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A LYAPUNOV-BASED PROOF OF THE QUADRATIC SEPARATION PRINCIPLE FOR SYSTEMS WITH NOISE-FREE MEASUREMENTS

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Abstract

A simple proof of the separation principle for systems with noise-free measurements and quadratic cost is presented. The proof is entirely deterministic in nature and is based on Lyapunov's theorem.

1 Introduction

The celebrated LQG separation principle has long been considered one of the cornerstones of modern control theory. In contrast to the simplicity of the theorem statement, the task of penetrating the existing proofs of this result in the continuous-time case can be daunting to all but a few specialists in the area of stochastic systems. This statement is particularly surprising in view of the fact that in a fundamental sense *the essential ingredients of the LQG problem are not inherently stochastic*. Indeed, it has been demonstrated (see e.g., [1]) that the LQG problem can be recast into an entirely equivalent deterministic setting. What has been lacking, however, is a corresponding deterministic proof. This is the point of our paper.

The starting point for our work is the proof of the Separation Theorem supplied by Russell (see [2], Chapter 6, page 377), which is stochastic, but based on Lyapunov theory. In his proof the optimal compensator is obtained relative to the class of observer-based systems with order equaling that of the plant. Using an approach similar to Russell's, we present in Section 4 a simple deterministic proof, establishing optimality with respect to the class of all linear, time-invariant, finite-dimensional systems of arbitrary order. In our paper we consider only the case of disturbance-free measurements, although an equally elegant proof can be constructed for the case of measurements with disturbance.

In preparing the ground work for our result, a number of side issues are discussed. For instance, an optimal estimation problem for systems with noise-free measurements is formulated and observed to be dual to a certain singular optimal control problem (see [3]). We believe that this duality approach to the estimation problem is new and distinct from the traditional treatments of the problem in the stochastic systems literature.

2 Problem Formulation

Consider the class of completely controllable and observable Linear Time-Invariant (LTI) systems

$$\begin{aligned} \dot{x} &= Ax + Bu \; ; \quad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, and $y \in \mathbf{R}^p$. Assume that C has maximal rank i.e., $\rho[C] = p$, and is already in the form $[0 \; I_p]$. Note that this is not a restriction since an appropriate similarity transformation can always yield this form.

Given nonzero initial conditions x_0, we desire to regulate the system (1) in such a way that the quadratic cost functional

$$\tilde{J} = \int_0^\infty x^T Q x + u^T R u \, dt$$

is minimized. Since, minimizing this cost functional gives rise to optimal compensators which parametrically depend on the unknown initial conditions, we consider the normalized cost functional

$$J = \int_{\Omega_0} \int_0^\infty x^T Q x + u^T R u \, dt \, dV_n(x_0) \tag{2}$$

$$\Omega_0 = \{x_0 \in \mathbf{R}^n : x_0^T \Sigma x_0 \leq 1\} \, .$$

This cost functional may be interpreted as the "average value" of the quadratic cost $\tilde{J}$ as x_0 is confined to a Σ-weighted euclidean ball of radius one, with $\Sigma = \Sigma^T > 0$. Note that in (2) $dV_n(x_0)$ is the differential volume in $\mathbf{R}^n$.

The class of compensators considered are the completely controllable and observable LTI dynamic systems

$$\begin{aligned} \dot{z} &= Ez + Ly \; ; \quad z(0) = 0 \\ u &= Gz + Hy \end{aligned} \tag{3}$$

where $z \in \mathbf{R}^l$ and l is any positive integer. Note that the minimality assumption on (3) is not a restriction since the order l is not fixed. It is further assumed that $Q = Q^T \geq 0$ and $R = R^T > 0$. Moreover, the pair $(A, \sqrt{Q})$ is required to be detectable. It is possible to write the cost J in a more explicit form. To do this, we note that the closed-loop system equations are given by the unforced system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \mathcal{A} \begin{bmatrix} x \\ z \end{bmatrix} \; ; \quad \text{i.c.} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \tag{4}$$

$$\mathcal{A} \doteq \begin{bmatrix} A + BHC & BG \\ LC & E \end{bmatrix}$$

which has the solution

$$\begin{bmatrix} x \\ z \end{bmatrix} = e^{\mathcal{A}t}\mathcal{B}x_0 \qquad (5)$$

where $\mathcal{B}^T = [I_n \ 0]$. Incorporating (5) in the expression for J, we may write

$$J = \int_{\Omega_0}\int_0^\infty x_0^T\mathcal{B}^Te^{\mathcal{A}^Tt}\Lambda e^{\mathcal{A}t}\mathcal{B}x_0 \, dt \, dV_n(x_0)$$

$$\Lambda = \begin{bmatrix} Q + C^TH^TRHC & C^TH^TRG \\ G^TRHC & G^TRG \end{bmatrix}. \qquad (6)$$

Moreover, using the trace operator, we write

$$J = Tr\left[\int_0^\infty \mathcal{B}^Te^{\mathcal{A}^Tt}\Lambda e^{\mathcal{A}t}\mathcal{B} \, dt \int_{\Omega_0} x_0x_0^T \, dV_n(x_0)\right].$$

This expression for J can be simplified further by noting that

$$\int_{\Omega_0} x_0x_0^T \, dV_n(x_0) = \beta_n\Sigma^{-1}$$

where β_n is a real positive constant given by

$$\beta_n = \frac{2\pi^{\frac{n}{2}}}{(\det\Sigma)^{\frac{1}{2}}n(n+2)\Gamma(\frac{n}{2})}; \quad n = 1, 2, 3, \ldots$$

and $\Gamma(\cdot)$ is the gamma function (see [4], Section 5.5 for details).

Note that β_n is simply a positive scalar and can be dropped from the cost without altering the optimization problem. Finally, we may (after some algebra) write the cost as

$$J = Tr\,[M\Gamma] \qquad (7)$$

$$\mathcal{A}^TM + M\mathcal{A} + \Lambda = 0 \qquad (8)$$

$$\Gamma = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \qquad (9)$$

and V denotes Σ^{-1}.

The fact that the cost (7) is finite and nonnegative can be established by the following argument. The controllability of (1) implies that there exists a control $u_{[0,\infty)}$ which steers x_0 to the origin. Indeed, this steering can be done in a way that $(x, u) \in L^2_{n+m}$. Moreover, by the obervability assumptions on both the plant and the compensator, we have that the pair $(\mathcal{A}, \mathcal{C})$ is observable where

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ HC & G \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \doteq \mathcal{C}\begin{bmatrix} x \\ z \end{bmatrix}.$$

Hence, we conclude that the pair $(x, z) \in L^2_{n+l}$ which implies $\mathcal{A}$ is a stable matrix. Moreover, Λ and Γ are symmetric positive semidefinite matrices, respectively.

2.1 Statement of the Problem

In view of the above formulation, we are now ready to state the precise optimization problem. The problem we consider is that of minimizing the cost J over all stabilizing compensators given by (3) subject to the constraint (8); i.e.,

$$\min_{(l,E,L,G,H)} J = Tr\,[M\Gamma]$$

subject to (8).

For future reference, we name the above problem the *Deterministic Linear Quadratic (DLQ) Problem.* Our strategy for obtaining the solution of this problem will be as follows: First, we present the solutions to two distinct problems-namely an optimal control and an optimal estimation. Subsequently, we demonstrate via a Lyapunov-based separation principle that the separate solutions of these two problems joinly provide a mininimizing solution to the DLQ problem.

3 Candidate Optimal Compensator

In this section, we discuss two different optimization problems. The first problem considered is an optimal control problem whereby an optimal gain matrix is sought. The second problem consists of an optimal estimation problem where an optimal estimate of the system state is required. Subsequently, we present a candidate optimal compensator which is based on these two solutions.

3.1 Optimal Control Problem

The first problem is a version of the well-known linear quadratic state feedback problem, in which it is assumed that the complete state x is available to the controller u; i.e., consider the completely controllable LTI system

$$\dot{x} = Ax + Bu; \quad x(0) = x_0$$

with the controller u given by $u = Fx$, $F \in \mathbf{R}^{m\times n}$. The required optimal control problem is then formulated by considering the cost functional

$$J_c = \int_{\Omega_0}\int_0^\infty x^TQx + u^TRu \, dt \, dV_n(x_0)$$

and minimizing it with respect to the gain matrix F. It is easy to show that the unique optimal gain F^* is given by

$$F^* = -R^{-1}B^TK \qquad (10)$$

where K is the unique positive semidefinite solution of

$$A^TK + KA - KBR^{-1}B^TK + Q = 0. \qquad (11)$$

Moreover, the optimal cost is given by $J_c^* = Tr\,(KV)$. Indeed, it can be shown that the cost J_c can be manipulated in the same way the cost J was so that

$J_c = Tr\ [M_c V]$ where

$$(A+BF)^T M_c + M_c(A+BF) + Q + F^T RF = 0\ .$$

Then the fact that F^* is the unique minimizing solution follows by noting that

$$(A+BF)^T(M_c - K) + (M_c - K)(A+BF) + \tilde{F}^T R\tilde{F} = 0$$

where $\tilde{F} \doteq F - F^*$. Moreover, $(A+BF)$ is a stable matrix. Hence, by Lyapunov's theorem [5], $(M_c - K) \geq 0$. Hence, we have

$$J_c - J_c^* = Tr[(M_c - K)V] > 0\ \ \forall\, F \neq F^*\ .$$

3.2 Optimal Estimation Problem

The second problem that we consider is an optimal estimation problem where an optimal estimate of the state of a LTI system is sought. It is shown that this problem is dual to a singular optimal control problem [3].

To state the precise problem, consider the completely observable LTI system

$$\begin{aligned} \dot{x} &= Ax\ ;\quad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{12}$$

where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^p$. As before, we assume that C has maximal rank and is already in the form $[0\ I_p]$. Note that systems with a known input term appearing in either part of (12), or with $\rho[C] < p$, may be reduced to the form (12) by redefining the output y. Also, it is important to note that the observability assumption is not restrictive. Indeed, if system (12) is not observable, one can always transform (12) to observability canonical form

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \qquad \tilde{C} = [\ \tilde{C}_1\quad 0\ ]$$

where the pair $(\tilde{A}_{11}, \tilde{C}_1)$ is observable, and simply proceed to state the problem in terms of the observable subsystem.

Furthermore, attention is restricted only to estimates given by

$$\hat{x} = H(y)$$

where, $H : D_p'^+ \rightarrow D_n'^+$ belongs to the class of LTI causal and contiuous estimation operators. Here, $D_k'^+$ denotes the $k-$dimensional space of distributions with support in $[0, \infty)$ (see e.g., [6]).

It is shown in [7] that this class of operators is precisely the set of convolution operators with kernels in the space of distributions. Hence,

$$\hat{x} = H * y$$

where $*$ denotes the convolution operator as defined in [6] and $H \in D_{n\times p}'^+$.

Just how well we estimate the state $x(t)$ may be quantified by means of the cost functional

$$J_e = \int_{\Omega_0} \int_0^\infty e^T e\ dt\ dV_n(x_0) \tag{13}$$

$$e = exp(A)x_0\ -\ H * Cexp(A)x_0$$

$$exp(A) = \begin{cases} 0 & \text{for } t < 0 \\ e^{At} & \text{for } t \geq 0\ . \end{cases}$$

Similar to our previous demonstration, the cost J_e may be written in the form

$$J_e\ =\ Tr \int_0^\infty EVE^T\ dt$$

$$E = exp(A)\ -\ H * Cexp(A)\ . \tag{14}$$

Finally, we define the optimal estimation problem as the minimization of the cost J_e when H ranges over $D_{n\times p}'^+$, subject to (14). Note that we assign $+\infty$ to J_e when $E \in D_{n\times n}'^+ - L_{n\times n}^2$. This is well-justified in view of Proposition 1 in [8].

3.3 Duality

We are now in a position to expose a dual relation between the optimal estimation problem that was stated above and a certain optimal control problem. To begin, we observe the following relabellings of the quantities used in the formulation of this problem; i.e., $\hat{A} \doteq A^T$; $\hat{B} \doteq C^T$; $U \doteq -H^T$; $X \doteq E^T$; $\hat{Q} \doteq V$. Thus, we have

$$J_e\ =\ Tr \int_0^\infty X^T \hat{Q} X\ dt$$

$$X\ =\ exp(\hat{A})\ +\ exp(\hat{A})\hat{B} * U\ . \tag{15}$$

Evidently, minimizing the cost J_e as U ranges over $D_{p\times n}'^+$, subject to (15) is dual to solving the primal optimal estimation problem. It turns out that this problem is related to a problem that we are more familiar with. To clarify, let $x_{0i}^T = [0, \ldots, 0, 1, 0, \ldots, 0]$, where the i^{th} entry is 1. Also, partition the matrix U as

$$U\ =\ [\ u_1,\ \ u_2,\ \ \ldots\ \ u_n\ ]$$

where $u_i \in \mathbf{R}^{p\times 1}$. Then, we can write

$$X\ =\ [\ x_1,\ \ x_2,\ \ \ldots\ \ x_n\ ]$$

$$x_i\ =\ exp(\hat{A})\ x_{0i}\ +\ exp(\hat{A})\hat{B} * u_i$$

for all $i = 1, 2, \ldots, n$. With some matrix algebra, we may write the cost J_e as

$$\begin{aligned} J_e &= \sum_{i=1}^n J_i \\ J_i &= \int_0^\infty x_i^T \hat{Q} x_i\ dt \\ \dot{x}_i &= \hat{A}x_i\ +\ \hat{B}u_i\ ;\quad x_i(0)\ =\ x_{i0}\ . \end{aligned} \tag{16}$$

Hence, minimizing J_e with respect to U can be reduced to minimizing each J_i with respect to u_i. Observe that each optimization is independent; hence, it suffices to find the general solution of (16).

At this point, we simply present the results of the optimal estimation problem. These results are obtained by first solving the dual singular optimal control problem and then simply translating the results (via the duality assignments) to obtain the solution to the primal estimation problem. The full detail of this analysis is interesting in its own right and will be reported in a future paper.

3.4 Optimal Estimation Results

In order to present the results, we partition the matrix A into 4 blocks such that $A_{11} \in \mathbf{R}^{(n-p)\times(n-p)}$, $A_{22} \in \mathbf{R}^{p\times p}$. Similarly, we partition the V matrix into 4 blocks such that V_{11} and V_{22} have the same dimensions as A_{11} and A_{22}, respectively. Then given that $z \in \mathbf{R}^{n-p}$, we have

$$\dot{z} = A_s z + E_s y \; ; \quad z(0) = 0 \tag{17}$$
$$\hat{x} = G_s z + H_s y \tag{18}$$

$$A_s = A_{11} + \Theta_s A_{21} \tag{19}$$
$$E_s = A_{12} + \Theta_s A_{22} - A_s \Theta_s \tag{20}$$
$$G_s = \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix} \tag{21}$$
$$H_s = \begin{bmatrix} -\Theta_s \\ I_p \end{bmatrix} \tag{22}$$
$$\Theta_s = -(P_1 A_{21}^T + V_{12})V_{22}^{-1} \; . \tag{23}$$

Moreover, P_1 is obtained as the unique positive definite solution of

$$A_{11}P_1 + P_1 A_{11}^T + V_{11} - \Theta_s V_{22} \Theta_s^T = 0 \; . \tag{24}$$

3.5 Candidate DLQ Compensator

Based on the two previous solutions, we present a candidate for the optimal compensator. To do this we need to partition the B matrix into 2 column blocks such that the first block $B_1 \in \mathbf{R}^{n-p\times m}$, and the second block $B_2 \in \mathbf{R}^{p\times m}$. Then, we have that

$$l^* = n - p \tag{25}$$
$$E^* = A_s + B_s G^* \tag{26}$$
$$L^* = A_{12} + \Theta_s A_{22} - A_s \Theta_s + B_s H^* \tag{27}$$
$$G^* = F^* G_s \tag{28}$$
$$H^* = F^* H_s \tag{29}$$
$$B_s = B_1 + \Theta_s B_2 \; . \tag{30}$$

The above formulas describe the complete structure of the candidate optimal compensator. Evidently, the compensator consists of a minimal-order Luenberger observer and a constant gain state estimate controller.

4 Separation Principle

Finally, in this section, we show that the candidate compensator given in the previous section is indeed an optimal solution to the DLQ problem. The essence of this verification lies in the following important theorem.

Theorem 4.1 *The indiviual solutions of the optimal control and optimal estimation given by* (10), (11) *and* (17) – (24), *jointly provide a minimizing solution given by* (25) – (30) *to the overall DLQ problem.*

Proof: Recall that the DLQ cost was given by

$$J = Tr[M\Gamma]$$

where M satisfies (8). Write the cost J as

$$J = Tr\,[(M - Y + Y)\Gamma]$$
$$\mathcal{A}^T Y + Y\mathcal{A} + \mathcal{F}^T R \mathcal{F} = 0$$
$$\mathcal{F} = \left[\; (F^* - HC) \quad -G \;\right] .$$

Let $\Delta = M - Y$ and note that

$$\mathcal{A}^T \Delta + \Delta\, \mathcal{A} + (\Lambda - \mathcal{F}^T R \mathcal{F}) = 0 \tag{31}$$

where $F^* = -R^{-1}B^T K$ and K is the unique positive semidefinite solution of the (11). Moreover, observe that we can write

$$\mathcal{A}^T \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} \mathcal{A} + (\Lambda - \mathcal{F}^T R \mathcal{F}) = 0 \; . \tag{32}$$

Subtracting (31) from (32), recalling that $\mathcal{A}$ is a stable matrix and invoking Lyapunov's theorem [5], we conclude that

$$\Delta = \begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} .$$

Hence, we have

$$J = Tr\,[KV] + Tr\,[Y\Gamma]$$

which may be written in the equivalent form

$$J = Tr\,[KV] + Tr\,[\tilde{Y}\mathcal{F}^T R \mathcal{F}]$$
$$\mathcal{A}\tilde{Y} + \tilde{Y}\mathcal{A}^T + \Gamma = 0 \; . \tag{33}$$

Furthermore, it is easily verified that

$$J = Tr\,[KV + PF^{*T}RF^*] + Tr \begin{bmatrix} \tilde{Y}_{11} - P & \tilde{Y}_{12} \\ \tilde{Y}_{12}^T & \tilde{Y}_{22} \end{bmatrix} \mathcal{F}^T R \mathcal{F}$$
$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$P_1 \in \mathbf{R}^{(n-p)\times(n-p)}$ and is the unique positive definite solution of (24). Note that (24) may be written in terms of P; i.e.,

$$AP + PA^T + V - H_s V_{22} H_s^T = 0 \; .$$

Also, the fact that $CP = 0$, enables us to write

$$\mathcal{A}\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\mathcal{A}^T + \Psi = 0 \tag{34}$$
$$\Psi = \begin{bmatrix} V - H_s V_{22} H_s^T & 0 \\ 0 & 0 \end{bmatrix} .$$

Subtracting (34) from (33), we obtain

$$\mathcal{A}X + X\mathcal{A}^T + \tilde{W} = 0$$

$$\tilde{W} = \begin{bmatrix} H_s V_{22} H_s^T & 0 \\ 0 & 0_{l\times l} \end{bmatrix}$$

and

$$X = \begin{bmatrix} \tilde{Y}_{11} - P & \tilde{Y}_{12} \\ \tilde{Y}_{12}^T & \tilde{Y}_{22} \end{bmatrix} \geq 0$$

by Lyapunov's theorem [5]. Note that so far, we have transformed the cost J into the form

$$J = Tr\,[KV + PF^{*T}RF^*] + Tr\,[X\mathcal{F}^T R\mathcal{F}]$$

where the first term in J is simply a constant and thus independent of the compensator parameters; hence only the second term is to be minimized with respect to these parameters.

To finish off the proof, we need only observe that $Tr\,[X\mathcal{F}^T R\mathcal{F}] \geq 0$ and $Tr\,[X\mathcal{F}^T R\mathcal{F}] = 0$ when evaluated at $(l^*, E^*, L^*, G^*, H^*)$. The first part of this observation is clear. To see the second part, we define the transformation

$$\mathcal{T} = \begin{bmatrix} I_n & 0 \\ T & -I_{l^*} \end{bmatrix} ; \quad T = [\; I_{l^*} \quad \Theta_s \;] .$$

Clearly, $\mathcal{T}$ is nonsingular and $\mathcal{T}^{-1} = \mathcal{T}$. Hence, after evaluating the second term in the cost at the $*$ quantities, we can write

$$\begin{aligned} Tr\,[X^*\mathcal{F}^{*T}R\mathcal{F}^*] &= Tr\,[X^*\mathcal{T}^T\mathcal{T}^T\mathcal{F}^{*T}R\mathcal{F}^*\mathcal{T}\mathcal{T}] \\ &= Tr\,[\hat{X}^*\mathcal{T}^T\mathcal{F}^{*T}R\mathcal{F}^*\mathcal{T}] \end{aligned}$$

where $\hat{X}^* \doteq \mathcal{T}X^*\mathcal{T}^T$. Moreover, it is easy to verify that

$$\hat{\mathcal{A}}^*\hat{X}^* + \hat{X}^*\hat{\mathcal{A}}^{*T} + \tilde{W}^* = 0$$

where we have denoted $\hat{\mathcal{A}}^* \doteq \mathcal{T}\mathcal{A}^*\mathcal{T}$. Again, note that we may write

$$Tr\,[\hat{X}^*\mathcal{T}^T\mathcal{F}^{*T}R\mathcal{F}^*\mathcal{T}] = Tr\,[\tilde{X}^*\tilde{W}^*]$$

$$\hat{\mathcal{A}}^{*T}\tilde{X}^* + \tilde{X}^*\hat{\mathcal{A}}^* + \mathcal{T}^T\mathcal{F}^{*T}R\mathcal{F}^*\mathcal{T} = 0 . \qquad (35)$$

Furthermore, straight forward algebra reveals that

$$\begin{aligned} \mathcal{T}^T\mathcal{F}^{*T}R\mathcal{F}^*\mathcal{T} &= \begin{bmatrix} 0_{n\times n} & 0 \\ 0 & G^{*T}RG^* \end{bmatrix} \\ \hat{\mathcal{A}}^* &= \begin{bmatrix} A + BF^* & -BG^* \\ 0 & A_s \end{bmatrix} . \end{aligned}$$

Now provided A_s is a stable matrix, from (35) and Lyapunov's theorem [5], we have

$$\tilde{X}^* = \begin{bmatrix} 0_{n\times n} & 0 \\ 0 & \tilde{X}_{22}^* \end{bmatrix} \geq 0 \qquad (36)$$

$$A_s^T\tilde{X}_{22}^* + \tilde{X}_{22}^*A_s + G^{*T}RG^* = 0 .$$

However, the stability of A_s can be seen by rewriting the equation (24) in terms of A_s; i.e.,

$$A_s P_1 + P_1 A_s^T + \Phi = 0 \qquad (37)$$

$$\Phi = P_1 A_{21}^T V_{22}^{-1} A_{21} P_1 + V_{11} - V_{12} V_{22}^{-1} V_{12}^T$$

and noting that Φ is positive definite and hence by Lyapunov's theorem [5], A_s is a stable matrix.

At this point, it is a trivial matter to see that $Tr\,[\tilde{X}^*\tilde{W}^*] = 0$, and as a consequence, $Tr\,[X^*\mathcal{F}^{*T}R\mathcal{F}^*] = 0$. Therefore, we conclude that

$$\begin{aligned} J &= Tr\,[KV + PF^{*T}RF^*] + Tr\,[X\mathcal{F}^T R\mathcal{F}] \\ &\geq Tr\,[KV + PF^{*T}RF^*] \end{aligned}$$

for all (l, E, L, G, H). Hence, $(l^*, E^*, L^*, G^*, H^*)$ is indeed a minimizing solution. Q.E.D.

5 Conclusion

An optimal compensation problem for deterministic LTI systems with noise-free measurements is formulated. Two subproblems of optimal control and optimal estimation are introduced. Based, on the solution of these two subproblems a candidate solution to the overall problem is suggested. Subsequently, it is proved that this candidate is in fact a minimizing solution. This separation theorem is proved entirely by using Lyapunov's theorem and some matrix manipulations. The arguments used in the proof are inspired by the proof of Russell [2]. Our proof is viewed as an extension of his result to noise-free cases. Furthermore, our result is stronger, since the class of compensators allowed in our formulation is larger than that considered by Russell.

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