High-Gain State Feedback Analysis Based on Singular System Theory*

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Abstract

We consider linear, time-invariant state-space systems under high-gain state feedback. The analysis is couched in terms of singular system theory and Grassman manifolds. Our work is distinguished from that of other authors by the fact that we do not allow a gain-dependent state coordinate change. Simple necessary and sufficient conditions are proven under which a singular system is a high-gain limit of a given state-space system. It is shown that the feedback matrix achieves a limit on an appropriate Grassmanian, so infinite gains constitute well-defined mathematical objects. The special cases of minimum-order stable and zeroth-order limits are studied in depth, including an analysis of solution behavior. Finally, the classical "cheap control" problem is interpreted within the context of our results.

1 Introduction

Consider the linear, time-invariant state-space system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. For any $K \in \mathbb{R}^{m\times n}$, we may apply state feedback

$$u = -Kx + v, \tag{2}$$

yielding the closed loop system

$$\dot{x} = (A - BK)x + Bv. \tag{3}$$

In this paper, we are interested in the "high-gain limits" of (3) as $\|K\| \to \infty$. We seek a characterization of all such limits for a given system (1). In addition, we will specialize our results to certain important classes of limits, and develop conditions under which a limit of (2) constitutes a well-defined system in its own right. We will then apply our results to the classical "cheap control" problem.

Numerous references deal with the issue of high gain limits under state feedback. For example, early papers such as [2] treat high gain in a classical singular perturbation context. Much of this work can be viewed largely as a special case of our results. The details will be provided in Sections 4-6.

More recent efforts, such as [3], [4], and [5], study high gain limits in great depth. However, this body of work is fundamentally different from ours in that a K-dependent coordinate change is allowed, while our approach admits no coordinate change. The consequences of the two approaches are strikingly different. Indeed, consider the 1st-order system $\dot{x} = u$ with feedback $u = -kx + v$. Our analysis (and that of [2]) dictates that the closed-loop system be written $-(1/k)\,\dot{x} = x - (1/k)\,v$, yielding $x = 0$ in the limit. Note that controllability is progressively weakened as k increases, and lost entirely for $k = \infty$. This is precisely the effect one would observe in practice, with the variable x representing the fixed (i.e. *K-independent*) state of the plant.

On the other hand, the analyses in [3], [4], and [5] allow a *K-dependent* coordinate change. In this case, the kth closed-loop system becomes $p_k q_k \dot{z} = -p_k k q_k z + p_k v$, where $x = q_k z$, and p_k, q_k are arbitrary nonzero sequences. For any $g \neq 0$, setting $p_k = 1$ and $q_k = 1/(kg)$ yields the controllable limit $z = gv$. The problem here is that the loss of controllability is masked by the coordinate change $z = kgx$, which scales the physical state x progressively higher as $k \to \infty$.

Another phenomenon that can occur with a k-

*This paper is a condensed version of the SICON article [1]. See [1] for the proofs of theorems.

dependent coordinate change is illustrated by the example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad (4)$$
$$u = -\begin{bmatrix} k^2 & 1 \end{bmatrix} x.$$

Let $x = Q_k z$ and premultiply (4) by P_k, where P_k, Q_k are nonsingular. Then

$$P_k Q_k \dot{z} = P_k \begin{bmatrix} 0 & 1 \\ -k^2 & -1 \end{bmatrix} Q_k z, \qquad (5)$$

which is equivalent to a system of the form

$$X_k \dot{z} = z. \qquad (6)$$

If $Q_k = I$,

$$X_k = \begin{bmatrix} -\frac{1}{k^2} & -\frac{1}{k^2} \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad (7)$$

irrespective of P_k. On the other hand, setting $P_k = I$ and

$$Q_k = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$$

yields

$$X_k = \begin{bmatrix} -\frac{1}{k^2} & -\frac{1}{k} \\ \frac{1}{k} & 0 \end{bmatrix} \to 0. \qquad (8)$$

Substituting (7) and (8) into (6) produces vastly different results. In particular, (7) produces impulses, while (8) does not. (See [16], Ch. 22.) Losing track of the impulsive behavior in (8) is again due to the progressive redefinition of the state.

Our approach disallows coordinate changes of the state x. A moment's reflection indicates that, in our setting, the high-gain limits of (3) form a subset of those in [3], [4], and [5]. Nevertheless, characterization of these "fixed coordinate" limits requires an independent analysis. Although the limits we obtain must satisfy the necessary conditions proven in ([3]) and ([4]), we will establish alternative conditions, which are arguably simpler and both necessary and sufficient. We will also conduct a careful analysis of stable and "zeroth order" limits, which have heretofore not been explicitly studied in the literature, at least at this level of generality.

One of our objectives is to establish results which are dual to those we developed for observers in [7]. To this end, much of our work relies on the theory of differentiable manifolds. (See e.g. [10].)

Throughout the paper, we assume for convenience that $\operatorname{rank} B = m$. For a system where this is not the case, an input coordinate change $\widehat{u} = Tu$ can be used to reduce the problem to our framework.

2 Preliminaries

Before we can talk about the limits of (1), we need some elementary results from singular system theory. Consider the matrix differential equation

$$E\dot{x} = Fx + Gu, \qquad (9)$$

where $E, F \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times m}$. We assume the matrix pencil (E, F) is *regular* – i.e. $\Delta(s) = |sE - F| \not\equiv 0$.

Since premultiplication of (9) by a nonsingular matrix M does not affect the dynamics of (9), it is natural to identify systems (9) related by such a transformation. On the other hand, right multiplication of E and A amounts to a coordinate change, so we avoid such transformations, retaining the coordinate-dependent nature of conventional state-space theory. We claim that this approach leads to a simpler theory overall.

With these ideas in mind, we couch our problem in terms the *Grassman manifold* $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, denoting points in $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$ by $[E, F, G]$. Setting

$$\mathcal{L}(n,m) = \left\{ [E,F,G] \in \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right) \,\middle|\, \Delta \not\equiv 0 \right\}$$

is consistent with the quotient structure of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, since premultiplication of $\begin{bmatrix} E & F & G \end{bmatrix}$ by a nonsingular M scales Δ by a nonzero constant. Since $\mathcal{L}(n,m)$ is an open, dense submanifold of $\mathcal{G}_n\left(\mathbb{R}^{2n+m}\right)$, it is an analytic manifold of dimension $n(n+m)$. We studied $\mathcal{L}(n,m)$ in [6].

We will make frequent use of the Weierstrass Decomposition ([9], pp. 24-28): For any regular pencil (E, F), there exists nonsingular M, N such that

$$MEN = \begin{bmatrix} I & 0 \\ 0 & E_f \end{bmatrix} \quad MFN = \begin{bmatrix} F_s & 0 \\ 0 & I \end{bmatrix}, \qquad (10)$$

where E_f is nilpotent. E_f and F_s are unique up to similarity. Define the *order* of (E, F) to be $\operatorname{ord}(E,F) = \deg \Delta$ (i.e. the dimension of F_s) and the *index* $\operatorname{ind}(E,F)$ to be the smallest integer $q \geq 1$ such that $E_f^q = 0$. The functions ord and ind may be consistently applied to points in $\mathcal{L}(n,m)$:

$$\operatorname{ord}[E,F,G] = \operatorname{ord}(E,F),$$
$$\operatorname{ind}[E,F,G] = \operatorname{ind}(E,F).$$

We will need to consider solutions of (9). To this end, we review some basic facts from the theory of distributions. (See e.g. [11]). Let $\mathcal{D}$ be the space of C^∞ functions $\phi : \mathbb{R} \to \mathbb{R}$ with bounded support, and let $\mathcal{D}'$ denote the dual space of $\mathcal{D}$. A *distribution* f is any member of $\mathcal{D}'$. Each locally L^1 function f (i.e. L^1 on bounded intervals) may be considered a distribution, since it determines a functional $\phi \to \int f\phi$. The unit impulse δ is defined to be the evaluation functional $< \delta, \phi >= \phi(0)$. Every distribution has a derivative defined by $< \dot{f}, \phi >= - < f, \dot{\phi} >$; thus $< \delta^{(i)}, \phi >= (-1)^i \phi^{(i)}(0)$. A sequence of distributions f_k is said to converge *weak** to f if $\langle f_k, \phi\rangle \to \langle f, \phi\rangle$ for every $\phi \in \mathcal{D}$. One advantage of working with distributions is that differentiation is a weak*-continuous operation. Besides weak* convergence, we will sometimes refer to uniform convergence $f_k \to f$ on an interval in $\mathcal{I} \subset \mathbb{R}$. This simply means that there exist locally L^1 functions g_k, g defined on $\mathcal{I}$ such that $< f_k, \phi >=< g_k, \phi >, <$

$f,\phi>=<g,\phi>$ for all ϕ with support in $\mathcal{I}$ and $g_k \to g$ uniformly. Let $U \subset \mathbb{R}$ be the largest open set such that $\operatorname{supp}\phi \subset U$ implies $<f,\phi>=0$. The *support* of f is $\operatorname{supp} f = U^c$. Let $\mathcal{D}'_+$ be the distributions with support in $[0,\infty)$.

In order to apply arbitrary initial conditions x_0 to (9), it is convenient to consider the augmented system

$$E\dot{x} = Fx + Gu + \delta E x_0, \tag{11}$$

which yields a unique solution $x \in \mathcal{D}'_+$. (See [16], Ch.22 for details.). Let $\exp(F_s) : \mathbb{R} \to \mathbb{R}^{\deg\Delta \times \deg\Delta}$ be given by

$$\exp(F_s)t = \begin{cases} e^{tF_s}, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

and define the *state-transition matrix*

$$\Phi = N \begin{bmatrix} \exp(F_s) & 0 \\ 0 & -\sum_{i=0}^{q-1} \delta^{(i)} E_f^i \end{bmatrix} M. \tag{12}$$

The state transition matrix relates to the system (11) as follows:

Theorem 1 *1)* $E\dot{\Phi} = A\Phi + \delta I$
2) The solution of (11) is $x = \Phi E x_0 + \Phi * Gu$.
3) The system (11) is asymptotically stable iff ΦE *is bounded and decays asymptotically to* 0.

3 The Manifold of Closed-Loop Systems

The present paper closely follows the development of [7], where the dual problem of the limiting behavior of state observers under high gain feedback was studied. One might speculate that the state feedback case should be obtained from [7] merely be taking the "transpose" of all theorems. While some theorems do transfer over in this way, much of the state feedback theory is different. One way to see that this must be true is to observe that, in both cases, systems are identified when they are related by left multiplication by a nonsingular M. In contrast, pure transposition of the observer problem would require *right* multiplication by M, leading to a K-dependent coordinate change, which we explicitly avoid.

The closed-loop systems (3) for a given plant (1) imbed naturally into $\mathcal{L}(n,m)$ via the map $K \to [I, A-BK, B]$. We denote the image of $\mathbb{R}^{m\times n}$ under this map by $\mathcal{C}_r$. We further denote the closure of $\mathcal{C}_r$ in $\mathcal{L}(n,m)$ by $\mathcal{C}$ and consider the set $\mathcal{C}_s = \mathcal{C}$ - $\mathcal{C}_r$. $\mathcal{C}$ may be regarded as the set all limits of (3), $\mathcal{C}_r$ the full-order limits (i.e. ordinary state space systems) and $\mathcal{C}_s$ the singular limits (i.e. generalized state space systems).

Theorem 2 *1)* $\mathcal{C} = \{[X, XA-Y, XB] \in \mathcal{L}(n,m) \mid \operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m\}$
2) $\mathcal{C}$ *is a regular submanifold of* $\mathcal{L}(n,m)$ *with dimension* nm.
3) $\mathcal{C}_r$ *is a (relatively) open, dense submanifold of* $\mathcal{C}$
4) $[X, XA-Y, XB] \in \mathcal{C}_s$ *iff* $\operatorname{rank}\begin{bmatrix} XB & Y \end{bmatrix} = m$ *with* X *singular.*

4 Stable and Zeroth Order Limits

In this section, we study certain subsets of $\mathcal{C}$ which have special significance. In particular, we examine those systems in $\mathcal{C}$ which are stable (i.e. all eigenvalues satisfy $\operatorname{Re}\lambda < 0$) and those with order 0. We begin with a discussion of an important submanifold of $\mathcal{C}$, which will help simplify the development. Let

$$\mathcal{C}_I = \{[X, I, XB] \in \mathcal{C}\}.$$

$\mathcal{C}_I$ is simply the set of points in $\mathcal{C}$ with no eigenvalue at 0. Each point in $\mathcal{C}_I$ corresponds to a system

$$X\dot{x} = x + XBv + \delta X x_0 \tag{13}$$

with state transition matrix determined by $X\dot{\Phi} = \Phi + \delta I$. From Theorem 2, part 1), we obtain

$$\mathcal{C}_I = \{[X, I, XB] \in \mathcal{G}_n\left(\mathbb{R}^{2n+m}\right) \mid \operatorname{rank}\begin{bmatrix} XB & XA-I \end{bmatrix} = m\}.$$

The next result gives several alternative characterizations of $\mathcal{C}_I$.

Theorem 3 *For any* $X \in \mathbb{R}^{n\times n}$, *the following are equivalent:*
1) $\operatorname{rank}\begin{bmatrix} XB & XA-I \end{bmatrix} = m$
2) $\operatorname{Ker}\begin{bmatrix} X & I \end{bmatrix} \subset \operatorname{Im}\begin{bmatrix} B & A \\ 0 & -I \end{bmatrix}$
3) $\operatorname{Im}(AX-I) \subset \operatorname{Im} B$
4) There exists $U \in \mathbb{R}^{m\times n}$ *such that* $AX + BU = I$.

Theorem 3, part 4) indicates that $\mathcal{C}_I$ is nonempty iff $\begin{bmatrix} A & B \end{bmatrix}$ has full rank – i.e. iff 0 is a controllable mode of (1). In this case, the affine set

$$\mathcal{W} = \left\{\begin{bmatrix} X \\ U \end{bmatrix} \in \mathbb{R}^{2n\times n} \,\middle|\, AX + BU = I\right\}$$

will prove central to our theory. The next result gives a precise relationship between $\mathcal{C}_I$ and $\mathcal{W}$.

Theorem 4 *1)* $[X, I, XB] \in \mathcal{C}_I$ *iff there exists* $U \in \mathbb{R}^{m\times n}$ *such that* $\begin{bmatrix} X \\ U \end{bmatrix} \in \mathcal{W}$. *In this case,* U *is unique.*
2) Let $K_k \in \mathbb{R}^{m\times n}$. *Then* $[I, A-BK_k, B] \to [X, I, XB] \in \mathcal{C}_I$ *as* $k \to \infty$ *iff* $A - BK_k$ *is nonsingular for large* k *and* $(A-BK_k)^{-1} \to X$. *In this case,* $-K_k(A-BK_k)^{-1} \to U$.
3) $\mathcal{C}_I$ *is a (relatively) open, dense submanifold of* $\mathcal{C}$, *diffeomorphic to* $\mathcal{W}$.

Since closed-loop systems in $\mathcal{C}_I$ (or, alternatively, $\mathcal{W}$) have no eigenvalue at 0, $\mathcal{C}_I$ contains all stable limits and all zeroth order limits.

Restricting to $\mathcal{C}_I$ yields a surprising result related to controllability of the closed-loop system (13).

Theorem 5 *Let $[X, I, XB] \in \mathcal{C}_I$. Then $\operatorname{rank} X \geq n - m$ with equality iff $XB = 0$.*

Theorem 5 states that high gain limits of (3) where the rank of X degenerates maximally have the unfortunate property that the input v exerts no control whatsoever on the system. This is undoubtedly a limitation for control problems where closed-loop tracking to a reference input is required.

Now we consider the special cases of minimum-order stable and zeroth order limits. By applying essentially the same arguments as in [6], several results are obtained immediately. These are summarized in Theorems 6 and 7. The first is based on the following construction. Choose any nonsingular matrix T such that

$$T^{-1}B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{14}$$

and let

$$\begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} = T^{-1}AT, \tag{15}$$

where $\widetilde{A}_{22} \in \mathbb{R}^{m \times m}$. If (A, B) is stabilizable,

$$\operatorname{rank} \begin{bmatrix} \lambda I - \widetilde{A}_{11} & -\widetilde{A}_{12} & 0 \\ -\widetilde{A}_{21} & \lambda I - \widetilde{A}_{22} & I \end{bmatrix} = n$$

for every λ with $\operatorname{Re} \lambda \geq 0$. Hence, $\operatorname{rank} \begin{bmatrix} \lambda I - \widetilde{A}_{11} & \widetilde{A}_{12} \end{bmatrix} = n - m$ (i.e. $\left(\widetilde{A}_{11}, \widetilde{A}_{12}\right)$ is stabilizable). We may thus choose Λ such that $\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda$ is stable, and set

$$X = T \begin{bmatrix} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \\ -\Lambda \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & 0 \end{bmatrix} T^{-1}, \tag{16}$$

$$U = \begin{bmatrix} -\left(\widetilde{A}_{21} - \widetilde{A}_{22}\Lambda\right)\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1} & I \end{bmatrix} T^{-1}. \tag{17}$$

By direct calculation, $AX + BU = I$, so $\begin{bmatrix} X \\ U \end{bmatrix} \in \mathcal{W}$ and $\xi = [X, I, 0] \in \mathcal{C}_I$. Note that $\operatorname{ind} \xi = 1$ and $\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)^{-1}$ is stable, so ξ is stable. From Theorem 1, part 1), the state transition matrix is

$$\Phi = T \begin{bmatrix} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)\exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \\ -\Lambda\left(\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right)\exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) + \delta I\right) & -\delta I \end{bmatrix} T^{-1}, \tag{18}$$

so

$$\Phi X = T \begin{bmatrix} \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \\ -\Lambda \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) & 0 \end{bmatrix} T^{-1}. \tag{19}$$

Letting

$$\begin{bmatrix} \widetilde{x}_{01} \\ \widetilde{x}_{02} \end{bmatrix} = T^{-1}x_0,$$

we obtain the solution of (13):

$$x = T \begin{bmatrix} I \\ -\Lambda \end{bmatrix} \exp\left(\widetilde{A}_{11} - \widetilde{A}_{12}\Lambda\right) \widetilde{x}_{01}.$$

Theorem 6 *1) $\mathcal{C}_s$ contains a stable point iff (A, B) is stabilizable.*
2) If $\xi \in \mathcal{C}_s$ is stable, then $\operatorname{ord} \xi \geq n - m$ with equality iff $\xi = [X, I, 0]$, where X has the structure (16).

We are also interested in the zeroth order closed-loop limits

$$\mathcal{C}_0 = \left\{ \xi \in \mathcal{C} \,\middle|\, \operatorname{ord} \xi = 0 \right\}.$$

$\mathcal{C}_0$ corresponds precisely to those $\xi = [X, I, XB] \in \mathcal{C}_I$ with X nilpotent. From Theorem 1, part 1), the state transition matrix is

$$\Phi = -\sum\nolimits_{i=0}^{q-1} \delta^{(i)} X^i, \tag{20}$$

so the solution of (13) is

$$x = \Phi X x_0 + \Phi * v = -\sum_{i=0}^{n-1} X^{i+1} B v^{(i)} - \sum_{i=1}^{n-1} \delta^{(i-1)} X^i x_0.$$

The system corresponds to successive differentiation of the input v plus a "noise" term.

Theorem 7 *1) $\mathcal{C}_0$ is nonempty iff (A, B) is controllable.*
2) If (A, B) is controllable and $m = 1$, $\mathcal{C}_0$ is a singleton.
3) If (A, B) is controllable, $m = 1$, $\xi_k \in \mathcal{C}_r$, and all eigenvalues λ_{ik} of ξ_k satisfy $|\lambda_{ik}| \to \infty$, then ξ_k converges to the unique point in $\mathcal{C}_0$.
4) If (A, B) is controllable and $m > 1$, $\mathcal{C}_0$ is uncountable and unbounded (as a subset of $\mathcal{W}$).
5) Every $\xi \in \mathcal{C}_0$ satisfies $\operatorname{ind} \xi \geq \frac{n}{m}$.

Next, we consider $\mathcal{C}_r$ approximations $[I, A - BK_k, B]$ to certain points in $\mathcal{C}_s$. This is important in applications, since points with singular X can only be achieved as limits as $\|K_k\| \to \infty$ in (3). In view of (11), the closed-loop system (3) can be written equivalently as

$$(A - BK_k)^{-1} \dot{x} = x + (A - BK_k)^{-1} Bv + \delta (A - BK_k)^{-1} x_0, \tag{21}$$

yielding state transition matrix

$$\Phi_k = (A - BK_k) \exp (A - BK_k) \tag{22}$$

and solution

$$x_k = \Phi_k \left(A - BK_k\right)^{-1} x_0 + \Phi_k * Bv. \tag{23}$$

We are interested in finding a sequence $\{K_k\}$ that yields not only convergence of $[I, A - BK_k, B]$ in $\mathcal{C}$, but also the strongest possible convergence of the forced and natural response in (23).

We begin by consider stable systems.

Theorem 8 *Let $\xi \in \mathcal{C}_s$ be stable with $\operatorname{ord}\xi = n - m$, and let*

$$\begin{aligned} K_k &= \begin{bmatrix} \widetilde{A}_{21} + k\Lambda & \widetilde{A}_{22} + kI \end{bmatrix} T^{-1}, \\ \xi_k &= [I, A - BK_k, B]. \end{aligned} \tag{24}$$

Then
1) $\xi_k \to \xi$,
2) $\Phi_k (A - BK_k)^{-1}$ is uniformly bounded,
3) $\Phi_k \to \Phi$ uniformly on $[\varepsilon, \infty)$ for every $\varepsilon > 0$,
4) $\Phi_k \to \Phi$ weak,*
where Φ is given by (18).

The results of [2] can be interpreted in terms of Theorems 6 and 8. In [2], the special case

$$K_\mu = -\left(1/\mu\right) K \tag{25}$$

is considered, where K is a fixed matrix and $\mu > 0$ is small. Adopting (14) and (15) and setting

$$\begin{bmatrix} \widetilde{K}_1 & \widetilde{K}_2 \end{bmatrix} = KT,$$

it is assumed in [2] (equations (32) and (33)) that $\widetilde{K}_2$ and $\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{K}_2^{-1}\widetilde{K}_1$ are stable. Under these conditions, (25) constitutes an alternative to (24). Indeed, define

$$\Gamma_\mu = \mu\widetilde{A}_{22} + \widetilde{K}_2, \quad \Delta_\mu = \widetilde{A}_{11} - \widetilde{A}_{12}\Gamma_\mu^{-1}\left(\mu\widetilde{A}_{21} + \widetilde{K}_1\right),$$

and note that Γ_μ and Δ_μ are stable for small $\mu > 0$. Block matrix inversion reveals

$$\begin{aligned} X &= T \begin{bmatrix} \Delta_\mu^{-1} & -\mu\Delta_\mu^{-1}\widetilde{A}_{12}\Gamma_\mu^{-1} \\ -\Gamma_\mu^{-1}\left(\mu\widetilde{A}_{21} + \widetilde{K}_1\right)\Delta_\mu^{-1} & \mu\left(\Gamma_\mu^{-1} + \Gamma_\mu^{-1}\left(\mu\widetilde{A}_{21} + \widetilde{K}_1\right)\Delta_\mu^{-1}\widetilde{A}_{12}\Gamma_\mu^{-1}\right) \end{bmatrix} T^{-1} \\ &\to T \begin{bmatrix} \left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{K}_2^{-1}\widetilde{K}_1\right)^{-1} & 0 \\ -\widetilde{K}_2^{-1}\widetilde{K}_1\left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{K}_2^{-1}\widetilde{K}_1\right)^{-1} & 0 \end{bmatrix} T^{-1}, \end{aligned}$$

which is the same as (16) with $\Lambda = \widetilde{K}_2^{-1}\widetilde{K}_1$. Note that, in [2], only asymptotic stability for each $\mu > 0$ is actually proven.

Now consider zeroth order systems $\xi \in \mathcal{C}_0$. Theorem 7, part 1), guarantees that (A, B) is controllable. From [15], pp. 342-343, there exist $\widetilde{K} \in \mathbb{R}^{m \times n}, w \in \mathbb{R}^m$ such that $\left(A - B\widetilde{K}, Bw\right)$ is controllable with $A - B\widetilde{K}$ nilpotent. Thus there exists a nonsingular N such that

$$N^{-1}\left(A - B\widetilde{K}\right)N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix},$$

$$N^{-1}Bw = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 9 *Let*

$$\beta_{ik} = \begin{pmatrix} n \\ i \end{pmatrix} k^{n-i}, \quad \widehat{K}_k = \begin{bmatrix} \beta_{0k} & \cdots & \beta_{n-1,k} \end{bmatrix},$$

$$\begin{aligned} K_k &= \widetilde{K} + w\widehat{K}_k N^{-1}, \\ \xi_k &= [I, A - BK_k, B]. \end{aligned}$$

Then
1) ξ_k converges to a point in $\mathcal{C}_0$,
2) $\Phi_k \to \Phi$ uniformly on $[\varepsilon, \infty)$ for every $\varepsilon > 0$,
3) $\Phi_k \to \Phi$ weak,*
where Φ is given by (20).

Note that, in Theorem 9, boundedness of the natural response matrix $\Phi_k (A - BK_k)^{-1}$ was dropped. This is a consequence of the appearance of impulses in Φ when $\xi \in \mathcal{C}_0$ and $X \neq 0$. We can, in fact, prove a stronger result, which demonstrates the disastrous effect of driving the system to a limit with $\operatorname{ord}\xi < n - m$.

Theorem 10 *Let $m < n$, $1 < p \leq \infty$, and $\xi_k \in \mathcal{C}$ be stable for all k. If the eigenvalues λ_{ik} of ξ_k satisfy $\max_i \{|\lambda_{ik}|\} \to \infty$ as $k \to \infty$, then $\|\Phi_k X_k\|_p \to \infty$.*

5 The Limiting Compensator

The state feedback law (2) may be written

$$\begin{bmatrix} I & K \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} = v. \tag{26}$$

This suggests that compensators of the form (2) are naturally identified with points $[I, K]$ in the Grassmanian $\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$. Consider the open, dense submanifolds $\mathcal{F} = f^{-1}(\mathcal{C})$ and $\mathcal{F}_r = f^{-1}(\mathcal{C}_r)$ of $\mathcal{G}_m\left(\mathbb{R}^{m+n}\right)$, and let $\mathcal{F}_s = f^{-1}(\mathcal{C}_s)$. The next result establishes basic properties of state feedback (26).

Theorem 11
1) $\mathcal{F}_r = \left\{[I, K] \in \mathcal{G}_m\left(\mathbb{R}^{m+n}\right) \,\middle|\, K \in \mathbb{R}^{m \times n}\right\}$
2) $\mathcal{F}_s = \left\{[Z_1, Z_2] \in \mathcal{F} \,\middle|\, \det Z_1 = 0\right\}$

The properties of f guarantee that, if K_k is any sequence of feedback matrices such that the closed-loop systems (3) converge in $\mathcal{C}$, then the sequence $[I, K_k]$ also converges in $\mathcal{G}_m(\mathbb{R}^{m+n})$. By Theorem 11, degeneration of (3) to a point in $\mathcal{C}_s$ occurs iff $[I, K_k]$ converges to a point in $\mathcal{F}_s$. In other words, the limiting compensator always exists, and it is singular iff the limiting closed-loop system is singular. Compensators in $\mathcal{F}_s$ are not physically realizable, since they correspond to feedback laws of the form $Z_1 u = -Z_2 x + v$ with Z_1 singular. Yet, as a mathematical object, each compensator in $\mathcal{F}$ determines a well-defined closed-loop system.

For the special case of minimum-order stable limits, as in Theorem (6), we can obtain the form of Z_1 and Z_2 explicitly.

Theorem 12 *If $\xi = [X, I, XB]$ is given by (16), then* $f^{-1}(\xi) = \left[0, \left[\begin{array}{cc} \Lambda & I \end{array}\right]\right]$.

We conclude this section by examining behavior of the input function u under high-gain feedback. For simplicity, we will only consider the case where $v = 0$. If we apply the feedback gains K_k to (3), then both u and x depend on k, and are related by the feedback law $u_k = K_k x_k$. In Theorems 8 and 9, we established cases under which the state-transition matrix Φ_k converges in two different topologies. More generally, consider the linear subspace

$$\mathcal{D}_0' = C[0,\infty) + \operatorname{span}\left\{\delta, \dot{\delta}, \ddot{\delta}, \ldots\right\} \subset \mathcal{D}_+',$$

where $C[0,\infty)$ is the set of continuous functions on $\mathbb{R}$ with support in $[0,\infty)$. Both weak* convergence and uniform convergence on every $[\varepsilon,\infty)$ correspond to specific topologies on $\mathcal{D}_0$. It is easy to show that both make $\mathcal{D}_0$ a topological vector space.

Theorem 13 *Suppose $\mathcal{D}_0$ is given a topology that makes it a topological vector space. If $[I, A - BK_k, B] \to [X, I, XB] \in \mathcal{C}_I$ and $\Phi_k \to \Phi$ in $\mathcal{D}_0$, then $u_k \to U\Phi x_0$ in $\mathcal{D}_0$.*

Theorem 13 can be extended to $v \neq 0$ through choice of an appropriate space of inputs v and exploiting the properties of the convolution operator. We leave the details to the reader.

6 Application to Cheap Control

A classical problem in the theory of linear-quadratic optimal control is the "cheap control" problem, where an input function $u^*(t)$ is sought to minimize the cost

$$J(\varepsilon) = \int_0^\infty x^T x + \varepsilon u^T u dt$$

subject to (1), with fixed initial condition x_0 and small $\varepsilon \geq 0$. For $\varepsilon > 0$, this problem has been extensively studied (e.g. see [13], [8], [12], [14]). The solution is obtained by constructing the unique positive definite symmetric solution $P(\varepsilon)$ of the algebraic Riccati Equation

$$P(\varepsilon)A + A^T P(\varepsilon) - \frac{1}{\varepsilon} P(\varepsilon) BB^T P(\varepsilon) + I = 0.$$

Then, for each x_0, the optimal u and x are related by the feedback law $u^* = -(1/\varepsilon) B^T P(\varepsilon) x^*$,yielding the closed-loop system

$$\left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} \dot{x}^* = x^* + \delta \left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} x_0$$

(cf. (21)).

For $\varepsilon = 0$, we adopt (14) and (15), let

$$\left[\begin{array}{cc} \widetilde{Q}_{11} & \widetilde{Q}_{12} \\ \widetilde{Q}_{12}^T & \widetilde{Q}_{22} \end{array}\right] = T^T T,$$

and let Γ be the unique positive definite symmetric solution of the reduced Riccati equation'

$$\Gamma\left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T\right) + \left(\widetilde{A}_{11} - \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T\right)^T \Gamma$$
$$-\Gamma \widetilde{A}_{12}\widetilde{Q}_{22}^{-1}\widetilde{A}_{12}^T \Gamma + \widetilde{Q}_{11} - \widetilde{Q}_{12}\widetilde{Q}_{22}^{-1}\widetilde{Q}_{12}^T = 0.$$

Setting

$$\Lambda = \widetilde{Q}_{22}^{-1}\left(A_{12}^T \Gamma + \widetilde{Q}_{12}^T\right) \tag{27}$$

leads to values of X, U, and Φ according to (16), (17), and (18). It is shown in [14], Corollary 2.6.1, that $J(0)$ is minimized, subject to (1), by $x^* = \Phi x_0$ and $u^* = U\Phi x_0$. Furthermore, [14], Theorem 2.7.1 indicates that

$$\left(A - \frac{1}{\varepsilon} BB^T P(\varepsilon)\right)^{-1} \to X$$

as $\varepsilon \to 0^+$. These facts are now interpreted in the context of the present paper.

Theorem 14 *For each $\varepsilon \geq 0$, let $\xi_\varepsilon^* \in \mathcal{C}_r$ be the optimal closed-loop system in the cheap control problem. Then $\xi_\varepsilon^* \to \xi_0^*$ in $\mathcal{C}$ as $\varepsilon \to 0^+$, where ξ_0^* is stable and $\operatorname{ord} \xi_0^* = n - m$. The limiting system ξ_0^* is determined uniquely by the singular compensator $\left[0, \left[\begin{array}{cc} \Lambda & I \end{array}\right]\right] \in \mathcal{G}_m(\mathbb{R}^{m+n})$ as in Theorem 12, where Λ is given by (27).*

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