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ON THE ROBUSTNESS OF COMPENSATOR DESIGNS BASED EXCLUSIVELY ON INPUT-OUTPUT INFORMATION

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Abstract

We are interested in the effects of unmodelled parasitics or high-frequency phenomena on the closed-loop performance of a given control system. We show that, if a compensator design is based entirely on input-output characteristics of the open-loop system, it is always possible that such unmodelled aspects of the plant may destabilize the closed-loop configuration. The proof is based on Hurwitz conditions.

1. Introduction

We wish to investigate the effects of high-frequency phenomena or parasitics on the behavior of closed-loop system characteristics. Parasitic effects are an inherent source of uncertainty in any mathematical model since no modelling process can capture all dynamic phenomena which determine system behavior. High-frequency effects are often treated using singular perturbation techniques (see [1], [2]); we also adopt this approach but in a more general setting.

The issue we specifically wish to address is that of the relationship between open-loop model accuracy and accuracy of the corresponding closed-loop model, obtained by applying a given conpensator. We will give a precise meaning to the term "accuracy" in Section 2, but loosely speaking we are referring to the degree to which a model predicts actual behavior of the physical system under all possible inputs and initial conditions. Conventional wisdom suggests that, whenever a model predicts system behavior to within some small degree of error, the resulting closed-loop model should be of comparable accuracy. This idea has been shown to be erroneous by examples such as the one contained in [3] which we now briefly review:

$$\begin{bmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} u \qquad (1)$$

The system (1) with $\varepsilon = 0$ is to be viewed as the open-loop model and has transfer function $P(s) = \frac{2}{s-1}$. The actual physical system is more exactly portrayed by (1) with $\varepsilon > 0$ small but not specifically known. Hence (1) represents a first-order model with parasitic uncertainty. It is easily shown (see [4]) that, as $\varepsilon \to 0^+$, the solutions of (1) converge in the sense of distributions and uniformly on compact subintervals of $(0,\infty)$ for each initial condition and input. In this sense, then, any fixed degree of accuracy might be achieved by the open-loop model (1) (with $\varepsilon=0$) if ε in the actual system is sufficiently small.

It is shown in [3] that, if feedback $u = ky$ is applied to (1), $k > -2$ yields convergent solutions as $\varepsilon \to 0^+$ and $k < -2$ yields divergent solutions. Hence, we may say that closed-loop model accuracy is at a tolerable level only when $k > -2$. When $k < -2$, one eigenvalue tends to $+\infty$ as $\varepsilon \to 0^+$; this phenomenon is not seen in the first-order closed-loop model.

We intend to further explore these issues but with an emphasis on input-output representations instead of the state-space approach of [3]. We will see that the inherent loss of information regarding internal structure that one faces in dealing with input-output models changes the picture dramatically.

2. Convergence of Transfer Functions

We need to state more precisely what we mean by the accuracy of models, especially with respect to input-output representations. Let (P_ε) be a sequence of $p\times m$ rational matrices with degree less than or equal to n (see [5]). We say that $P_\varepsilon \to P$ if there exist sequences $E_\varepsilon \to E$, $A_\varepsilon \to A$, $B_\varepsilon \to B$, and $C_\varepsilon \to C$ such that

1) $\det(sE-A) \not\equiv 0$

2) $P_\varepsilon(s) = C(sE_\varepsilon-A_\varepsilon)^{-1}B_\varepsilon \;\forall\; \varepsilon > 0$

3) $P(s) = C(sE-A)^{-1}B$.

4) the solutions of $E_\varepsilon \dot{x} = A_\varepsilon x + B_\varepsilon u$

for each initial condition and input converge to that of $E\dot{x} = Ax + Bu$ in the sense of distributions and uniformly on compact subintervals of $(0,\infty)$.

Condition 1) guarantees that the differential equations will have unique solutions for sufficiently small ε. A transfer matrix sequence (P_ε) thus converges if and only if it has a convergent sequence of realizations with convergent solutions. For example, the sequence

$$P_\varepsilon(s) = \frac{\varepsilon(2\varepsilon-1)s^2+6\varepsilon s+2}{(\varepsilon s+1)^2(s-1)}$$

converges to

$$P(s) = \frac{2}{s-1}$$

since it has the realization (1).

The reason we insist on the existence of convergent realizations is that convergence of

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coefficients in P_ε above is not sufficient to guarantee that the inverse transform of $P_\varepsilon(s)U(s)$ is well-behaved in ε for every input U. For instance $P_\varepsilon(s) = \frac{1}{\varepsilon s+1}$ converges in our sense, since $E_\varepsilon = \varepsilon$, $A_\varepsilon = B_\varepsilon = C_\varepsilon = 1$ satisfy condition 4). On the other hand, $P_\varepsilon(s) = \frac{1}{\varepsilon s-1}$ does not and, for $u(t) = 1$, $P_\varepsilon(s)U(s)$ has inverse transform $e^{t/\varepsilon}-1$. Here, just as with state-space models, we need to ensure that time-domain behavior is accurately predicted.

3. Further Examples

We begin to show in this section that the potentially bad situation illustrated by (1) is in fact worse when we are restricted to designs based entirely on input-output information. Suppose, instead of the model (1) with $\varepsilon = 0$, we are merely given a transfer function $P(s) = \frac{2}{s-1}$. On the basis of this information we proceed to design a compensation scheme, say $u = ky$, in order to achieve some desired specifications. There are infinitely many realizations of P, one being

$$\begin{bmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \frac{3}{k} \end{bmatrix}$$

where k is assumed to have been already chosen, and $\varepsilon = 0$ for the moment. The closed-loop system is

$$\begin{bmatrix} 1 & & \\ & \varepsilon & \\ & & \varepsilon \end{bmatrix} \dot{x} = \begin{bmatrix} 1+2k & 0 & 6 \\ 0 & 0 & -1 \\ k & 1 & 1 \end{bmatrix} x \qquad (2)$$

again with $\varepsilon = 0$. Now, if ε is set slightly positive, (2) has characteristic polynomial

$$\det(sE_\varepsilon-(A_\varepsilon+B_\varepsilon kC_\varepsilon))=\varepsilon^2 s^3-\varepsilon(1+\varepsilon(1+2k))s^2 +(1+\varepsilon(1+2k))s-6\varepsilon k-1-2k$$

Not only is the polynomial not Hurwitz for any k, but it has at least one root with real part tending to $+\infty$. Furthermore, direct calculation shows that the solutions of (2) diverge as $\varepsilon \to 0^+$. We conclude, therefore, that for the open-loop transfer function $P(s) = \frac{2}{s-1}$ and any compensator gain k, there exists a sequence $P_\varepsilon \to 0$ in the sense of Section 2 such that

$$\frac{P_\varepsilon}{1-kP_\varepsilon} \not\to \frac{P}{1-kP}$$

In other words, there exist systems arbitrarily close to our open-loop model which are not accurately characterized in the closed-loop configuration. This fact is in striking contrast with the situation (1) where internal structural information is included in the model. There the closed-loop model fails to be accurate only for those k in a subinterval of **R**. Here, no k is guaranteed to yield reasonable closed-loop performance. Dropping internal structural information has destroyed our ability to accurately predict closed-loop behavior.

One might be tempted to say that the use of a static conpensator is the source of the apparent difficulties. A strictly proper compensator would induce additional high-frequency roll-off in the system loop gain and so should reduce the problems caused by unmodelled parasitics. The following example illustrates that this is not necessarily true either. Suppose the open-loop model is again $P(s) = \frac{2}{s-1}$, and a strictly proper compensator $C(s) = \frac{a}{s+b}$ is designed to meet some specifications. The perturbation (P_ε) of P, where P_ε is realized by

$$\begin{bmatrix} 1 & & & \\ & \varepsilon & & \\ & & \varepsilon & \\ & & & \varepsilon^4 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2\varepsilon \\ 0 & \varepsilon & 1 & -3\varepsilon^2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (3)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & \frac{4}{a} \end{bmatrix} x$$

satisfies conditions 1) - 4) of Section 2. Again by direct calculation it can be seen that, for any realization of C(s), the closed-loop system has at least one eigenvalue with real part tending to $+\infty$ and solutions diverging as $\varepsilon \to 0^+$. We therefore have shown that, for any first-order strictly proper compensator C, there exist systems which are adequately modelled by P in an open-loop configuration but not closed-loop.

It is important to note that, in order to generate an example where input-output information is insufficient to reliably compensate a system, we had to resort to the use of a nonstandard perturbation, characterized by the ε^4 term on the left side of (3) and various perturbation terms on the right. Such nonstandard perturbations are far from completely understood but certainly must be included in a thorough perturbational analysis.

4. Main Result

Rather than continuing to generate more and more general examples, we will state and prove our main theorem.

Theorem

Let the rational functions P and C be strictly proper and proper, respectively. There exists a sequence $P_\varepsilon \to P$ in the sense of Section 2 such that $\frac{P_\varepsilon}{1-P_\varepsilon C}$ has at least one pole with real part tending to $+\infty$ and divergent solutions.

Sketch of Proof. Suppose P is realized by a controllable, observable triple (F,G,H), and let

$$E_\varepsilon = \begin{bmatrix} I & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon \\ & & & & \varepsilon^{m^2} \end{bmatrix}$$

$$A_\varepsilon = \begin{bmatrix} F & & & & & \\ & 0 & \cdots & & 0 & -1 \\ & 1 & \ddots & & \vdots & -2\varepsilon^{m-1} \\ & \varepsilon^{m-1} & \ddots & \ddots & \vdots & -3\varepsilon^{2(m-1)} \\ & \vdots & \ddots & \ddots & 0 & \vdots \\ & \varepsilon^{(m-1)^2} & \cdots & \varepsilon^{m-1} & 1 & -(m+1)\varepsilon^{m(m-1)} \end{bmatrix}$$

$$B_\varepsilon = \begin{bmatrix} G \\ \hline 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_\varepsilon = [H \,|\, 0 \cdots 0 \; \tfrac{m+2}{\alpha_p}]$$

where

$$C(s) = \frac{\sum_{i=0}^{p} \alpha_i s^i}{s^q + \sum_{i=0}^{n-1} \beta_i s^i}$$

and $m = q - p + 1$. Applying the transformations

$$M^{-1} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ \varepsilon^{m-1} & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \varepsilon^{m(m-1)} & \cdots & & \varepsilon^{m-1} & 1 \end{bmatrix}$$

$$N^{-1} = \begin{bmatrix} 0 & & & -1 \\ 1 & \ddots & & -\varepsilon^{m-1} \\ & \ddots & \ddots & \vdots \\ & & 0 & \vdots \\ & & 1 & -\varepsilon^{m(m-1)} \end{bmatrix}$$

yields an equivalent system $(ME_\varepsilon N, MA_\varepsilon N, MB_\varepsilon, C_\varepsilon N)$ with

$$ME_\varepsilon N = \begin{bmatrix} I & & & & \\ & -\varepsilon^m & \varepsilon & & \\ & & \ddots & \ddots & \\ & & & \ddots & \varepsilon \\ & & & & -\varepsilon^m \end{bmatrix}, \qquad MA_\varepsilon N = \begin{bmatrix} F & 0 \\ 0 & I \end{bmatrix}$$

$$MB_\varepsilon = \begin{bmatrix} G \\ \hline 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad C_\varepsilon N = [H \,|\, -\tfrac{m+2}{\alpha_p} \; 0 \cdots 0]$$

Direct calculation reveals that, when any realization of $C(s)$ is applied to the transformed system, divergent solutions result. Also, the $(n+q+1)$th coefficient can be shown to be negative for sufficiently small ε, so at least one root must be in the right-half plane. Furthermore, shifting the polynomial by any $M < \infty$ results in a polynomial whose $(n+q+1)$th coefficient is still negative. Thus, at least one root must have real part tending to $+\infty$. It is easy to see that at $\varepsilon = 0$ either of the two given forms has transfer function $P(s)$, so the system realizes a perturbation $P_\varepsilon \to P$ where convergence is consistent with Section 2. □

The theorem asserts that a transfer function does not contain enough information for robust compensation with respect to parasitic effects. Given any input-output model P and any compensator C, there exist systems P_ε arbitrarily close to P in the sense of Section 2 such that the closed-loop model $\frac{P}{1-PC}$ does not adequately predict the actual closed-loop system performance. Worse yet, the increasing instability of the perturbed closed-loop system illustrates that the actual system may, at least potentially, be widly unstable even when the model predicts stability.

This result has serious consequences for conventional identification techniques where system characteristics are determined simply by applying various inputs and viewing the system responses at the output. The most crucial issue here involves the class of parasitics one is to allow. By restricting the class of plant perturbation (P_ε) sufficiently, one avoids the theorem's negative conclusion. For example, by considering only two-time scale systems as characterized by [1], one can easily show that a strictly proper compensator guarantees accuracy of the closed-loop model. On the other hand, extending the class of allowed perturbations to those described in Section 2 shows that <u>any</u> compensator may have robustness problems. The question that needs to be answered is that of describing the class of perturbations that are actually "physically" realizable. This will require further study.

One important point to observe is that in all constructions we have considered the open-loop model, determined by the 4-tuple (E,A,B,C), is not fast controllable and fast observable (see [6]). It is possible to show that this is an essential feature of all examples of this type. We are convinced that it is the existence of high-frequency modes of the system, which are only weakly controllable or observable and are actually "hidden" in the system model, that cause the undersirable effects we have observed. The essential difference between the information contained in the input-output model P and that contained in (1) with $\varepsilon = 0$ is that, in passing to a transfer function representation, information describing uncontrollable or unobservable modes is lost. The theorem states that the loss of information describing uncontrollable or unobservable modes at infinity is, in fact, crucial information if one expects to do robust compensation.

5. Conclusion

The central issue that still needs to be addressed is that of the physical realizability of different classes of parasitics. One possible approach would be to compare the types of perturbations used in the proof of our main theorem to, say, the class of parasitics realizable in a thoroughly understood family of physical systems such as the passive (or active) RLC networks. If this class of perturbations could be shown to be broad enough to allow the proof of the theorem to still carry through, we would have a compelling argument for maintaining at least partial internal structural information when performing compensator design.

Besides such theoretical arguments, it is our belief that the high-frequency phenomena we have described in this paper do crop up in practice and can be exhibited and studied systematically in an experimental setting. Further theoretical research is required, however, before such a program can be undertaken.

References

[1] P. V. Kokotovic, R. E. O'Malley, P. Sannuti, "Singular Perturbations and Order Reduction in Control Theory--An Overview," Automatica, 12, 1976, 123-132.

[2] V. R. Saksena, J. O'Reilly, P. V. Kokotovic, "Singular Perturbations and Time-Scale Methods in Control Theory: Survey 1976-1983," Automatica, 20, May 1984.

[3] H. K. Khalil, "A Further Note on the Robustness of Output Feedback Control Methods to Modelling Errors," IEEE Trans. Auto. Control, AC-29, September 1984.

[4] S. L. Campbell, Singular Systems of Differential Equations, Research Notes in Mathematics, Vol. 40, Pitman, 1980.

[5] C. T. Chen, Introduction to Linear Systems Theory, Holt-Rinehart and Winston, Inc., 1970.

[6] J. D. Cobb, "Controllability, Observability, and Duality, in Singular Systems," IEEE Trans. Auto. Control, AC-29, December 1984.