CONDITIONS FOR EXISTENCE OF AN OPTIMAL CONTROL IN THE LQ REGULATOR PROBLEM FOR SINGULAR SYSTEMS

J. DANIEL COBB
Department of Electrical and Computer Engineering
University of Wisconsin
1415 Johnson Drive
Madison, WI 53706-1691

INTRODUCTION

We consider the problem of minimizing the cost functional

$$J(x,u) = \int_0^\infty [x^T \; u^T] \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \tag{1}$$

subject to the singular system constraint

$$E\dot{x} = Ax + Bu \tag{2}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, E is singular, $R > 0$, and $\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geq 0$.

Certain fundamental questions need to be answered concerning the relationship between the matrices Q, N, R, P, E, A, and B and the optimal pairs (x^*, u^*). The most obvious of these follow.

1) Under what conditions does there exist at least one optimal (x^*, u^*) and when is it unique?
2) To what function spaces do the (x^*, u^*) belong?
3) When can an optimal control u^* be implemented according to a feedback law on x^*?
4) How can one compute the optimal pairs efficiently?

For the case $N = 0$ and $Q > 0$, we answered questions 1)-3) in [1]. Question 4) is treated for the general case by Bender and Laub in [2] using necessary conditions derived by Jonchkeere [3]. Jonchkeere also investigates conditions under which his necessary conditions exhibit existence and uniqueness of solutions. The question of existence of an optimal pair (x^*, u^*) when $N \neq 0$ or when Q is singular has remained open and motivates our present work.

As a first attempt at a solution, we reformulate the optimization problem by "completing the square" in $J(x,u)$. Let $\hat{u} = u + R^{-1}N^Tx$, $\hat{Q} = Q - NR^{-1}N^Tx$, and $\hat{A} = A - BR^{-1}N^T$. Then we may rewrite the cost (1) and system constraint (2) as

$$J(x,\hat{u}) = \int_0^\infty x^T\hat{Q}x + \hat{u}^TRu \; dt, \quad E\dot{x} = \hat{A}x + B\hat{u} \tag{3}$$

This yields an equivalent form which can be handled by the methods of [1] as long as $\hat{Q} > 0$ and $\det(sE - \hat{A}) \not\equiv 0$. If either condition fails to hold, some new theory is required.

HILBERT SPACE FORMULATION

We first note that, as in [1], we must take $J = \infty$ whenever the integrand of (3) is not in L^2 (e.g. if it contains impulses). Let $\mathcal{D}_+$ be the set of all distributions with support in $[0,\infty)$ and let $\mathcal{D}_0$ be the subspace of $\mathcal{D}_+^{n+m}$ consisting of all elements z satisfying $Cz = 0$, where $C^TC = \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix}$. Also let $\bar{\mathcal{D}}$ be the quotient space of $\mathcal{D}_+^{n+m}$ with respect to $\mathcal{D}_0$,

$$<[z_1],[z_2]> = \int_0^\infty z_1{}^T C^T C z_2 \, dt \qquad (4)$$

where $[z] \in \bar{\mathcal{D}}$ is the equivalence class determined by z, and $\mathcal{L}_2[0,\infty) = \{d \in \bar{\mathcal{D}} \,|\, \bar{J}(q) < \infty\}$. We can prove the following

<u>Lemma</u> Equipped with the inner product (4), $\mathcal{L}_2[0,\infty)$ is a Hilbert space.

Consider now the system equation (2). We define $\mathcal{N}(x_0) \in \bar{\mathcal{D}}$ to be the equivalence class generated by the natural response of (2) due to the initial condition x_0. Also let

$$\Lambda = \{[z] \,\Big|\, z = \begin{bmatrix} x \\ u \end{bmatrix} \text{ and } x \text{ is the forced response of (2) due to } u\}$$

The original optimization problem is easily seen to be equivalent to the problem of minimizing $\bar{J}(w) = <w,w>$ over $\mathcal{N}(x_0) + \Lambda$.

<u>Theorem</u> If the system (2) is stabilizable and impulse controllable (see [ ]), $(\mathcal{N}(x_0) + \Lambda) \cap \mathcal{L}_2[0,\infty) \neq \phi$ and $(\mathcal{N}(x_0) + \Lambda) \cap \mathcal{L}_2[0,\infty)$ is a closed linear variety in $\mathcal{L}_2[0,\infty)$.

The Hilbert space projection theorem thus implies that a unique optimum exists in $\mathcal{L}_2[0,\infty)$ under the same conditions as stated in [1]. Hence, an optimal pair (x^*,u^*) exists for the original problem under these same conditions. The pair (x^*,u^*) in general is not unique, however, since the optimum $w^* \in \mathcal{L}_2[0,\infty)$ may be generated by many solutions pairs of (2).

[1] J. D. Cobb, "Descriptor Variable Systems and Optimal State Regulation", *IEEE Transactions on Automatic Control*, Vol. 28, No. 5, May 1983, 601-611.

[2] D. J. Bender, A. J. Laub, "The Linear-Quadratic Optimal Regulator for Descriptor Systems," *IEEE Transactions on Automatic Control*, in press.

[3] E. Jonckheere, "Variational Calculus for Descriptor Systems," *IEEE Transactions on Automatic Control*, in press.