State Feedback Impulse Elimination for Singular Systems over a Hermite Domain*

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Abstract

We reduce the problem of impulse elimination via state feedback in singular differential equations to algebra. Our results are developed for systems over an arbitrary Hermite domain. We show that the established theories for the time-invariant and the real analytic time-varying settings can be unified in this way. Besides the constant and real analytic functions, several other function rings are considered. Our algebraic theory is applied to these cases, providing solutions to the impulse elimination problem for classes of systems not previously studied. In particular, our work allows the restriction of the feedback matrix to certain function rings.

1 Introduction

We are interested in the problem of designing a state feedback law $u = K(t)\,x$ for a time-varying singular differential equation

$$E(t)\,\dot{x} = A(t)\,x + B(t)\,u \tag{1}$$

such that the closed-loop system

$$E(t)\,\dot{x} = (A(t) + B(t)\,K(t))\,x \tag{2}$$

exhibits no impulsive transients. The matrices E, A, and B are assumed to have entries in an appropriate set of functions on $\mathbb{R}$ (possibly constant) with $E(t), A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $K(t) \in \mathbb{R}^{m\times n}$. This problem has been treated in a variety of contexts over the past 25 years [9], [15], [11], [12], [5]. For example, we originally posed and solved the problem for the time-invariant (i.e. constant matrix) case in [9].

For time-invariant systems, the fact that solutions of (2) can exhibit impulsive behavior was originally established in [13] and [14], Ch.22. One method of analysis is based on the Weierstrass decomposition ([8], p.28, Theorem 3):.Given E, A with $\det(sE - A) \not\equiv 0$, there exist nonsingular $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix},$$

where N is nilpotent. If $N \neq 0$, the solution of (1) contains an impulsive term of the form

$$z = -\sum \delta^{(k-1)} N^k z_o. \tag{3}$$

More generally, when $E(t)$ and $A(t)$ are analytic functions, it is shown in [4] that an expression similar to (3) holds under mild assumptions

Since impulses must be interpreted as unbounded, conventional notions of closed-loop stability dictate that K be chosen to make (2) impulse free. For the time-invariant case, we established a necessary and sufficient condition ([9], Theorem 6) under which such a matrix K exists. This condition can be written

$$\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = \mathbb{R}^n.$$

Since then, two alternative proofs of this result have appeared. (See [11], Theorem 2.5.1 and [12], Theorem 3-2.1.)

The work of Campbell and Petzold [4] extended the theory of singular systems (1) to the time-varying setting, where E, A, and B are matrices over the real analytic functions on $\mathbb{R}$. More recently, the corresponding impulse elimination problem has been solved by Wang in ([5], Theorem 4.1). In this case, necessary and sufficient conditions for impulse elimination are

$$\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n \quad \forall t,$$

$$\operatorname{rank} E(t) = \text{constant}.$$

Our contention is that the impulse elimination problem is primarily a problem in algebra. Indeed, after careful examination (and some modification), the arguments in [5] can be reduced to algebraic manipulations over a

*This paper is a condensed version of the SICON article [1]. See [1] for the proofs of theorems.

certain class of rings. Pursuing this idea not only leads to a unification of the time-invariant and analytic time-varying theories, but also yields a more general framework in which the impulse elimination problem for other classes of time-varying systems can be solved with little extra effort.

An important consequence of our approach is that it allows the entries of K to be restricted to certain function rings (although E, A, and B must share the same restriction). Hence, we are able to solve a wide variety of constrained feedback problems which have not been considered in the literature.

Our algebraic theory is the subject of Sections 2 and 3. In Section 4, we apply our results to various types of time-varying singular systems.

2 Algebraic Preliminaries

Let R be a commutative ring (with identity). If $x_1, \ldots, x_k \in R$, a *Bezout identity* is an equation of the form $\sum a_i x_i = 1$ ($a_i \in R$). For a matrix $M \in R^{p \times q}$, let

$$\operatorname{rank} M = \max\{k \mid M \text{ has a nonzero } k\text{th-order minor}\} \tag{4}$$

and

$$\rho M = \{\max k \mid \text{the } k\text{th-order minors of } M \text{ satisfy a Bezout identity}\}. \tag{5}$$

Obviously, $\operatorname{rank} M \geq \rho M$ for any M. It can be shown that $\operatorname{rank} M$ and ρM are invariant under left and right unimodular transformations. (See [2], p.25.) If $R = \mathbb{R}$, then $\operatorname{rank} M = \rho M$. We denote this common value by $\operatorname{rank}_{\mathbb{R}} M$.

Consider the set G of all triples (P, Q, D), where $P, Q, D \in R^{n \times n}$ and P, Q are unimodular. Define the binary operation

$$(P_1, Q_1, D_1) * (P_2, Q_2, D_2) = (P_2 P_1, Q_1 Q_2, D_1 Q_2 + Q_1 D_2).$$

It is routine to verify that G has the structure of a group. Now consider pairs (E, A), where $E, A \in R^{n \times n}$. We may define a right group action on the set of all (E, A) according to

$$(E, A) \cdot (P, Q, D) = (PEQ, P(AQ + ED)). \tag{6}$$

The *orbit* of particular (E, A) is the set of all pairs $\left(\widetilde{E}, \widetilde{A}\right)$ such that $\left(\widetilde{E}, \widetilde{A}\right) = (E, A) \cdot (P, Q, D)$ for some P, Q, D. It is easy to verify that the set of all orbits forms a partition of $R^{n \times n} \times R^{n \times n}$.

Following the terminology of Campbell and Petzold [4], we say (E, A) is in *standard canonical form*, if

$$E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \tag{7}$$

where N is strictly upper triangular with E, A identically partitioned. Similar to their notion of "analytic solvability" for systems (1), we say $(E, A) \in R^{n \times n} \times R^{n \times n}$ is *algebraically solvable*, if its orbit under (6) contains a member in standard canonical form. (The degenerate cases (I, X) and (N, I) are also allowed.) We say that (E, A) has *unit index* if the orbit of (E, A) contains a member in standard canonical form with $N = 0$. It is clear from the definitions that algebraically solvability are invariant under the group action (6).

The question arises whether a unit index orbit can contain a member in standard canonical form with $N \neq 0$. Fortunately, the next result answers this question in the negative.

Theorem 1 *Suppose (E, A) has unit index and $(E, A) \cdot (P, Q, D)$ is in standard canonical form (7). Then $N = 0$.*

In practice, algebraic solvability may be difficult to establish, so we introduce a more direct condition that will suit our purposes just as well. We say that (E, A) is *presolvable* if **any one** of the following conditions holds:
PS1) $\operatorname{Im} E + A \operatorname{Ker} E = R^n$,
PS2) $\operatorname{Im} E \cap A \operatorname{Ker} E \neq 0$,
PS3) $\operatorname{Ker} E \cap \operatorname{Ker} A \neq 0$.
Algebraic solvability and standard canonical form are related to existence and uniqueness of solutions of (1), as discussed in [4]. However, presolvability is a purely algebraic condition, having no simple connection to the dynamics of (1). Nevertheless, we can prove the following.

Theorem 2 *1) Algebraic solvability implies presolvability.*
2) Presolvability is invariant under the group action (6).

If (E, A) has unit index, it turns out that the matrix D plays no essential role in establishing standard canonical form. This is made precise in the next theorem.

Theorem 3 *If (E, A) has unit index, then there exists a unimodular $Q \in R^{n \times n}$ such that,* ***for every*** *$D \in R^{n \times n}$, there exists a unimodular $P \in R^{n \times n}$ which yields standard canonical form (7) with $N = 0$.*

For an arbitrary commutative ring R, we can establish necessary conditions under which (E, A) has unit index.

Theorem 4 *If (E, A) has unit index, then*
1) $\operatorname{rank} E = \rho E$,
2) $\operatorname{Im} E + A \operatorname{Ker} E = R^n$,
3) (E, A) is presolvable.

Let $B \in R^{n \times m}$. The group action (6) may be extended to triples (E, A, B) according to

$$(E, A, B) \cdot (P, Q, D) = (PEQ, P(AQ + ED), PB). \tag{8}$$

In [15] we introduced the concept of "impulse controllability", which is fundamental to the study of state feedback in singular systems. We can adapt this idea to the algebraic setting by taking its feedback characterization as the definition. We say that $K \in R^{m\times n}$ is *impulse eliminating*, if $(E, A+BK)$ has unit index. The triple (E, A, B) is *impulse controllable*, if there exists an impulse eliminating K.

Theorem 5 *Impulse controllability is invariant under (8).*

Theorem 6 *If (E, A, B) is impulse controllable, then*
1) $\operatorname{rank} E = \rho E$,
2) $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$.

3 Pencils over an Hermite Domain

We say R is an *Hermite domain* if it is an integral domain and, for every $a, b \in R$, there exist $u, v, x, y \in R$ such that $ux + vy = 1$ and $ax + by = 0$ ([3], p.469). It should be noted that the definition of an Hermite domain varies in the literature. For example, [16] gives a definition (p.345) which is different from, but is implied by, the one given in [3]. In particular, every Bezout domain is Hermite ([3], Theorem 3.2), and, therefore, every principal ideal domain, field, etc. is also an Hermite domain. *For the remainder of this section, our standing assumption is that R is an Hermite domain (as in [3]).*

One advantage of working in an Hermite domain is that matrices over R can be triangularized: For any $M \in R^{p\times q}$ $(p \neq q)$, there exists a lower triangular $L \in R^{\min\{p,q\}\times\min\{p,q\}}$ and a unimodular $Q \in R^{q\times q}$ such that

$$MQ = \begin{cases} \begin{bmatrix} L & 0 \end{bmatrix}, & p < q \\ \begin{bmatrix} L \\ 0 \end{bmatrix}, & p > q \end{cases}.$$

A similar result, in which $\operatorname{Ker} M$ plays a special role, was established for real analytic functions in [6]. The arguments used in [6] are essentially algebraic and can be adapted to any Hermite domain.

Theorem 7 *Let $M \in R^{p\times q}$. If $\operatorname{rank} M = k > 0$, then there exist $L \in R^{p\times k}$ with $\operatorname{rank} L = k$ and a unimodular $Q \in R^{q\times q}$ such that*

$$MQ = \begin{bmatrix} L & 0 \end{bmatrix}. \tag{9}$$

Corollary 8 *Let $M \in R^{p\times q}$.*
1) If $\rho M = p$, then there exists a unimodular Q such that

$$MQ = \begin{bmatrix} I & 0 \end{bmatrix}.$$

2) If $\operatorname{rank} M = k$, then there exist $L \in R^{k\times k}$ with $\operatorname{rank} L = k$ and unimodular P and Q such that

$$PMQ = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

3) If $\operatorname{rank} M = \rho M$, then there exist unimodular P and Q such that

$$PMQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

4) If $\operatorname{rank} M = \rho M = p$, then there exists $L \in R^{(q-p)\times q}$ such that $\begin{bmatrix} M \\ L \end{bmatrix}$ is unimodular.

Another advantage of working in an integral domain is that, if $M \in R^{p\times p}$, $x \in R^p$, and $Mx = 0$, then either $x = 0$ or $\det M = 0$, since $(\det M)x = (\operatorname{adj} M)Mx = 0$.We will make frequent use of this fact in developing our main results.

The next result is complimentary to Theorem 4, part 2).

Theorem 9 *If $\operatorname{Im} E + A \operatorname{Ker} E = R^n$, then (E, A) has unit index.*

The next theorem, complementary to Theorem 6, is our main result.

Theorem 10 *If*
1) $\operatorname{rank} E = \rho E$,
2) $\operatorname{Im} E + A \operatorname{Ker} E + \operatorname{Im} B = R^n$,
3) (E, A) is presolvable,
then (E, A, B) is impulse controllable.

Let $P_1, P_2, Q, Q_1, Q_2, A_{22}, B_2, \widehat{A}, \widehat{B}$ be given. Then, for any K_1, W, Y, T, V with V and

$$U = \begin{bmatrix} \widehat{A} & \widehat{B} \\ W & Y \end{bmatrix}$$

unimodular, we set

$$K_2 = Q_2 \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T & V \end{bmatrix} Q_1^{-1}.$$

It follows that

$$\begin{aligned} P_2P_1(A_{22} &+ B_2K_2)Q_1 \\ &= \begin{bmatrix} \widehat{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widehat{B} \\ I & 0 \end{bmatrix} \begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} \\ &= U \begin{bmatrix} I & 0 \\ T & V \end{bmatrix} \end{aligned}$$

is unimodular. Setting $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} Q^{-1}$ guarantees that $(E, A+BK)$ has unit index.

We note that the map $\pi(K_1, W, Y, T, V) = K$ is one-to-one. Indeed, if we choose K in the range of π, then K_1 is uniquely determined, and setting $L = Q_2^{-1}K_2Q_1$ yields

$$\begin{bmatrix} W & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ T & V \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

so

$$T = L_{21}, \quad V = L_{22}, \quad Y = L_{12}L_{22}^{-1},$$
$$W = L_{11} - L_{12}L_{22}^{-1}L_{21}.$$

Hence, π may be considered a parametrization of the set of all impulse eliminating K with unimodular V (i.e. the 2,2 block of $Q_2^{-1}K_2Q_1$). Unfortunately, this may not be a complete parametrization of $\mathcal{I}$, as the example

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

illustrates. Here, direct calculation shows that $\mathcal{I}$ consists of all matrices of the form

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

with k_{12} a unit. However, π only yields those matrices of the form

$$K = \begin{bmatrix} W + YT & YV \\ T & V \end{bmatrix}$$

with V and Y units. Although π does predict that $k_{12} = YV$ must be a unit, it does not allow $k_{22} = V$ to be a non-unit, in spite the admissibility of such values. Hence, the range of π is a proper subset of $\mathcal{I}$.

4 Applications to Time-Varying Singular Systems

In this section, we consider time-varying differential equations

$$E(t)\dot{x} = A(t)x + B(t)u, \tag{10}$$

where the entries of E, A, and B belong to a ring of real-valued functions on $\mathbb{R}$. We assume $E(t), A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$. The interesting case occurs when $E(t)$ is singular on a subset of $\mathbb{R}$. Such systems have been studied at length under the assumption that E, A, and B are either constant [7] or real analytic [4], [5]. We will show that these cases fit into our algebraic framework, and examine certain additional classes of functions that can be treated in our setting. Our work does not apply to problems where E, A, B, and K are allowed to have arbitrary entries in C^n, since C^n is not Hermite.

In studying (10), it is useful to consider a change of variables of the form $x = Q(t)z$, where $Q(t)$ is everywhere nonsingular and where both Q and Q^{-1} belong to a given class of functions. Assuming differentiability of Q, direct substitution yields the equivalent system

$$\begin{aligned} P(t)E(t)Q(t)\dot{z} &= P(t)(A(t)Q(t) \\ &\quad -E(t)\dot{Q}(t))z + P(t)B(t)u, \end{aligned} \tag{11}$$

where $P(t)$ is also nonsingular for every t. (Note the relationship of (11) to the group action (8).)

Another important consideration in working with any kind of differential equation is that of solvability. Roughly, this means that (10) exhibits existence and uniqueness of solutions over a large class of forcing functions u. In the case of equations based on matrices over the real analytic functions $\mathcal{A}(\mathbb{R})$, Campbell and Petzold [4] define (E, A) to be *analytically solvable* if, for every C^n function u, the system

$$E(t)\dot{x} = A(t)x + u \tag{12}$$

has at least one C^1 solution x on $\mathbb{R}$ and no two distinct solutions coincide for any t. They then proceed to show that analytic solvability is equivalent to the existence of analytic nonsingular matrices P and Q that put (11) into standard canonical form. Hence, analytic solvability is equivalent to algebraic solvability.

In the time-invariant setting, analytic solvability of (10) reduces to the condition that the matrix pencil (E, A) be *regular* – i.e.

$$\det(sE - A) \not\equiv 0. \tag{13}$$

(See [8], pp.45-49.) From [8], p.28, Theorem 3, (13) is equivalent to the existence of nonsingular $P, Q \in \mathbb{R}^{n\times n}$ that put the pencil into *Weierstrass canonical form*:

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}, \tag{14}$$

where N is nilpotent. Since $\dot{Q} = 0$, (14) and (7) are the same, so (13) is equivalent to algebraic solvability.

In addition to solvability, we note that the unit index property is a natural concept in both the constant and real analytic settings, occurring iff $N \equiv 0$.

In order to study the impulsive behavior of singular systems, we must adopt a more sophisticated viewpoint based on distribution theory. In (12) we may investigate the consequences of applying an input u, which is arbitrary C^1 up to time $t = t_0$ and drops abruptly to 0 at t_0. As discussed in [14], Chapter 22, the resulting solution exists as a distribution and is, in fact, the unique distribution x satisfying $x(t) = 0$ for $t < t_0$ and

$$E(t)\dot{x} = A(t)x + \delta_{t_0}E(t_0)x_0, \tag{15}$$

where δ_{t_0} is the unit impulse and $x_0 = \lim_{t\to t_0^-} x(t)$. Equation (15) gives a precise meaning to the natural response of (10) with arbitrary initial conditions.

Our principal objective is to find a matrix $K(t)$, whose entries reside in the *same ring of functions as the entries of* E, A, *and* B, and such that the state feedback law $u = K(t)x$ yields a unit index closed-loop system

$$E(t)\dot{x} = (A(t) + B(t)K(t))x + \delta_{t_0}E(t_0)x_0. \tag{16}$$

Thus we are simultaneously treating a wide variety of constrained feedback problems, which have not been considered in the literature.

In order to apply our results to (10), we first need to identify a function ring R that satisfies the conditions that 1) R is an Hermite domain, 2) R is closed under differentiation, 3) solvability in the classical sense implies presolvability, and 4) the analytic and algebraic notions of the unit index property coincide. Note that it follows from 4) that the analytic and algebraic notions of impulse controllability must also coincide. Once these conditions are established, we are guaranteed that the results of Sections 2 and 3 apply to systems over R. In particular, Theorems 6 and 10 give necessary and sufficient algebraic conditions under which (10) is impulse controllable. It remains only to translate conditions 1) and 2) from Theorems 6 and 10 into analytic terms.

For the remainder of this paper, we restrict ourselves to subrings R (with identity) of $\mathcal{A}(\mathbb{R})$. Properties 1) and 2) will have to be established case-by-case. On the other hand, 3) and 4) hold automatically for $\mathcal{A}(\mathbb{R})$ as a consequence of previous results. Indeed, condition 3) may be established by examining the proof of Theorem 2 in [4]. In the light of our Theorem 7 and its corollaries, the arguments used by Campbell and Petzold carry over verbatim to R, demonstrating that analytic solvability of (E, A) guarantees algebraic solvability and, therefore, presolvability. To establish 4), suppose (E, A) is analytically (and algebraically) solvable. If $N \equiv 0$, then (E, A) has unit index in the algebraic sense with $D = -\dot{Q}$. Conversely, suppose (E, A) has algebraic unit index. Then, from Theorem 3, we may choose Q such that setting $D = -\dot{Q}$ yields P that achieves (14) with $N = 0$. Hence, the two notions of unit index coincide. This establishes that our algebraic theory applies to any Hermite subring of $\mathcal{A}(\mathbb{R})$ which is closed under differentiation.

Time-Invariant Systems: To treat time-invariant systems

$$E\dot{x} = Ax + Bu,$$

set $R = \mathbb{R}$. Since $\mathbb{R}$ is a field, it is Hermite. Viewing $\mathbb{R}$ as the set of constant functions, it is closed under differentiation. We therefore conclude that Theorems 6 and 10 specialize to the characterization of time-invariant impulse controllability first established in [15]. The constructions used in proving Theorems 6 and 10 thus constitute an alternative to the known proofs of this result, as presented in [9] Theorem 6, [11], Theorem 2.5.1, and [12], Theorem 3-2.1.

General Analytic Systems: It is easily shown that, for $R = \mathcal{A}(\mathbb{R})$, [6], $\mathcal{A}(\mathbb{R})$ is Hermite. (In fact, it is shown in [10], Theorem 1.19 that $\mathcal{A}(\mathbb{R})$ is a Bezout domain.) R is closed under differentiation, so conditions 1) and 2) of Theorems 6 and 10 are necessary and sufficient for impulse controllability. It remains to link the algebraic conditions to analytic conditions on $E(t)$, $A(t)$, and $B(t)$.

Theorem 11 *Conditions 1) and 2) of Theorems 6 and 10 hold for* $R = \mathcal{A}(\mathbb{R})$ *iff* $\operatorname{rank}_{\mathbb{R}} E(t)$ *is constant and* $\operatorname{Im} E(t) + A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) = \mathbb{R}^n$ *for every* $t \in \mathbb{R}$.

Theorem 11 shows that Theorems 6 and 10 specialize to Theorem 4.1 of [5] for systems over the real analytic functions.

Now we apply our theory to classes of time-varying singular systems (10) which have not been previously studied.

Polynomial Systems: Let $R = \mathbb{R}[t]$ be the polynomials on $\mathbb{R}$ with real coefficients. $\mathbb{R}[t]$ is a subring of $\mathcal{A}(\mathbb{R})$ containing 1 and a principal ideal domain, so it is Hermite. $\mathbb{R}[t]$ is closed under differentiation. Theorem 11 applies to $\mathbb{R}[t]$ without modification.

Periodic Systems: Let $\mathcal{P}(\tau)$ be the analytic functions on $\mathbb{R}$ with period $\tau > 0$. (τ need not be the fundamental period.) $\mathcal{P}(\tau)$ is a subring of $\mathcal{A}(\mathbb{R})$ containing 1 and is closed under differentiation.

Theorem 12 $\mathcal{P}(\tau)$ *is a Bezout domain.*

It follows from Theorem 12 that $\mathcal{P}(\tau)$ is an Hermite domain. It can be further shown that $\mathcal{P}(\tau)$ is a principal ideal domain. Theorem 11 applies to $\mathcal{P}(\tau)$ without modification.

Systems Analytic at ∞: Let $\mathcal{A}_\infty(\mathbb{R})$ be the subring of $\mathcal{A}(\mathbb{R})$ consisting of all functions analytic at ∞. (x analytic at ∞ means that $x\left(\frac{1}{t}\right)$ is analytic at 0.) From the chain rule,

$$\dot{x}\left(\frac{1}{t}\right) = -t^2 \frac{d}{dt}\left(x\left(\frac{1}{t}\right)\right),$$

so $\mathcal{A}_\infty(\mathbb{R})$ is closed under differentiation.

Theorem 13 $\mathcal{A}_\infty(\mathbb{R})$ *and* $\mathcal{P}(\tau)$ *are isomorphic.*

It follows from Theorems 12 and 13 that $\mathcal{A}_\infty(\mathbb{R})$ is an Hermite domain.

The conditions of Theorem 11 must be augmented to handle analyticity at ∞.

Theorem 14 *Conditions 1) and 2) of Theorems 6 and 10 hold for* $R = \mathcal{A}_\infty(\mathbb{R})$ *iff*

$$\operatorname*{rank}_{\mathbb{R}} E(t) = \operatorname*{rank}_{\mathbb{R}} E(\infty),$$

$$\begin{aligned} \operatorname{Im} E(t) &+ A(t) \operatorname{Ker} E(t) + \operatorname{Im} B(t) \\ &= \operatorname{Im} E(\infty) + A(\infty) \operatorname{Ker} E(\infty) + \operatorname{Im} B(\infty) \\ &= \mathbb{R}^n \end{aligned}$$

for every $t \in \mathbb{R}$.

5 Conclusion

Our work demonstrates that the solutions of the state feedback impulse elimination problem, as originally developed for the time-invariant and time-varying cases in [9] and [5], share a common algebraic basis. Once exposed, this structure lends itself naturally to numerous generalizations, requiring only a small amount of analytic effort to turn the problem into algebra. The rings discussed in this paper are only a few of the many possibilities. For example, it is easy to show that similar conclusions hold for the real analytic functions with an isolated singularity at ∞, those with a pole or removable singularity at ∞, those with a zero of order at least k at a fixed point in $\mathbb{R} \cup \{\infty\}$, rational functions with no pole in $\mathbb{R}$, etc. Perhaps the greatest challenge is to fully exploit our theory by proposing an Hermite domain which is not PID, Bezout, etc. We leave this question for further research.

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