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A Necessary and Sufficient Condition for High-Frequency Robustness of Non-Strictly-Proper Feedback Systems*

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Abstract

We consider stability and robustness of feedback systems, where plant and compensator need not be strictly proper. In an earlier paper [1] we described a functional R_∞ which, when negative, guarantees closed-loop instability as a result of parasitic interactions in the feedback loop. In our main result, Theorem 5, we prove that, when $R_\infty > 0$, there exist perturbations of plant and compensator from a narrow class which result in closed-loop stability and convergence. Hence, we may view $R_\infty > 0$ as a necessary and sufficient condition for closed-loop robustness in non-strictly-proper feedback loops.

1 Introduction

Consider the multivariable feedback system in Figure 1, where $\mathbf{P}(s)$ and $\mathbf{C}(s)$ are matrices of rational functions

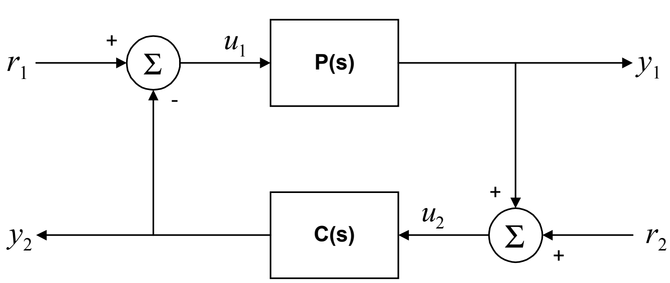

Figure 1: Feedback System

with real coefficients. In [1] we discuss stability and robustness of such systems when $\mathbf{P}$ and $\mathbf{C}$ are not assumed to be strictly proper. Our analysis hinges on the return difference

$$R = \det(I + \mathbf{CP})$$

and its high-frequency limit

$$R_\infty = \lim_{\sigma \to \infty} R(\sigma) \in [-\infty, \infty].$$

The closed-loop system in Figure 1 is governed by the transfer function matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{P}(I + \mathbf{CP})^{-1} & -\mathbf{P}(I + \mathbf{CP})^{-1}\mathbf{C} \\ (I + \mathbf{CP})^{-1}\mathbf{CP} & (I + \mathbf{CP})^{-1}\mathbf{C} \end{bmatrix}.$$

Henceforth, we adopt the assumption that $\mathbf{H}$ is BIBO stable. In particular, this implies that H is proper, so

$$(I + \mathbf{CP})^{-1} = I - \mathbf{C}\left(\mathbf{P}(I + \mathbf{CP})^{-1}\right)$$

is proper, R is not strictly proper, and $R_\infty \neq 0$.

A natural approach to studying stability and robustness of Figure 1 is to examine strictly proper perturbations of $\mathbf{P}$ and $\mathbf{C}$. Then conventional feedback theory can be utilized. For technical reasons, we need additional assumptions. We say that $\mathbf{P}_k \to \mathbf{P}$ *weakly*, if there exists $\sigma < \infty$ such that

W1) $\mathbf{P}_k$ has no pole in $[\sigma, \infty)$ for large k,
W2) $\mathbf{P}_k \to \mathbf{P}$ pointwise on $[\sigma, \infty)$.

Suppose we construct weakly convergent sequences $\mathbf{P}_k \to \mathbf{P}$ and $\mathbf{C}_k \to \mathbf{C}$. Letting $R_k = \det(I + \mathbf{C}_k \mathbf{P}_k)$, it is obvious from the definition of weak convergence that $R_k \to R$ weakly. The zeros of R are poles of $\mathbf{H}$, and the zeros of R_k are poles of the perturbed closed-loop system $\mathbf{H}_k$. In [1] we prove the following result.

Theorem 1 *If $\mathbf{P}_k$ and $\mathbf{C}_k$ are strictly proper, $\mathbf{P}_k \to \mathbf{P}$ and $\mathbf{C}_k \to \mathbf{C}$ weakly, and $R_\infty < 0$, then there exist $\sigma_k \in \mathbb{R}$ such that $\sigma_k \uparrow \infty$ and $R_k(\sigma_k) = 0$ for every k.*

Theorem 1 says that, under very mild assumptions, $R_\infty < 0$ guarantees that $\mathbf{H}_k$ has a high-frequency pole σ_k on the positive real axis, guaranteeing extreme instability of the closed-loop system. Our objective in this paper is to show that, when $R_\infty > 0$, we have the opposite situation − viz. that the closed-loop system is robust to certain reasonable perturbations of $\mathbf{P}$ and $\mathbf{C}$.

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2 Preliminaries

We begin by recalling some basic facts about rational matrices and their state-space realizations. The *characteristic polynomial* Δ_p of a rational matrix $\mathbf{P}$ is the least common denominator of all minors of $\mathbf{P}$. If $\mathbf{P}$ is strictly proper, its *McMillan degree* is $\nu(\mathbf{P}) = \deg \Delta_p$. Δ_p is also the characteristic polynomial of any state-space realization of minimal dimension (i.e. any controllable and observable realization). Appropriate extensions of realization theory to the case of non-strictly-proper $\mathbf{P}$ are developed in [3] and summarized in [4], Theorem 1.2. If $\mathbf{P}$ is non-strictly-proper, we may perform entrywise polynomial division to obtain $\mathbf{P} = \mathbf{P}_s + \mathbf{P}_f$, where $\mathbf{P}_s$ is strictly proper and $\mathbf{P}_f$ is polynomial. Let $\mathcal{R}$ be the operator on the space of rational matrices defined by

$$\mathcal{R}(\mathbf{P})(s) = -\frac{1}{s}\mathbf{P}\left(\frac{1}{s}\right).$$

Then $\mathcal{R}(\mathbf{P}_f)$ is strictly proper and we may define the *degree* of $\mathbf{P}$ according to

$$\mu(\mathbf{P}) = \nu(\mathbf{P}_s) + \nu(\mathcal{R}(\mathbf{P}_f)).$$

Our analysis hinges on state-space realizations of $\mathbf{P}$ and $\mathbf{C}$. Suppose $\mathbf{P}$ has realization

$$\begin{aligned} E\dot{x} &= Ax + Bu_1 \\ y_1 &= Cx \end{aligned} \tag{1}$$

with minimal dimension. (See [8] for basic information on singular systems.) Then $\mathbf{P}(s) = C(sE-A)^{-1}B$ and, from [3], (E, A, B, C) is a controllable and observable 4-tuple with $\mu(\mathbf{P})$ states. The characteristic polynomial of (1) is

$$\Delta_p(s) = \det(sE - A).$$

It can be shown that

$$\deg \Delta_p \leq \operatorname{rank} E \tag{2}$$

with equality iff $\mathbf{P}$ is proper. Applying the Weierstrass decomposition to (1) (see [5], Ch.12), we obtain

$$\begin{aligned} M_pEN_p &= \begin{bmatrix} I_{n_{ps}} & 0 \\ 0 & A_f \end{bmatrix},\ M_pAN_p = \begin{bmatrix} A_s & 0 \\ 0 & I_{n_{pf}} \end{bmatrix}, \\ M_pB &= \begin{bmatrix} B_s \\ B_f \end{bmatrix},\ CN_p = \begin{bmatrix} C_s & C_f \end{bmatrix}, \end{aligned}$$

where M_p and N_p are nonsingular and A_f is nilpotent. Letting

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = N_p^{-1}x$$

leads to the decoupled state-space system

$$\begin{bmatrix} I_{n_{ps}} & 0 \\ 0 & A_f \end{bmatrix}\begin{bmatrix} \dot{x}_s \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & I_{n_{pf}} \end{bmatrix}\begin{bmatrix} x_s \\ x_f \end{bmatrix} + \begin{bmatrix} B_s \\ B_f \end{bmatrix}u_1$$

$$y_1 = \begin{bmatrix} C_s & C_f \end{bmatrix}\begin{bmatrix} x_s \\ x_f \end{bmatrix}.$$

Then (1) has transfer function matrix

$$\mathbf{P}(s) = C_s(sI - A_s)^{-1}B_s + C_f(sA_f - I)^{-1}B_f,$$

and characteristic polynomial

$$\Delta_p(s) = \alpha \det(sI - A_s)\det(sA_f - I) \tag{3}$$

for some constant $\alpha \neq 0$. Note that $\mathbf{P}$ is proper iff $A_f = 0$.

Similar statements can be made about $\mathbf{C}$, yielding a realization

$$\begin{aligned} J\dot{z} &= Fz + Gu_2 \\ y_2 &= Hz, \end{aligned} \tag{4}$$

a Weierstrass decomposition

$$\begin{aligned} M_cJN_c &= \begin{bmatrix} I_{n_{cs}} & 0 \\ 0 & F_f \end{bmatrix},\ M_cFN_c = \begin{bmatrix} F_s & 0 \\ 0 & I_{n_{cf}} \end{bmatrix} \\ M_cG &= \begin{bmatrix} G_s \\ G_f \end{bmatrix},\ HN_c = \begin{bmatrix} H_s & H_f \end{bmatrix}, \end{aligned} \tag{5}$$

and characteristic polynomial

$$\Delta_c(s) = \beta \det(sI - F_s)\det(sF_f - I). \tag{6}$$

Then $\mathbf{H}$ has minimal realization

$$\begin{bmatrix} E & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & -BH \\ GC & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & H \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix},$$

which can be written

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & A_f & 0 \\ 0 & 0 & 0 & F_f \end{bmatrix}\begin{bmatrix} \dot{x}_s \\ \dot{z}_s \\ \dot{x}_f \\ \dot{z}_f \end{bmatrix} = \begin{bmatrix} A_s & -B_sH_s & 0 & -B_sH_f \\ G_sC_s & F_s & G_sC_f & 0 \\ 0 & -B_fH_s & I & -B_fH_f \\ G_fC_s & 0 & G_fC_f & I \end{bmatrix}\begin{bmatrix} x_s \\ z_s \\ x_f \\ z_f \end{bmatrix} + \begin{bmatrix} B_s & 0 \\ 0 & G_s \\ B_f & 0 \\ 0 & G_f \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_s & 0 & C_f & 0 \\ 0 & H_s & 0 & H_f \end{bmatrix}\begin{bmatrix} x_s \\ z_s \\ x_f \\ z_f \end{bmatrix},$$

The closed-loop characteristic polynomial is

$$\Delta_{cl}(s) = \det \begin{bmatrix} sI - A_s & B_s H_s & 0 & B_s H_f \\ -G_s C_s & sI - F_s & -G_s C_f & 0 \\ 0 & B_f H_s & sA_f - I & B_f H_f \\ -G_f C_s & 0 & -G_f C_f & sF_f - I \end{bmatrix}. \tag{7}$$

BIBO stability of $\mathbf{H}$ implies properness of $\mathbf{H}$; so, as in (2), $\deg \Delta_{cl} = n_{ps} + n_{cs} + \rho$, where

$$\rho = \operatorname{rank} A_f + \operatorname{rank} F_f.$$

From [6], p.159, we know that

$$R = \frac{\Delta_{cl}}{\Delta_p \Delta_c}. \tag{8}$$

Let

$$\gamma_\rho s^\rho + ... + \gamma_0 = \det \begin{bmatrix} I - sA_f & -B_f H_f \\ G_f C_f & I - sF_f \end{bmatrix}. \tag{9}$$

Recalling

$$\det(sA_f - I) = (-1)^{n_{pf}}, \ \det(sF_f - I) = (-1)^{n_{cf}},$$

we obtain

$$R_\infty = \begin{cases} \gamma_\rho, & \mathbf{P} \text{ and } \mathbf{C} \text{ proper}, \\ \gamma_\rho \cdot \infty, & \mathbf{P} \text{ or } \mathbf{C} \text{ improper}. \end{cases}$$

Note that $\mathbf{P}$ and $\mathbf{C}$ proper implies

$$R_\infty = \det \begin{bmatrix} I & -B_f H_f \\ G_f C_f & I \end{bmatrix} = \det(I + B_f H_f G_f C_f). \tag{10}$$

3 Sufficiency of $R_\infty > 0$

Merely showing that $R_\infty > 0$ guarantees closed-loop robustness to certain weak perturbations would not be an acceptable result, since the class of weak perturbations is so large. To obtain a better result, we limit our analysis to the narrowest perturbation class normally encountered in singular perturbation problems. As an initial step, we consider rational functions

$$\begin{aligned} f_k(s) &= \frac{b_{qk}s^q + ... + b_{0k}}{a_{rk}s^r + ... + a_{0k}} \\ f(s) &= \frac{b_m s^m + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0} \end{aligned}$$

and say that $f_k \to f$ *parametrically* if

$$\begin{aligned} q &\geq m, \quad r \geq n \\ a_{ik} &\to a_i; \quad i = 0, ..., n-1 \\ a_{nk} &\to 1 \\ a_{ik} &\to 0; \quad i = n+1, ..., r \\ b_{ik} &\to b_i; \quad i = 0, ..., m \\ b_{ik} &\to 0; \quad i = m+1, ..., q. \end{aligned}$$

For matrices, we say $\mathbf{P}_k \to \mathbf{P}$ *parametrically* if each entry converges parametrically. We say $\mathbf{P}_k \to \mathbf{P}$ *strongly*, if

S1) $\mathbf{P}_k \to \mathbf{P}$ parametrically,

S2) $\mu(\mathbf{P}_k) = \mu(\mathbf{P})$ for large k,

S3) there exists $\varepsilon > 0$ and $K < \infty$ such that, when $k > K$, no finite pole λ_{ik} of $\mathbf{P}_k$ satisfies both $|\lambda_{ik}| > \frac{1}{\varepsilon}$ and $|\arg \lambda_{ik}| < \frac{\pi}{2} + \varepsilon$.

Condition S3) is equivalent to saying that the divergent poles of $\mathbf{P}_k$ lie in a fixed left half-plane sector. S1) and S3) together imply weak convergence. Furthermore, it is shown in [4] that S1) guarantees that $\mathbf{P}_k$ and $\mathbf{C}_k$ have realizations of the form (1) with convergent matrices. Let $\mathcal{L}^{-1}$ denote the inverse Laplace transform operator. Then, from S3) and [7], Theorem 1, $\mathcal{L}^{-1}\{\mathbf{P}_k\} \to \mathcal{L}^{-1}\{\mathbf{P}\}$ and $\mathcal{L}^{-1}\{\mathbf{C}_k\} \to \mathcal{L}^{-1}\{\mathbf{C}\}$ as distributions. (See [2] for a discussion of distributional convergence.) In addition, we can show that the inverse transforms converge uniformly on compact subintervals of $(0, \infty)$. Hence, strong convergence embodies all the properties that are normally encountered in classical singular perturbation problems. We will eventually prove that, for $R_\infty > 0$, the closed-loop system is robust to certain strong, strictly proper plant and compensator perturbations.

Next we study a class of perturbations of $\mathbf{P}$ and $\mathbf{C}$ obtained by choosing $A_{fk} \to A_f$ and $F_{fk} \to F_f$ and substituting A_{fk} and F_{fk} into (**??**) and (5) in place of A_f and F_f. Recall that the *index* $\operatorname{ind} A$ of a square matrix A is the smallest integer $p \geq 1$ such that $\operatorname{rank} A^p = \operatorname{rank} A^{p+1}$. It is easy to show that $\operatorname{ind} A = 1$ is equivalent to having $\operatorname{rank} A$ nonzero eigenvalues in A, counting multiplicities.

Lemma 2 *Let* $A_{fk} \to A_f$ *and* $F_{fk} \to F_f$, *where* $\operatorname{rank} A_{fk} = \operatorname{rank} A_f$, $\operatorname{rank} F_{fk} = \operatorname{rank} F_f$, $\operatorname{ind} A_{fk} = \operatorname{ind} F_{fk} = 1$, *and every nonzero eigenvalue* λ_{ik} *of* A_{fk} *and* F_{fk} *satisfies* $\operatorname{Re} \lambda_{ik} < 0$ *for large* k. *Then* $\mathbf{P}_k$, $\mathbf{C}_k$, *and* $\mathbf{H}_k$ *are proper and* $R_{k\infty} \to R_\infty$.

Proof. If $\mathbf{P}$ and $\mathbf{C}$ are proper, then $A_f = A_{fk} = 0$ and $F_f = F_{fk} = 0$, so $\mathbf{P}_k = \mathbf{P}$, $\mathbf{C}_k = \mathbf{C}$, $\mathbf{H}_k = \mathbf{H}$, and $R_{k\infty} = R_\infty$. Suppose $\mathbf{P}$ and $\mathbf{C}$ are not both proper. Since A_{fk} and F_{fk} have unit index, the corresponding $\mathbf{P}_k$ and $\mathbf{C}_k$ are proper. Despite the fact that A_{fk} and F_{fk} may not be nilpotent, we may substitute them for A_f and F_f in (3), (6), and (7), yielding Δ_{pk}, Δ_{ck}, and Δ_{clk}. Applying (2) to (7), we obtain properness of $\mathbf{H}_k$. Applying (8) to the perturbed system, we obtain

$$R_{k\infty} = \lim_{\sigma \to \infty} \Gamma_k(\sigma),$$

where

$$\Gamma_k(s) = \frac{\det \begin{bmatrix} sA_{fk} - I & B_f H_f \\ -G_f C_f & sF_{fk} - I \end{bmatrix}}{\det(sA_{fk} - I)\det(sF_{fk} - I)} = \frac{\gamma_{\rho k}s^\rho + ... + \gamma_{0k}}{\prod_{i=1}^{\rho}(1 - \lambda_{ik}s)} \tag{11}$$

with $\gamma_{ik} \to \gamma_i$. Thus

$$R_{k\infty} = \frac{\gamma_{\rho k}}{\prod_{i=1}^{\rho} (-\lambda_{ik})}. \tag{12}$$

Since the denominator of (12) is positive, real, and converging to 0, $R_{k\infty} \to \gamma_\rho \cdot \infty = R_\infty$. ■

Strongly convergent sequences $\mathbf{P}_k$ and $\mathbf{C}_k$ satisfying the conditions of Lemma 2 are easily constructed. For example, let $\mathbf{P}$ have minimal realization (**??**), and suppose A_f has Jordan form

$$T^{-1}A_fT = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_l \end{bmatrix}, \; J_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}. \tag{13}$$

Let

$$\begin{aligned} A_{fk} &= T \begin{bmatrix} J_{1k} & & \\ & \ddots & \\ & & J_{lk} \end{bmatrix} T^{-1}, \qquad (14) \\ J_{ik} &= \begin{bmatrix} -\frac{1}{k} & 1 & & \\ & \ddots & \ddots & \\ & & -\frac{1}{k} & 1 \\ & & & 0 \end{bmatrix}. \end{aligned}$$

Lemma 3 *Suppose A_{fk} and F_{fk} are constructed as in (13) and (14). Then $\mathbf{P}_k$, $\mathbf{C}_k$, and $\mathbf{H}_k$ are proper, $\mathbf{P}_k \to \mathbf{P}$, $\mathbf{C}_k \to \mathbf{C}$, and $\mathbf{H}_k \to \mathbf{H}$ strongly, and $R_{k\infty} \to R_\infty$.*

Proof. The conditions of Lemma 2 obviously hold, guaranteeing properness of $\mathbf{P}_k$, $\mathbf{C}_k$, and $\mathbf{H}_k$ and convergence of $R_{k\infty}$. Also, S1) and S2) are obvious for $\mathbf{P}_k$, $\mathbf{C}_k$, and $\mathbf{H}_k$. The divergent poles of $\mathbf{P}_k$ and $\mathbf{C}_k$ are just $\lambda_{ik} = -k$, so S3) holds. From (2) and (7),

$$n_{ps} + n_{cs} \geq \deg \Delta_{clk} \geq \deg \Delta_{cl} = n_{ps} + n_{cs}$$

so $\deg \Delta_{clk}$ is constant. Hence $\mathbf{H}_k$ has no divergent poles and S3) holds vacuously. ■

Before we state our main theorem, we need one more preliminary result.

Lemma 4 *A square matrix M is the product of two stable matrices iff $\det M > 0$ and $M \neq -\alpha I$ for any $\alpha > 0$.*

Proof. (Necessary) Let $M = \Sigma\Pi$, where Σ and Π are stable with eigenvalues $\{\sigma_i\}$ and $\{\pi_i\}$, respectively. Then

$$\det M = \left(\prod \sigma_i\right)\left(\prod \pi_i\right) > 0.$$

If $M = -\alpha I$ with $\alpha > 0$, then $\Sigma^{-1} = -\frac{1}{\alpha}\Pi$ is unstable. Stability of Σ yields a contradiction.

(Sufficient) A proof of the converse is too long to present here in detail. The general idea is to first construct a nonsingular matrix T such that every leading principal minor of $T^{-1}MT$ is positive. Second, find a lower triangular triangular $\tilde{\Sigma}$ and an upper triangular $\tilde{\Pi}$ such that $T^{-1}MT = \tilde{\Sigma}\tilde{\Pi}$. These constructions can be performed using standard matrix manipulations and an inductive argument. ■

Theorem 5 *If $R_\infty > 0$, then there exist strictly proper sequences $\mathbf{P}_k$ and $\mathbf{C}_k$ such that $\mathbf{P}_k \to \mathbf{P}$, $\mathbf{C}_k \to \mathbf{C}$, and $\mathbf{H}_k \to \mathbf{H}$ strongly and $\mathbf{H}_k$ is proper for large k.*

Proof. Our construction proceeds in four stages. First, we assume $\mathbf{P}$ is strictly proper and $\mathbf{C}$ is proper. This means $n_{pf} = 0$ and $F_f = 0$. Let $\mathbf{P}_k = \mathbf{P}$ and $\mathbf{C}_k$ be determined by setting $F_{fk} = \frac{1}{k}\Sigma_1$, where Σ_1 is any stable matrix. Then $\mathbf{C}_k$ and $\mathbf{H}_k$ are obviously strictly proper and satisfy S1) and S2). Since the divergent poles of $\mathbf{C}_k$ are just the eigenvalues of $k\Sigma_1^{-1}$, S3) follows for $\mathbf{C}_k$. From [9], Corollary, 2.1, $\mathbf{H}_k$ satisfies S3).

Now assume that $\mathbf{P}$ and $\mathbf{C}$ are proper and that $I + B_fH_fG_fC_f \neq -\alpha I$ for any $\alpha > 0$. Then, by Lemma 4 and (10), there exist stable matrices Σ_2 and Π such that $I + B_fH_fG_fC_f = \Sigma_2\Pi$, so $\Sigma_2^{-1}(I + B_fH_fG_fC_f)$ is stable. Let $A_{fk} = \frac{1}{k}\Sigma_2$ and fix $F_f = 0$. Then $\mathbf{P}_k$ is strictly proper and $\mathbf{P}_k \to \mathbf{P}$ strongly. Letting $\mathbf{C}_k = \mathbf{C}$, [9], Corollary 2.1, again implies $\mathbf{H}_k \to \mathbf{H}$ strongly with $\mathbf{H}_k$ proper. Based on the construction in the preceding paragraph, for each k we can find a sequences $\mathbf{P}_{kj}$ $(= \mathbf{P}_k)$ and $\mathbf{C}_{kj}$ such that $\mathbf{C}_{kj}$ is strictly proper and $\mathbf{C}_{kj} \to \mathbf{C}_k$ and $\mathbf{H}_{kj} \to \mathbf{H}_k$ strongly as $j \to \infty$ for every k. Hence, there exists a sequence of integers $j_k \uparrow \infty$ such that $\mathbf{P}_{kj_k}$, $\mathbf{C}_{kj_k}$, and $\mathbf{H}_{kj_k}$ are strongly convergent.

Next, assume $\mathbf{P}$ and $\mathbf{C}$ are proper, but $I + B_fH_fG_fC_f = -\alpha I$ for some $\alpha > 0$. Then there exists a sequence $B_{fk} \to B_f$ such that $I + B_{fk}H_fG_fC_f \neq -\beta_k I$ for any $\beta_k > 0$. Fix $A_f = 0$ and $F_f = 0$, but use B_{fk} in place of B_f in (**??**). Then $\mathbf{P}_k$ is proper, $\mathbf{C}_k = \mathbf{C}$, and $\mathbf{P}_k \to \mathbf{P}$ strongly, since its poles are constant. As in the proof of Lemma 3, $\deg \Delta_{clk}$ is constant; hence, $\mathbf{H}_k \to \mathbf{H}$ strongly with $\mathbf{H}_k$ proper. As in the preceding paragraph, we may construct $\mathbf{P}_{kj}$, $\mathbf{C}_{kj}$, and j_k such that $\mathbf{P}_{kj_k}$, $\mathbf{C}_{j_k}$, and $\mathbf{H}_{kj_k}$ are strongly convergent.

Finally, suppose $\mathbf{P}$ and $\mathbf{C}$ are not both proper. Then, from Lemma 3, the construction (13) and (14) yields strongly convergent proper sequences $\mathbf{P}_k$, $\mathbf{C}_k$, and $\mathbf{H}_k$ with $R_{k\infty} \to R_\infty$. From the preceding paragraph, we may construct $\mathbf{P}_{kj}$, $\mathbf{C}_{kj}$, and j_k such that $\mathbf{P}_{kj_k}$, $\mathbf{C}_{j_k}$, and $\mathbf{H}_{kj_k}$ are strongly convergent. ■

We note that condition S3) along with properness of $\mathbf{H}_k$ imply BIBO stability of $\mathbf{H}_k$ for large k. Hence, the construction in the proof of Theorem 5 yields strong perturbations of $\mathbf{P}$ and $\mathbf{C}$ under which the closed-loop system is robustly stable. This establishes sufficiency of $R_\infty > 0$.

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