OBSERVABILITY AND IMPULSE OBSERVERS IN DESCRIPTOR-VARIABLE SYSTEMS*

DANIEL COBB
Department of Electrical Engineering
University of Toronto
Toronto, Ontario, Canada M5S 1A4

ABSTRACT

A theory of observability for systems $E\dot{x}=Ax+Bu$, $y=Cx$, E singular, which complements that for controllability, already existing in the literature, is presented. A natural extension of the state-space duality theorem is shown to hold, and an observer, which detects impulsive behavior, is studied.

1. INTRODUCTION

In this paper we discuss generalizations of the concepts of controllability and observability to systems of the form

$$E\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where E and A are n×n matrices, B is n×n and C is k×n. According to the development in Gantmacher [1], (1) and (2) may be decomposed (after a change of basis) according to $x = \begin{bmatrix} x_s \\ x_f \end{bmatrix}$ where

$$\dot{x}_s = A_s x_s + B_s u \tag{3}$$

$$A_f \dot{x}_f = x_f + B_f u \tag{4}$$

$$y = C_s x_s + C_f x_f \tag{5}$$

If

$$r = \deg|Es-A| \tag{6}$$

then x_s is an r-vector and A_f is nilpotent. The unique solution of (4) is

$$x_f = -\sum_{i=0}^{q-1} A_f^i B_f u^i \tag{7}$$

where q is the index of A_f and u^i denotes the ith derivative.

Inevitably, when considering such a system, one is faced with the problem of interpreting "inconsistent" initial conditions, as they are termed by Campbell [2], which may be thought of as points in the state

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space through which no solution passes for a given u. It can be shown that, in practice, such initial values may be achieved if the system operates in the presence of disturbances. This situation might result, for example, from the addition of a disturbance input v to the right side of (1) which, in a worst case analysis, would be multiplied by the identity matrix. Hence, by assigning t=0 to the instant when the disturbance disappears, any initial condition can result. It can also be shown (see [2], for example) that inconsistent initial conditions may lead to impulses and their derivatives in the natural response. Specifically,

$$x_f = - \sum_{i=1}^{q-1} \delta^{i-1} A_f^i x_{0f} \tag{8}$$

where x_{0f} is the component of the initial condition corresponding to subsystem (4). The impulses that appear in this deterministic approach to disturbances, reflect the differentiating structure of the system as determined by (7). Analogously, in a stochastic approach, this structure would result in high frequency amplification of noise.

The first attempts at extending the concepts of controllability and observability to the system (1) were made by Rosenbrock [3] and later by Verghese et al. [4]. Theirs is a frequency domain approach, controllability and observability being described in terms of infinite input and output decoupling zeros. Our work is closer to that of Yip et al. [5], who takes a time-domain approach, but does not account for inconsistent initial conditions. This fact does not cause any significant problems when it comes to aspects of controllability, since there the reachability of states for t>0 is the key issue. Observability, on the other hand, depends heavily on initial conditions. Ignoring inconsistent initial conditions leads to a definition of observability which is not the algebraic dual of controllability. These difficulties will be dealt with in Section 2.

In Section 3 we will discuss the control and observation of the impulsive behavior of (1). This will lead to system properties which do not appear in state-space theory. The results in Section 3 will be applied in Section 4 to the problem of constructing a dynamic observer which reproduces the impulsive portion of the internal variable x, given only input and output information. Finally, we will discuss the observer's use in a feedback compensation scheme which eliminates impulsive behavior.

2. OBSERVABILITY

We begin by reviewing the definition of controllability as given in [5]. Let

$$\mathcal{R}_s = \sum_{i=0}^{r-1} \mathrm{Im}(A_s^i B_s) \tag{9}$$

$$\mathcal{R}_f = \sum_{i=0}^{q-1} \mathrm{Im}(A_f^i B_f) \tag{10}$$

$$\mathcal{R} = \mathcal{R}_s \oplus \mathcal{R}_f \tag{11}$$

$\mathcal{R}$ is the controllable subspace corresponding to (1). Let (1) have initial condition $x_0 = \begin{bmatrix} x_{0s} \\ x_{0f} \end{bmatrix}$ and let C^{q-1} be the space of q-1 times continuously differentiable functions on $[0,\infty)$.

Definition

System (1) is controllable if for every $\tau>0$ and x_0, $w \in \mathbb{R}^n$ there exists $u \in C^{q-1}$ such that $x(\tau)=w$.

Theorem 1 (Yip)

Let $\tau>0$ and x_0, $w \in \mathbb{R}^n$. There exists $u \in C^{q-1}$ such that $x(\tau)=w$ iff $w \in \mathcal{R}$.

Corollary

The system (1) is controllable iff $\mathcal{R}=\mathbb{R}^n$.

We would like to define observability so that the dual nature of controllability and observability is preserved. In order to achieve this and to account for all initial conditions, it becomes necessary to restrict the system under consideration to $[0,\infty)$. In the state-space case this would not change the form of the system equation since all solutions are continuous. However, in our case the situation is complicated by the existence of distributions.

Let x_+ and y_+ be the distributions which 1) are equal to x and y on $(0,\infty)$, 2) vanish on $(-\infty,0)$, and 3) contain at the origin the same impulses and their derivatives as x and y. Then x_+ and y_+ satisfy the equations

$$E\dot{x}_+ = Ax_+ + Bu + \delta Ex(0^-) \tag{12}$$

$$y_+ = Cx_+ \tag{13}$$

In fact, the solution of (12) is unique within the space of distributions which vanish on $(-\infty,0)$. This formulation allows us to speak of system trajectories resulting from various initial conditions without having to consider the disturbances that generate those initial conditions.

We are now in a position to define observability. Let

$$N_s = \bigcap_{i=0}^{r-1} \mathrm{Ker}(C_s A_s^i) \tag{14}$$

$$N_f = \bigcap_{i=0}^{q-1} \mathrm{Ker}(C_f A_f^i) \tag{15}$$

$$N = N_s \oplus N_f \tag{16}$$

N is the unobservable subspace corresponding to (1). Let C_p^{q-1} be the space of q-1 times piecewise continuously differentiable functions on $[0,\infty)$.

Definition

The system (1) is observable if knowledge of $u \epsilon C_p^{q-1}$, y_+, and $y(0^-)$ are sufficient to determine $x(0^-)$.

Theorem 2

Let u=0 in (12). Then $y_+=0$, $y(0^-)=0$ iff $x(0^-) \epsilon N$.

Corollary

The system (1) is observable iff $N=0$.

The theorem can be proven by combining (8) with the decomposition of (12) and (13). Since the system is linear, Theorem 2 is equivalent to the statement that, for any given u, two initial conditions, whose difference lies in N, give rise to the same output. Hence, the corollary follows easily. Later we will see that observability, as we have defined it, is the algebraic dual of controllability.

3. IMPULSE CONTROLLABILITY AND OBSERVABILITY

Since it is unique to systems with a singular E matrix, impulsive behavior in x warrants special attention. In this section we outline a theory which emphasizes the control and observation of impulses. To facilitate the development, we denote by $x[\tau]$ the impulsive part of x at $t=\tau$. Similarly, let $y[\tau]$ be the impulsive part of y at τ. Then

$$E\dot{x}[\tau] = Ax[\tau] - \delta_\tau E(x(\tau^+) - x(\tau^-)) \tag{17}$$

$$y[\tau] = Cx[\tau] \tag{18}$$

where δ_τ is the unit impulse at τ. It can be shown that the solution of (17) is unique within the space of distributions with point support at τ. Decomposition of (17) reveals that

$$x_f[\tau] = \sum_{i=0}^{q-1} \delta_\tau^{i-1} A_f^i(x(\tau^+) - x(\tau^-)) \tag{19}$$

Hence, x can only contain impulses of the form (19). We would like to know which impulses of the class determined by (19) can actually be generated by choosing u appropriately, and which can be detected given only output information.

Definition

System (1) is impulse controllable if for every $\tau \in \mathbb{R}$, $w \in \mathbb{R}^{n-r}$ there exists $u \in C_p^{q-1}$ such that

$$x[\tau] = \sum_{i=1}^{q-1} \delta_\tau^{i-1} A_f^i w$$

Let

$$I_r = \sum_{i=1}^{q-1} \operatorname{Im}(A_f^i B_f) \tag{20}$$

We call I_r the impulse controllable subspace. This name is motivated by the following result.

Theorem 3

There exists $u \in C_p^{q-1}$ such that $x[\tau] = \sum_{i=1}^{q-1} \delta_\tau^{i-1} A_f^i w$ iff $\sum_{i=1}^{q-1} \delta_\tau^{i-1} A_f^i w \in I_r$.

Corollary

The following statements are equivalent:

1) System (1) is impulse controllable.
2) $I_r = \operatorname{Im} A_f$.
3) There exists an m×n matrix K such that $\deg|Es-(A+BK)| = \operatorname{rank} E$.

Theorem 3 can be proven in a manner similar to Theorem 2. The equivalence of conditions 2) and 3) in the corollary, was established in [6] where the problem of eliminating impulses due to inconsistent initial conditions (or, equivalently, due to arbitrary disturbances) was explored. The corollary of Theorem 3 indicates then that all impulsive transients can be eliminated with linear feedback iff they can be generated with an appropriate u.

We now consider the dual of impulse controllability.

Definition

System (1) is impulse observable if, for every $\tau \in \mathbb{R}$, knowledge of $y[\tau]$ is sufficient to determine $x[\tau]$.

Let

$$I_n = \bigcap_{i=1}^{q-1} \mathrm{Ker}(C_f A_f^i) \tag{21}$$

I_n is the impulse unobservable subspace.

Theorem 4

$y[\tau]=0$ iff $x(\tau^+)-x(\tau^-) \in I_n$.

Corollary

The following statements are equivalent:

1) System (1) is impulse observable.
2) I_n = Ker A_f.
3) There exists an n×k matrix K such that $\deg|Es-(A+KC)|=\mathrm{rank}\ E$.

The equivalence of 1) and 3) can be proven by the following duality theorem. Consider the dual system

$$E'\dot{x} = A'x + C'u \tag{22}$$

$$y = B'x \tag{23}$$

Theorem 5

1) System (1) is controllable iff the dual system is observable.
2) System (1) is impulse controllable iff the dual system is impulse observable.

The proof of this theorem is somewhat complicated, involving the relationship among the subspaces R_s, R_f, N_s, etc. for the two systems. The duality result serves as partial motivation for the exact definitions which we chose for observability. In the next section we will see that there is in fact a more pragmatic justification for our choice of definitions.

4. AN IMPULSE OBSERVER

In this section we consider the problem of constructing a dynamic system which utilizes input and output information from (1) and generates impulses equal to those of x, as given by (19). In particular we are looking for an impulse observer of the form

$$E\dot{z} = Gz - Ky + Hu \tag{24}$$

such that the estimate z exhibits the same impulsive behavior as x. Hence, we would like to find values for G, K, and H so that the error

$$e = z - x \tag{25}$$

contains no impulses.

The error e is governed by

$$E\dot{e} = Gz - Ky - Ax + (H-B)u \tag{26}$$

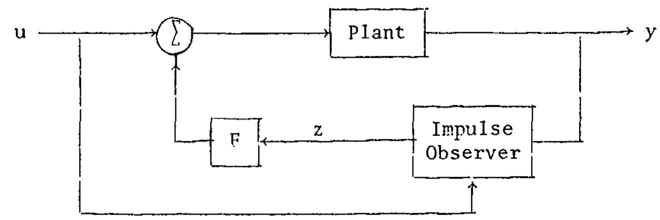
so if we choose H=B and G=A+KC then

$$E\dot{e} = (A+KC)e \tag{27}$$

From the corollary to Theorem 4, a matrix K can be found so that e has no impulses iff (1) is impulse observable.

The problem of stabilizing the observer still needs to be addressed. That this can be accomplished iff subsystem (3) is observable, follows from Theorem 5 and the results of [6]. Under mild conditions then a stable impulse observer can be constructed for the system (1) which identifies internal high frequency behavior in (1), due to random disturbances.

The results of [6], which deal with elimination of impulses via linear feedback when x is available, can now be augmented to handle the case where only y is available. To this end we consider the compensation scheme



where the plant is characterized by (1). The closed loop system may be modelled in terms of e and x yielding

$$\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BF & BF \\ 0 & A+KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \tag{28}$$

since u=Fz=Fx+Fe. Because of the triangular structure of (28), and the results of [6] and this paper, impulse controllability and impulse observability of (1) are necessary and sufficient to guarantee that (28) exhibits no impulsive behavior whatsoever. The eigenvalues of the closed-loop system are simply those of the plant and observer individually. They can be placed arbitrarily iff subsystem (3) is both controllable and observable.

5. CONCLUSIONS

The key to understanding the dual nature of controllability and observability clearly lies in the observation that all initial conditions may occur if the system operates in the presence of disturbances. This idea has led us to a natural extension of the duality theorem from state-space theory. We have also seen that, being unique to systems with a singular E matrix, impulsive behavior motivates the study of certain aspects of controllability and observability which do not appear when E is nonsingular. Finally, the problem of reorganizing system structure by applying feedback, so that internal differentiations are eliminated, has led to the concept of impulse observer. We have shown that such an observer can be used in a feedback loop in the same way that a standard observer is used in state-space theory. We believe that many other familiar results may be generalized in this way with novel interpretations occurring in many instances.

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