STRUCTURALLY STABLE FEEDBACK CONTROL OF SINGULAR SYSTEMS

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ABSTRACT

Optimal regulation of linear systems of the form $E\dot{x} = Ax+Bu$, with E singular and quadratic cost, is considered. It is shown that the nonuniqueness of the optimizing feedback matrix can be exploited to give a closed loop system which remains stable in the presence of small perturbations (structurally stable). Although the problem of structural stability is not fully understood in general, a complete solution is given for an important special case.

INTRODUCTION

Systems of the form

$$E\dot{x} = Ax+Bu \tag{1}$$

have been treated in [1]-[5]. In [4] it is demonstrated that (1) may be decomposed into two subsystems

$$\dot{x}_s = A_s x_s + B_s u \tag{2a}$$

$$A_f \dot{x}_f = x_f + B_f u \tag{2b}$$

of dimension r and n-r respectively, where A_f is nilpotent and $x = \begin{bmatrix} x_s \\ x_f \end{bmatrix}$. It is shown in [5] that subsystem (2b) may have impulses present in the unforced solution. We have proven in [2] that all impulses in (2b) may be eliminated by applying a feedback matrix to the system if and only if

$$\mathrm{Im} A_f + \mathrm{Ker}\, A_f + \mathrm{Im}\, A_f = \mathbb{R}^{n-r} \tag{3}$$

We have also proven in [3] that an input u* exists which minimizes

$$J = \int_0^\infty ||x(t)||^2 + ||u(t)||^2 dt \tag{4}$$

with respect to (1) if and only if (2a) is stabilizable and (3) holds. In this case, u* is unique and can be implemented with a feedback matrix K which is not unique. (Impulses must necessarily be eliminated for optimality.) We are interested in distinguishing among the various choices of K.

The following example illustrates the problem:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5}$$

Such a system may arise from a singular perturbation problem involving

$$\begin{bmatrix} -\varepsilon & 1 \\ 0 & -\varepsilon \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{6}$$

where $\varepsilon > 0$. For $u = 0$ and $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, (6) has the solution

$$x_\varepsilon(t) = \begin{bmatrix} -\frac{t}{\varepsilon^2} e^{-\frac{t}{\varepsilon}} \\ e^{-\frac{t}{\varepsilon}} \end{bmatrix} \tag{7}$$

As $\varepsilon \to o^+$, $x_\varepsilon \to \begin{bmatrix} -\delta \\ o \end{bmatrix}$. The limit of x_ε can be shown to be the solution of (5) for the same u^o and $x(o)$. Following [3], optimality is achieved for all feedback matrices of the form

$$K = [\alpha \quad \sqrt{2}\alpha - 1] \tag{8}$$

where $\alpha \neq o$. If we choose $\alpha > o$ the solution of the closed-loop system converges as $\varepsilon \to o^+$. However, for $\alpha < o$, the closed loop system has one eigenvalue tending to $+\infty$. Clearly, this is an undesirable situation. We would like to identify, in the general case, the class of feedback matrices which are not only optimal at $\varepsilon = o$, but which also yield convergence of solutions as $\varepsilon \to o^+$.

PROBLEM FORMULATION

Actually, we will consider a more general type of perturbation than the one described above. Our main assumption will be that, whatever perturbation of (1) exists, the corresponding solutions of (1) converge as $\varepsilon \to o^+$ for every possible value of $x(o)$. This, after all, is saying nothing more than that (1) is a "good" idealization of the physical system being modelled

Definition: We say that a feedback matrix K yields a structurally stable closed-loop system if the closed-loop solutions converge under all perturbations which guarantee convergence of solutions of (1).

Symbolically, the situation can be described as follows: Applying feedback K to (1) yields a system which can be decomposed as in (2) to give

$$\dot{x}_{sK} = A_{sK}x_{sK} + B_{sK}u \tag{9a}$$

$$A_{fK}\,\dot{x}_{fK} = x_{fK} + B_{fK}u \tag{9b}$$

The closed-loop system is structurally stable if and only if $e^{tA_{fK}^{-1}}$ is convergent whenver $e^{tA_f^{-1}}$ converges. (Here we are only considering perturbations which make E nonsingular. The general case can also be handled in this framework, but with increased notational complexity.)

The central problem is that of determining which values of K from the optimal class yield a structurally stable system. Structural instability is clearly unacceptable since the idealized closed-loop model would not accurately predict the behaviour of the physical system in question.

THE CASE OF CYCLIC A_f

When A_f in (2b) is cyclic at $\varepsilon = o$, a solution is readily obtained. We are able to prove the following series of results leading up to an algorithm for choosing K:

1) There exists a subspace S^* with $S^* \oplus \mathrm{Ker}E = \mathbb{R}^n$ such that the class of optimizing feedback matrices is the linear variety in $\mathbb{R}^{m\times n}$ formed by adding to any optimal K all matrices $\bar{K}$ satisfying

$$\mathrm{Ker}\ \bar{K} \supset S^* \tag{10}$$

2) As a result of 1), if the class of optimal feedback matrices is nonempty, it contains at least one member that yields structural stability.

3) If coordinates are chosen in (2b) so that A_f is in Jordan form at $\varepsilon = o$, a matrix K_f which is optimal for (2b) alone yields structural stability if and only if

$$b_{n-r}k_1 > o \quad \text{for } n\text{-}r \text{ odd} \tag{11a}$$

$$b_{n-r}k_1 < o \quad \text{for } n\text{-}r \text{ even} \tag{11b}$$

where $B_f = \begin{bmatrix} b_1 \\ \vdots \\ b_{n-r} \end{bmatrix}$, $K_f = [k_1 \cdots k_{n-r}]$.

4) If ε-dependent matrices E, A_1, and A_2 are given with $A_1(o) = A_2(o)$ and $E(o)\dot{x} = A_1(o)x$ a closed-loop optimal system, then the solutions of $E\dot{x} = A_1 x$ converge if and only if those of $E\dot{x} = A_2 x$ do also.

Utilizing 1) - 4), we have obtained an algorithm for finding an optimal feedback matrix which yields structural stability once <u>any</u> optimal matrix is known:

a) Starting with (1), change coordinates to decouple the system as in (2) with A_f in Jordan form at $\varepsilon = o$.

b) Alter the (r+1)th column of the given optimal K to satisfy (11) by adding an appropriate $\bar{K}$ satisfying (10). The altered K is also optimal and is guaranteed to yield structural stability.

c) Transform back to the original coordinates.

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