such that all (m_B-1)-dimensional faces of $\mathcal{P}$ are D-stable.

Sketch of Proof 1) Here the proof is essentially the same as in [4], except that all constructions must be shifted and scaled in the complex plane to compensate for the fact that D can have boundaries other than Re s = 0.

2) In this case we consider an open diamond-shaped region $d_\delta\subset D$, whose width δ may be small. It suffices to construct a polytope $\mathcal{P}$ containing a matrix with a pair of eigenvalues on the boundary of D, but with all m_B-dimensional faces consisting of matrices with all eigenvalues in d_δ. After shifting and scaling in $\mathbb{C}$, we need only consider d_δ=int conv{$\pm1,\pm i$}; we may also assume that the points $\pm i$ are on the boundary of D.

Consider the polytope

$$\mathcal{P}_\varepsilon = \left\{\begin{bmatrix} w & 1+x & y^T \\ -1+x & -w & z^T \\ y & z & 0 \end{bmatrix} \;\middle|\; \left\|\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right\| \le \varepsilon\right\}$$

where $y,z\in\mathbb{R}^{n-2}$. Consideration of the characteristic polynomial of points in $\mathcal{P}_\varepsilon$ along with a multivariable extension of L'Hospital's rule shows that, for every $\delta>0$, there exists an $\varepsilon>0$ such that $\mathcal{P}=\mathcal{P}_\varepsilon$ satisfies the desired properties. □

Note that Theorem 3.1 also implies that the half-planes constructed in Lemma 2.1 are maximal in the sense that there exists a D-unstable matrix M in $\mathbb{R}^{n\times n}$ such that every half-plane containing M of dimension greater than n^2-m_A or n^2-m_B must also contain a D-unstable matrix. Indeed, if this were not the case, the arguments in Theorem 3.1 could be used to prove that m_A and m_B are not minimal.

The construction used in the proof of Theorem 3.1 might be viewed as somewhat weak in three respects: 1) The polytope $\mathcal{P}$ contains only a single marginally D-unstable matrix (i.e. a matrix having all eigenvalues in D and at least one on the boundary of D). 2) The construction yields only a polytope of dimension m_A or m_B. 3) Arbitrary subpolytopes are not considered; thus it is not clear that checking all subpolytopes of dimension, say, m_A-1 or m_B-1 would not guarantee D-stability. The minimality proof would be more convincing if it could be extended to give a family of polytopes, each 1) containing a strictly D-unstable point (and, since the D-unstable set in $\mathbb{C}$ is necessarily open, infinitely many D-unstable points), 2) having arbitrary dimension k, and 3) having all $\min\{k-1,m_A-1\}$-dimensional or $\min\{k-1,m_B-1\}$-dimensional subpolytopes D-stable. In [4] we showed how such improvements over Theorem 3.1 can be made for the case where D satisfies Assumption A with $b=-\infty$ and $a=0$. Essentially the same arguments also apply for an arbitrary nonempty, open, convex, conjugate-symmetric D. We omit the details here and refer the reader to [4].

4. Conclusions

Our results demonstrate to what extent the techniques for checking polytope stability proposed in [1] and [3] can be extended to the case of n×n matrices. We have shown that, without further information describing the particular structure of a polytope, either (2n-4)-dimensional or (2n-2)-dimensional faces need to be checked for D-stability, depending on the structure of D. Since testing even one such face can be a formidable task when n is large, and since the number of (2n-4)-dimensional and (2n-2)-dimensional faces grow exponentially with n, more work needs to be done before a computationally tractable algorithm can be devised for checking D-stability. It is our hope, however, that our work will be useful as an integral part of some future coherent theory of robust stability.

REFERENCES

[1] A. C. Bartlett, C. V. Hollot, H. Lin, "Root Locations of an Entire Polytope of Polynomials: It Suffices to Check the Edges," *Proceedings of the American Control Conference*, 1611-1616, 1987.

[2] N. K. Bose, "A System-Theoretic Approach to Stability of Sets of Polynomials," *Contemporary Mathematics*, Vol. 47, 25-34, 1985.

[3] V. L. Kharitonov, "Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations," *Differential'nye Uravneniya*, Vol. 14, no. 11, 1483-1485, 1978.

[4] J. D. Cobb, C. L. DeMarco, "The Minimal Dimension of Stable Faces Required to Guarantee Stability of A Matrix Polytope," submitted.

[5] A. Bronsted, *An Introduction to Convex Polytopes*, Springer-Verlag, 1983.

[6] J. D. Cobb, "The Minimal Dimension of Stable Faces Required to Guarantee Stability of A Matrix Polytope: D-Stability," submitted.