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AN EXPLICIT CHARACTERIZATION OF DESTABILIZING PHENOMENA FOR AN ARBITRARY CONTROL SYSTEM[1]

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Abstract

An explicit characterization is obtained of unmodelled dynamic phenomena which can be present in any plant and compensator and which potentially can destabilize the closed-loop configuration. It is shown that such uncertainty can be so small as to be undetectable by any real-world system identification scheme. This paper generalizes the results of [1] for state-space systems to systems with algebraic constraints.

1. INTRODUCTION

The results of this paper represent the end-product of a research effort directed at characterizing the set of linear time-invariant compensators which stabilize a given plant in the presence of high-frequency unmodelled dynamics or parasitics. It is assumed that the parasitic elements are not known explicitly but reside in a large class of possibilities. Until recently it was our assumption that sufficiently accurate measurements can always be made in establishing system models so that the existence of robustly stabilizing compensators is guaranteed for any given plant. Our work was thus directed at 1) characterizing sufficient information for a robust design and 2) finding a convenient way of describing the compensators of interest.

Recent results have led us to the opposite conclusion: Contrary to what we view as conventional wisdom in control engineering practice, it is *not* possible to obtain sufficient information concerning the structure of a pair of physical systems (plant and compensator) on the basis of real-world measurements so as to guarantee stability, even within the linear, time-invariant framework. There always exists some (perhaps very small) probability that the unmodelled dynamic elements inherent in the plant and compensator will interact in a particularly undesirable way to produce instability in the closed-loop configuration. Another way of stating this conclusion is to say that open-loop measurements can never guarantee closed-loop stability. Inherent in our work is an explicit characterization of one kind of uncertainty which can lead to unpredicted instability.

Consider the linear, time-invariant model

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}u$$
$$y = [\tilde{C}_1 \quad \tilde{C}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{1}$$

where $\tilde{A}_{22}$ is nonsingular. Equation (1) is assumed to contain all known system information. In general, such information consists of dynamic relationships among the system variables as well as some algebraic constraints. Our intention is to drop the assumption of nonsingularity of $\tilde{A}_{22}$ in a later paper.

Beyond those system variables whose explicit characterization is known, there exist other physical quantities which, if modelled, would lead to an augmented system of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}u \tag{2}$$
$$y = [C_1 \quad C_2 \quad C_3]\begin{bmatrix} x_1 \\ x_2 \\ \xi \end{bmatrix}$$

where A_{33} and $A_{22} - A_{23}A_{33}^{-1}A_{32}$ are nonsingular, and

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix}A_{33}^{-1}[A_{31} \quad A_{32}] = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \tag{3}$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \begin{bmatrix} A_{13} \\ A_{23} \end{bmatrix}A_{33}^{-1}B_3 = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \tag{4}$$

$$[C_1 \quad C_2] - C_3A_{33}^{-1}[A_{31} \quad A_{32}] = [\tilde{C}_1 \quad \tilde{C}_2] \tag{5}$$

$$C_3A_{33}^{-1}B_3 = 0 \tag{6}$$

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Without loss of generality, we may further assume that A_{33} and $A_{22} - A_{23}A_{33}^{-1}A_{32}$ are stable matrices; indeed, if this is not initially true, premultiplication of (2) by

$$M = \begin{bmatrix} I & 0 & 0 \\ 0 & S(A_{22}-A_{23}A_{33}^{-1}A_{32})^{-1} & 0 \\ 0 & 0 & TA_{33}^{-1} \end{bmatrix}$$

where S and T are stable, achieves the desired result. It is known from the multi-rate singular perturbation literature (e.g., see [2]) that our nonsingularity assumptions and equations (3)-(6) together guarantee that (1) and (2) predict precisely the same behavior for x and y regardless of the initial condition and input.

Besides the unmodelled variables ξ, every system model also contains some uncertainty with regard to unmodelled dynamics. Such phenomena might be characterized as a small peturbation of (2) of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon I & 0 \\ 0 & 0 & \alpha_\varepsilon I \end{bmatrix} \begin{bmatrix} \dot{x}_{1\varepsilon} \\ \dot{x}_{2\varepsilon} \\ \dot{\xi}_\varepsilon \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \\ \xi_\varepsilon \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u \qquad (7)$$

$$y = [C_1 \quad C_2 \quad C_3] \begin{bmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \\ \xi_\varepsilon \end{bmatrix}$$

where $\varepsilon \to 0^+$ and $\alpha_\varepsilon \to 0^+$ at perhaps different rates. We are interested in the effects of perturbations of the augmented system (2) on closed-loop stability.

2. PHYSICAL MEASUREMENTS AND SYSTEM EQUIVALENCE

In this section we address the issue of which *mathematical* perturbations of the nominal system model (1) can in fact be *physically* present in the augmented system (2). Since we are primarily concerned with characterizing sufficient information for guaranteed stability, it would be interesting to know first if there exist perturabtions of (2) which simply cannot be detected by physical measurements of any sort and then to what extent such perturbations can alter the behavior of the closed-loop configuration. To achieve this, we need to set down some basic rules of measurement which must be adhered to in any real-world modelling scenario.

For example, since any measurement or control process occurs over a finite interval of time, we may assume that the input u to the systems (1), (2), and (7) is defined on some compact interval, say $[0,\tau]$. Also, since every actuator has a maximum value as well as a maximum rate of change of voltage, force, or other input quantity that it can generate, we may assume that u is of class C^1 and that there exist numbers $K_0, K_1 < \infty$, independent of u, such that

$\|u(t)\| < K_0$ and $\|\dot{u}(t)\| < K_1$ for all $t \in [0,\tau]$.

Finally, we may take t = 0 to correspond to the time when an input is first applied; hence, $u(0) = 0$. These conditions determine the class of admissible input functions $\mathcal{U}$.

As part of the modelling process, one might *initialize the system at a variety of initial conditions*. Just as with the input, there is a bound on the maximum possible value of initial condition which can be induced; i.e. the initial vector x_0 must lie within some ball $B(0,K_2)$ of radius $K_2 < \infty$, centered at the origin.

Associated with any measuring device is a number $\delta > 0$ such that no two quantities differing by an amount less than δ can be distinguished from one another. The space $\mathcal{U}$ and the numbers δ and K_2 taken together determine *an equivalence between systems*. Theorem 1 below makes this idea precise. It is assumed that the same input u and initial condition vector x_0 are applied to each system.

Theorem 1 Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_\varepsilon = \begin{bmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \end{bmatrix}$$

ξ, and ξ_ε be the solutions of (2) and (7), respectively. Then there exists a $K < \infty$ such that to each $\delta > 0$ there corresponds an $\varepsilon_0 > 0$ such that

1) $\|x_\varepsilon(t) - x(t)\| < K$, $\|\xi_\varepsilon(t) - \xi(t)\| < K \quad \forall\, t \in [0,\tau]$

2) $\|x_\varepsilon(t) - x(t)\| < \delta$, $\|\xi_\varepsilon(t) - \xi(t)\| < \delta \quad \forall\, t \in [\delta,\tau]$

3) $\|y_\varepsilon(t) - y(t)\| < \delta \quad \forall\, t \in [0,\tau]$

whenever $u \in \mathcal{U}$, $x_0 \in B(0,K_2)$, and $\varepsilon < \varepsilon_0$.

Theorem 1 states that the perturbed system (7) cannot be distinguished from the augmented system (2) (and, consequently, from the original system (1)) as long as the given rules of measurement involving $\mathcal{U}$, δ, and K_2 are adhered to and the perturbed elements ε and α_ε are sufficiently close to their nominal values. Indeed, ε_0 can be selected such that the differences between the matrix entries in (2) and (7) are smaller than the tolerance δ; furthermore, 2) and 3) indicate that the same is true of the state and output vectors of the nominal versus perturbed systems. A minor technicality concerning the natural response of the state vector is handled by 1): In general, some boundary layer peaking (see [3]) occurs in the perturbed system (7). Nevertheless, according to Theorem 1 the peaking is bounded uniformly in ε and the base of the initial spike can be made arbitrarily small. Hence, this transient behavior is undetectable for sufficiently small δ. We therefore conclude that the nominal system (2) and the perturbed system (7) are indistinguishable on the basis of any kind of measurement scheme.

The proof of Theorem 1 is based on the following result.

Lemma 1 Let $A_{22} - A_{23}A_{33}^{-1}A_{32}$ and A_{33} in (7) be nonsingular. Then there exist an $\varepsilon_0 > 0$ and nonsingular matrix-valued functions M_ε and N_ε defined on $[0,\varepsilon_0)$ such that

$$M_\varepsilon \begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon I & 0 \\ 0 & 0 & \alpha_\varepsilon I \end{bmatrix} N_\varepsilon = \begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon F_{1\varepsilon} & 0 \\ 0 & 0 & \alpha_\varepsilon F_{2\varepsilon} \end{bmatrix}$$

$$M_\varepsilon \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} N_\varepsilon = \begin{bmatrix} A_{s\varepsilon} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$M_\varepsilon \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} B_{s\varepsilon} \\ B_{1f\varepsilon} \\ B_{2f\varepsilon} \end{bmatrix}$$

$$[C_1 \quad C_2 \quad C_3] = [C_{s\varepsilon} \quad C_{1f\varepsilon} \quad C_{2f\varepsilon}] N_\varepsilon$$

where

$$A_{s\varepsilon} \longrightarrow A_{11} - [A_{12} \quad A_{13}] \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$$

$$B_{s\varepsilon} \longrightarrow B_1 - [A_{12} \quad A_{13}] \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$$

$$C_{s\varepsilon} \longrightarrow C_1 - [C_2 \quad C_3] \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix}$$

$$F_{1\varepsilon} \longrightarrow (A_{22} - A_{23}A_{33}^{-1}A_{32})^{-1}$$

$$B_{1f\varepsilon} \longrightarrow (A_{22}^{-1} - A_{23}A_{33}^{-1}A_{32})^{-1}$$

$$C_{1f\varepsilon} \longrightarrow C_2 - C_3A_{33}^{-1}A_{32}$$

$$F_{2\varepsilon} \longrightarrow A_{33}^{-1}$$

$$B_{2f\varepsilon} \longrightarrow B_3$$

$$C_{2f\varepsilon} \longrightarrow C_3A_{33}^{-1}$$

Lemma 1 yields the system decomposition

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon F_{1\varepsilon} & 0 \\ 0 & 0 & \alpha_\varepsilon F_{2\varepsilon} \end{bmatrix} \dot{\phi} = \begin{bmatrix} A_{s\varepsilon} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \phi + \begin{bmatrix} B_{s\varepsilon} \\ B_{1f\varepsilon} \\ B_{2f\varepsilon} \end{bmatrix} u$$

$$y = [C_{s\varepsilon} \quad C_{1f\varepsilon} \quad C_{2f\varepsilon}]\phi$$

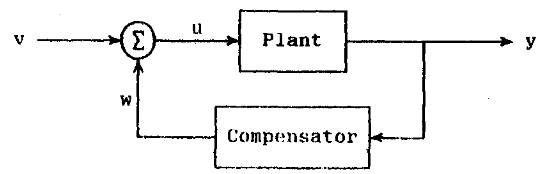
where

$$\phi = N_\varepsilon^{-1} \begin{bmatrix} x_{1\varepsilon} \\ x_{2\varepsilon} \\ \xi_\varepsilon \end{bmatrix}$$

Lemma 1 is proven using the general decomposition procedure for singularly perturbed systems outlined in [4]. Its proof may be viewed as a multiple time scale generalization of the two time scale techniques employed in [1].

3. CLOSED-LOOP INSTABILITY

Consider the feedback configuration



where the plant is nominally given by (1) and the compensator by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} y \tag{8}$$

$$w = [\tilde{H}_1 \quad \tilde{H}_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

As in (1), we assume nonsingularity of $\tilde{F}_{22}$. Furthermore, we suppose that the closed-loop system is nominally stable and that it exhibits no impulsive motion in its response. The latter condition is equivalent to nonsingularity of the matrix

$$\Sigma = \begin{bmatrix} \tilde{A}_{22} & \tilde{B}_2\tilde{H}_2 \\ \tilde{G}_2\tilde{C}_2 & \tilde{F}_{22} \end{bmatrix}$$

It is of primary interest to know whether the same closed-loop configuration of the perturbed systems (7) and

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon I & 0 \\ 0 & 0 & \alpha_\varepsilon I \end{bmatrix} \begin{bmatrix} \dot{z}_{1\varepsilon} \\ \dot{z}_{2\varepsilon} \\ \dot{\zeta}_\varepsilon \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} z_{1\varepsilon} \\ z_{2\varepsilon} \\ \zeta_\varepsilon \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} y \tag{9}$$

$$y = [H_1 \quad H_2 \quad H_3] \begin{bmatrix} z_{1\varepsilon} \\ z_{2\varepsilon} \\ \zeta_\varepsilon \end{bmatrix}$$

is guaranteed to be stable for sufficiently small ε. Here we are assuming that conditions analogous to (3)-(6) hold for the F_{ij}, G_i, and H_i. The perturbed closed-loop system is of the form

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \varepsilon I & 0 \\ 0 & 0 & \alpha_\varepsilon I \end{bmatrix} \dot{\psi} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \psi + \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} v \tag{10}$$

$$y = [Z_1 \quad Z_2 \quad Z_3]\psi$$

where

$$X_{22} = \begin{bmatrix} A_{22} & B_2H_2 \\ G_2C_2 & F_{22} \end{bmatrix}$$

$$X_{23} = \begin{bmatrix} A_{23} & B_2H_3 \\ G_2C_3 & F_{23} \end{bmatrix}, \; X_{32} = \begin{bmatrix} A_{32} & B_3H_2 \\ G_3C_2 & F_{32} \end{bmatrix} \tag{11}$$

$$X_{33} = \begin{bmatrix} A_{33} & B_3H_3 \\ G_3C_3 & F_{33} \end{bmatrix}$$

$$Y_3 = \begin{bmatrix} B_3 \\ 0 \end{bmatrix}, \; Z_3 = [C_3 \;\; 0]$$

The discussion of Section 2 can now be applied to the closed-loop system (10). If $X_{22} - X_{23}X_{33}^{-1}X_{32}$ and X_{33} are nonsingular, then from Lemma 1 it follows that corresponding to each eigenvalue λ of X_{33} there is an eigenvalue $\frac{1}{\varepsilon}\lambda_\varepsilon$ of (10) with $\lambda_\varepsilon \longrightarrow \lambda$ as $\varepsilon \longrightarrow 0^+$. Hence, if X_{33} is unstable, the closed-loop system (10) must also be unstable. We can in fact prove a stronger result.

<u>Theorem 2</u> Let $R < \infty$ and $\delta_1, \delta_2 > 0$.

1) There exist matrices X_{33}, Y_3, Z_3, X_{22}, X_{23}, and X_{32} of the form (11) such that a) A_{33} and F_{33} are stable, b) (X_{33}, Y_3, Z_3) is controllable and observable, c) X_{33} is nonsingular and unstable, d) $X_{22} - X_{23}X_{33}^{-1}X_{32}$ is nonsingular, and e) equations (3)-(6) and the analogous compensator equations are satisfied.

2) Under the conditions of part 1), there exists $\varepsilon_0 > 0$ such that the transfer function matrix of (10) has a pole p_ε with Re $p_\varepsilon > R$ whenever $0 < \varepsilon < \varepsilon_0$.

3) Under the conditions of part 1), there exists $\varepsilon_0 > 0$ such that corresponding to each $\varepsilon \in (0, \varepsilon_0)$ there exist a continuous function $u_\varepsilon : [0, \tau] \longrightarrow \mathbb{R}^m$ with $\|u_\varepsilon(t)\| < \delta_1$ for all $t \in [0, \tau]$ and a set $\Omega_\varepsilon \subset [0, \tau]$ with $m\Omega_\varepsilon < \delta_2$ such that the output of (10), subject to $\psi(0) = 0$ and $u \equiv u_\varepsilon$, satisfies $\|y_\varepsilon(t)\| > R$ for every $t \in [0, \tau] - \Omega_\varepsilon$. ($m$ denotes Lebesgue measure.)

The proof of Theorem 2 requires only a slight modification of the techniques appearing in [1]. We conclude from the theorem that any two systems of the form (1), which nominally determine a stable closed-loop configuration, can always contain unmodelled variables and corresponding high-frequency dynamics which are not discernable under any measurement scheme (Theorem 1) and which destabilize the closed-loop system (Theorem 2). Our result explicitly characterizes such destabilizing phenomena.

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