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SLOW AND FAST STABILITY IN SINGULAR SYSTEMS

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Abstract

We consider the problem of robust feedback control of LTI systems in the presence of parasitics. An essential related concept is that of fast stability -- a type of stability characterizing parasitics or high-frequency effects. A rigorous definition of fast stability is formulated in terms of topologies on a certain differentiable manifold and is connected with the robust feedback problem. A generic solution is given along with illustrative examples.

1. Introduction

Recently, a number of examples have come to light which demonstrate how little is known about the effects of parasitics on feedback systems. For example, Khalil [1] proposes the following: Consider the system

$$\dot{x}_1 = x_1 + x_2 + u$$
$$\varepsilon \dot{x}_2 = x_1 - x_2 + u \qquad (1)$$
$$y = x_1 + x_2$$

where $\varepsilon \geq 0$ is small, and adopt the feedback law

$$u = 2y \qquad (2)$$

For $\varepsilon=0$ the closed-loop system has one eigenvalue $\lambda_1 = -6$, but for $\varepsilon > 0$ the closed-loop system is second order with eigenvalues $\lambda_1 \to -6$ and $\lambda_2 \to +\infty$. Hence, although (1) with $\varepsilon = 0$ is certainly a good model of the open-loop system, we would not consider (1) and (2) together (with $\varepsilon = 0$) an appropriate model for the closed-loop system. This is true since, as $\varepsilon \to 0$, the closed-loop solutions do not look at all like those at $\varepsilon = 0$. Alternatively, we might say that the feedback law (2) is not robust with respect to certain parasitics.

Various researchers have informally pointed out that this problem might be dependent on the fact that the transfer function of (1) with $\varepsilon = 0$ is not strictly proper and, hence, some sort of positive feedback effect could be causing the difficulty (although such an observation far from trivializes the problem). In response, Khalil [2] has proposed a higher-dimensional example which illustrates the same non-robust behavior but which gives a strictly proper transfer function at $\varepsilon = 0$. We seek a systematic way of dealing with such examples.

Our aproach to singular perturbations differs somewhat from that typified by the form (1) which appears extensively in the literature [3], [4]. The form (1) is perfectly good for studying a large class of parasitics viewed one at a time; however, we feel that in many instances it makes sense to consider a broad class of perturbations simultaneously. For example, there are certainly cases where a large number of ill-defined parasitics are neglected in developing a model, the uncertain parasitics being a cause for concern. (Indeed, to some degree this problem is inherent in any modelling process.) The robust feedback problem then becomes one of finding feedback gains which preserve the accuracy of the model upon perturbation, the gains being independent of the perturbations which range over the largest "reasonable" class. These ideas will be made more precise in section 3.

2. A Manifold of Linear Systems

In order to formulate the perturbational problem more abstractly, we consider the set of LTI singular systems - viz. systems of the form

$$E\dot{x} = Ax + Bu$$
$$y = Cx \qquad (3)$$

where E and A are $n \times n$ matrices (E may be singular) with $\det(Es-A) \not\equiv 0$, B is $n \times m$, and C is $p \times n$. Evidently we are dealing with $\mathbf{R}^{n(2n+m+p)}$ minus an algebraic variety. Next, define an equivalence relation according to

$$(E_1, A_1, B_1, C_1) \approx (E_2, A_2, B_2, C_2)$$

if

$$[E_1 A_1 B_1] = M[E_2 A_2 B_2]$$

for some nonsingular $M \in \mathbf{R}^{n^2}$ and $C_1 = C_2$. Finally, form the corresponding quotient set $L(n,m,p)$ (or simply L) from $\mathbf{R}^{n(2n+m+p)}$.

We may further define a map

$$\Phi_{x_0 u} : \longrightarrow \mathcal{D}_+^{n+p}$$

where $\mathcal{D}_+$ is the space of distributions on $\mathbf{R}$ with support in $[0, \infty)$ and $x_0 \in \mathbf{R}^n$ and $u \in \mathcal{D}_+^m$ are respectively the initial condition and input in (3). $\Phi_{x_0 u}$ is determined by

$$\xi \longrightarrow (x,y)$$

where x and y form the solution of (3) in the sense of [5].

In a forthcoming publication we will show that L can be given the structure of a real analytic manifold in a natural way, the manifold satisfying certain completeness properties with respect to the map $\Phi_{x_0 u}$. Imposed on this manifold we have not only manifold topology but also a large class of other topologies, the most interesting of which are those determined by

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imposing a topology T on $\mathcal{D}_+^{n+p}$ and letting T' on L be the weakest topology which makes each $\Phi x_0 u$ continuous. It is a consequence of the completeness properties of L that T' is stronger than manifold topology whenever T is at least as strong as the standard toplogy on $\mathcal{D}_+^{n+p}$ (see [6]). The interplay of these various topologies is a central issue in the robustness problem. We explore this in the next section.

3. Fast Stability

We begin this section by recalling from [7] that a nonsingular transformation decomposes (3) into two subsystems

$$\dot{x}_s = A_s x_s + B_s u$$
$$A_f \dot{x}_f = x_f + B_f u \qquad (4)$$
$$y = C_s x_s + C_f x_f$$

where A_f is nilpotent. The subscripts s and f correspond to the "slow" and "fast" subsystems. Since the slow subsystem is just a state-space system, we use the term slow stability to refer to its stability properties as defined and characterized in state-space theory. Fast stability analogously refers to stability properties of the fast subsystem in the sense we are about to describe.

At this point we would like to allow a large class of possible perturbations of the given system. However, we certainly cannot get away with looking at all small perturbations since simply substituting $A_f + \epsilon I$ for A_f in (4) sends all fast eigenvalues to $+\infty$. Under such an approach, all systems with sigular E would be fast unstable. Fortunately, not all perturbations make physical sense. For example, small negative masses, inductances, etc. ordinarily do not exist in practice. One way to restrict the class of allowed parasitics is to simply agree on a system of neighborhoods of the point (3) on the manifold L. In fact, the system of neighborhoods along with the empty set forms a (rather weak) topology on L; conversely, any topology on L determines a system of neighborhoods of (3). Thus we may equivalently restrict perturbations by choosing some topology T'' on L - perferably one stronger than manifold topology.

Consdier a point ξ in L and choose a topology on $\mathcal{D}_+^{n+p}$. This induces a topology T' on L.

Definition: The pair (ξ, T'') is said to be fast stable (relative to T) if T'' is at least as strong as T'.

The definition says simply that a fast stable system is one where solutions for each x_0 and u converge in T under every perturbation of interest.

In order to avoid the difficulties in specifying the topology T'' explicitly, we could define it indirectly by picking a further topology on $\mathcal{D}_+^{n+p}$ and letting the maps $\Phi x_0 u$ induce T''. However, for the robust feedback problem we can escape these apparent complexities by again imposing T on $\mathcal{D}_+^{n+p}$; hence, $T'' = T'$. We then we have that $(\xi, T'')^+ = (\xi, T')$ is always fast stable.

At first glance, this seems to trivialize the whole construction, but we are really interested in stability of the closed-loop system formed by applying the feedback law

$$u = Ky$$

to (3). The matrix K maybe thought of as determining a map

$$K: L - I \longrightarrow L \qquad (5)$$

where

$$I = \{\xi \in L \,|\, (E,A,B,C) \text{ reprsents } \xi \Rightarrow$$
$$\deg|Es - (A+BKC)| = \text{rank } E\}$$

Hence K takes each system in its domain into a corresponding closed-loop system which exhibits no impulsive behavior (see [8]). The robust feedback problem for a given $\xi \in L$ is then just that of characterizing the class of all K which render the closed-loop system $(K(\xi), T')$ fast stable or, equivalently, which are continuous at ξ with respect to T'. All we have really done here is to assume that the open-loop system is fast stable and to initiate the search for feedback gains which retain fast stability. In other words, we have assumed that the open-loop model is "good" in the sense that its solutions approximate physical reality under small perturbations; for robustness we require the same from the closed-loop system.

4. A Generic Solution

Definition: A system (3) with singular E is said to be fast cyclic if, in the decomposition (4), A_f is cyclic.

Note that, since A_f is always nilpotent, (3) is fast cyclic if and only if A_f has unit rank degeneracy or, equivalently, rank E = n-1. Clearly, this is a generic condition on L since it is so on $R^{n(2n+m+p)}$ and since manifold topology on L is just quotient set topology inherited from $R^{n(2n+m+p)}$. In fact, the same is true on the topological subspace of L consisting of points corresponding to singular E.

An even stronger result can be proven:

Proposition: Let T be standard distributional topology on $\mathcal{D}_+^{n+p}$ and let T' on L be induced by the $\Phi x_0 u$. Then, with respect to T', the fast cyclic systems are open and dense in the topological subspace of points with singular E.

Proof: As noted previously, T' is stronger than manifold topology and the fast cyclic systems form an open set. To show density, substitute $A_f - \frac{1}{k} I$ for A_f in (4). Then

$$e^{t(A_f - \frac{1}{k} I)^{-1}} \longrightarrow \sum_{i=1}^{q-1} \delta^{(i-1)} A_f^i$$

follows from [5] in the T' sense where q is the index of nilpotency of A_f. Hence $\Phi x_0 u$ converges for each x_0, u and the corresponding sequence (ξ_k) converges in T'. □

Actually, genericity holds when even stronger topologies are chosen on $\mathcal{D}_+^{n+p}$. This demonstrates that the fast cyclic systems account for practically every case of interest in a very strong sense.

We now present a solution to the robust feedback problem for this generic case. For simplicity assume we are already in coordinates which decompose the system as in (4).

Theorem: If (4) is fast cyclic then K retains fast stability iff

$$A_f^{q-1}(I + B_f K C_f)|\text{Ker } A_f > 0 \qquad (6)$$

Sketch of Proof: Let ξ be represented by (4) and assume $(\xi, \mathcal{T}')$ is fast stable. From [9] we have that any $\mathcal{T}'$-convergent net (ξ_ε) with $\xi_\varepsilon \longrightarrow \xi$ can be represented by

$$\dot{x}_s = A_{s\varepsilon} x_s + B_{s\varepsilon} u$$

$$A_{f\varepsilon} \dot{x}_f = x_f + B_{f\varepsilon} u$$

$$y = C_{s\varepsilon} x_s + C_{f\varepsilon} x_f$$

where $(A_{s\varepsilon}), (B_{s\varepsilon})$, etc. all converge and $A_f = \lim A_{f\varepsilon}$ is nilpotent. If $A_f = 0$ then $A_{f\varepsilon}$ is scalar and fast stability requires $A_{f\varepsilon} \longrightarrow 0^-$. In order to keep the fast eigenvalue negative in the closed loop system, $I + B_f K C_f$ must be positive. Hence $\lambda_f \longrightarrow -\infty$ and each $\Phi_{x_0 u}$ converges. If $A_f \neq 0$ and we are in coordinates which put A_f into Jordan form, we already known from (5) and [8] that the lower left element of $I + B_f K C_f$ is nonzero. A simple argument involving the characteristic polynomial shows that $\lambda_f \longrightarrow -\infty$ if and only if that element (given by (6)) is actually positive. □

One approach to finding a complete solution to the robust feedback problem might be as follows:

Conjecture: Let $\mathcal{T}'$ be a topology induced on $\mathcal{L}$ by the $\Phi_{x_0 u}$. Then ξ is a point of continuity of the map K iff there exists a $\mathcal{T}'$-neighborhood U of ξ such that K is continuous at every fast cyclic point in U.

If the above conjecture is true, the analytical robustness problem would be reducible to one primarily algebraic in nature where the condition (6) is applied throughout an open, dense subset of a relative neighbood of ξ. It is no surprise that each topology $\mathcal{T}$ on $\mathcal{D}_+^{n+p}$ would change the neighborhood over which (6) is to be applied yielding a different class of robust feedback gains for each type of convergence considered.

5. Interpreting the Output Equations

In studying the problem of robust feedback in the presence of parasitics it immediately becomes clear that the physical interpretation of the output equation is a critical issue. For example, consider the system discussed in [2]:

$$\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} u \qquad (7)$$

$$y = [1 \quad 0 \quad 1]x \qquad (8a)$$

Alternatively, we might "solve" for x_2 and x_3 in terms of x_1 and u yielding

$$y = 2x_1 \qquad (8b)$$

Equation (7) could also be reduced, but we may wish to leave the "dummy" variables x_2 and x_3 in the model in order to study the effects of third-order perturbations. For the nominal model, (8a) and (8b) are equivalent, both giving the transfer function

$$H(s) = \frac{2}{s-1}$$

Suppose feedback $u = Ky$ with $K < -2$ is applied to (7). If (8b) is used, it can be shown that K is robust. However, [2] demonstrates that K is not robust with respect to (8a). Since the only difference between the two cases is in their output equations, we must conclude that the two descriptions (8a) and (8b) correspond to two very different internal structures. Physically, (8a) and (8b) have completely different meanings: In the first case sensors are physically attached to variables x_1 and x_3 feeding out their sum while in the second case one sensor measures x_1 and doubles it; x_3 is not directly sensed. Note that the equivalence of (8a) and (8b) breaks down when (7) is perturbed, the relationships among x_1, x_2, and x_3 becoming differential rather than algebraic. These more complex relationships are certainly more realistic for the physical system.

The preceding example illustrates how crucial it might be to use a singular representation rather than a state-space one. One could further reduce (7) to the form

$$\dot{x}_1 = x_1 + u$$

which, in conjunction with (8b), predicts that, say, $K = -3$ is a perfectly good feedback gain. Unfortunately, if the physics of the problem indicates that (7) and (8a) constitute a more appropriate model, we would be in danger of creating a horribly unstable closed-loop system with an eigenvalue near $+\infty$.

In conclusion, we have seen that, just as in state space theory, where a transfer function may not be adequate to describe all internal behavior of the (slow) system, in singular system theory there is a type of (fast) internal behavior that a system may have which is not captured by its transfer function. Moreover, such behavior might not even be visible in any state-space representation. This observation lends more validity to arguments in favor of the further development of singular system theory.

6. References

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