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THE MINIMAL DIMENSIONALITY OF STABLE FACES REQUIRED TO GUARANTEE STABILITY OF A MATRIX POLYTOPE

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ABSTRACT

We consider the problem of determining whether a polytope of n×n matrices is stable, by checking stability of low-dimensional faces of the polytope. We show that stability of all (2n-4)-dimensional faces guarantees stability of the entire set. Furthermore, we prove that, for any n and any $k \geq 2n-4$, there exists an unstable polytope of dimension k such that all its (2n-5)-dimensional subpolytopes are stable.

1. Background and Introduction

In this paper we consider the problem of ascertaining whether certain subsets of $\mathbb{R}^{n\times n}$ consist entirely of stable matrices. (Here we take stability of a matrix to mean that all its eigenvalues are in the open left half-plane).

First we need some definitions. A polytope $\mathcal{P}$ in a vector space V is the convex hull of any finite subset of V. The dimension of $\mathcal{P}$ is the dimension of the smallest linear variety containing $\mathcal{P}$. A vertex of $\mathcal{P}$ is any singleton of the form $\Lambda \cap \mathcal{P}$, where Λ is a supporting hyperplane of $\mathcal{P}$. A k-dimensional subpolytope of $\mathcal{P}$ is any k-dimensional polytope which is also the convex hull of some of the vertices of $\mathcal{P}$. A k-dimensional face is a k-dimensional subpolytope of $\mathcal{P}$ contained in the boundary of $\mathcal{P}$. Finally, a k-dimensional half-plane in $\mathcal{V}$ is any set of the form

$$\mathcal{H} = \{ f(x) \in V \mid x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^k,\ x_k \geq 0 \}$$

where $f:\mathbb{R}^k \to V$ is affine and one-to-one.

In the robust control literature, considerable interest has been generated recently by the problem of determining whether stability of a polytope in either $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ can be guaranteed simply by checking stability of low-dimensional faces. (Stability of a vector $x \in \mathbb{R}^n$ means that the polynomial $s^n + x_n s^{n-1} + \ldots + x_1$ is Hurwitz.) For example, for polynomial polytopes of a particularly simple structure ("interval polynomials") Kharitonov [1] showed that only four specially constructed vertices need be checked. A recent result of Bartlett, Hollot, and Lin [2] demonstrates that for an arbitrary polynomial polytope, checking all 1-dimensional faces ("edges") is sufficient to guarantee stability of $\mathcal{P}$. With respect to polytopes in $\mathbb{R}^{n\times n}$, Fu and Barmish [3] have shown that stability of all 1-dimensional subpolytopes is insufficient to guarantee stability of $\mathcal{P}$. DeMarco [4] has shown that, in fact, (n-2)-dimensional faces are insufficient, but that 2n-dimensional faces are sufficient.

In this paper we refine these bounds (for $n \geq 3$) and arrive at an integer m such that checking stability of all m-dimensional faces is sufficient to guarantee stability of $\mathcal{P}$. Furthermore, we show that for any n and $k \geq m$ there exists an unstable polytope of dimension k with all (m-1)-dimensional subpolytopes stable; hence m is minimal. We also show that, for $k < m$, there exists a k-dimensional unstable polytope with all k-dimensional subpolytopes stable.

2. Sufficiency of m=2n-4

We begin with a result characterizing the geometry of the set of unstable points in $\mathbb{R}^{n\times n}$.

Lemma For each unstable $A \in \mathbb{R}^{n\times n}$, there exists an (n^2-2n+4)-dimensional half-plane $\mathcal{H} \subset \mathbb{R}^{n\times n}$ such that 1) $A \in \mathcal{H}$ and 2) $B \in \mathcal{H}$ implies B is unstable.

Proof Case I: A has a real eigenvalue $\lambda_0 \geq 0$. Let $T=[v\ W]$, where v is an eigenvector corresponding to λ_0 and W is chosen to make T nonsingular. Consider the (n^2-n+1)-dimensional half-plane

$$\mathcal{H} = \{ T \begin{bmatrix} \lambda & y \\ 0 & Z \end{bmatrix} T^{-1} \mid \lambda \geq \lambda_0,\ y \in \mathbb{R}^{1\times n-1},\ Z \in \mathbb{R}^{n-1\times n-1} \}$$

Then $A \in \mathcal{H}$ and every matrix in $\mathcal{H}$ is unstable. Also, $n^2-n+1 \geq n^2-2n+4$.

Case II: A has a complex eigenvalue pair $\alpha_0 \pm i\beta_0$ with $\alpha_0 \geq 0$. Let $T=[v\ w\ X]$, where v+iw is an eigenvector corresponding to $\alpha_0 + i\beta_0$ and X is chosen to make T nonsingular. Consider the (n^2-2n+4)-dimensional half-plane

$$\mathcal{H} = \{ T \begin{bmatrix} U & Y \\ 0 & Z \end{bmatrix} T^{-1} \mid \mathrm{tr}U \geq 2\alpha_0,\ Y \in \mathbb{R}^{2\times n-2},\ Z \in \mathbb{R}^{n-2\times n-2} \}$$

$\mathcal{H}$ is unstable, since $\mathrm{tr}U \geq \alpha_0$ implies U has at least one eigenvalue λ with $\mathrm{Re}\lambda \geq \alpha_0$. Also, $A \in \mathcal{H}$, since our choice of T guarantees that A has

$$U = \begin{bmatrix} \alpha_0 & \beta_0 \\ -\beta_0 & \alpha_0 \end{bmatrix}$$ □

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Corollary Stability of every matrix in every (2n-4)-dimensional face of $\mathcal{P}$ guarantees stability of every matrix in $\mathcal{P}$.

Proof If $\mathcal{P}$ contains an unstable A, there exists an (n^2-2n-4)-dimensional half-plane $\mathcal{H}$ consisting entirely of unstable points and containing A. From dimensionality arguments, such a plane must intersect a (2n-4)-dimensional face of $\mathcal{P}$. (See [4] for details.) □

3. Minimality of m=2n-4

In this section we show that, for every integer n, there exists a polytope $\mathcal{P}\in\mathbb{R}^{n\times n}$ containing an unstable point and such that all (2n-5)-dimensional subpolytopes of $\mathcal{P}$ are stable. Hence, we conclude that checking stability of k-dimensional subpolytopes of $\mathcal{P}$, for any k<2n-4 is, in general, not sufficient to guarantee stability of $\mathcal{P}$.

Consider the polytope

$$\mathcal{P}=\left\{\begin{bmatrix} 0 & 1 & -x^T \\ -1 & 0 & -y^T \\ x & y & -I \end{bmatrix} \;\middle|\; \left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1\right\}$$

A routine calculation shows that $\mathcal{P}$ has characteristic polynomial $p(s)=(s+1)^{n-4}\Delta(s)$, where

$$\Delta(s)=s^4+2s^3+(2+x^Tx+y^Ty)s^2 +(2+x^Tx+y^Ty)s+1+x^Txy^Ty-(x^Ty)^2$$

From the Schwartz inequality, it is clear that all coefficients are strictly positive. The corresponding 4×4 Hurwitz matrix has its leading principal 3×3 minor equal to

$$M_3(x,y)=4x^Tx+4y^Ty+4(x^Ty)^2+(x^Tx-y^Ty)^2$$

Clearly, $M_3\ge 0$ with equality iff x=y=0. Thus, $\mathcal{P}$ consists entirely of stable points, except for the relative interior point corresponding to x=y=0. We conclude that checking (2n-5)-dimensional faces (in this case the entire boundary of $\mathcal{P}$) is insufficient to guarantee stability.

Comments 1) The preceding example can be strengthened by adding εI to $\mathcal{P}$, where ε is sufficiently small. This yields a polytope with stable (2n-5)-dimensional boundary, but containing a ball of strictly unstable points.

2) Since the union of all (2n-5)-dimensional subpolytopes is nowhere dense, shifting the parameter set $\left\|\begin{bmatrix} x \\ y \end{bmatrix}\right\|_\infty \le 1$ by an arbitrarily small vector yields an unstable polytope with all (2n-5)-dimensional subpolytopes stable.

3) The polytope $\mathcal{P}$ described above can be transformed into a similar example with any given dimension either by removing parameters or by using $\mathcal{P}$ as a face of a higher dimensional polytope.

Note that the constructions described in 1), 2), and 3) can be carried out simultaneously to given a stronger but algebraically messy version of the minimality proof offered above.

4. Conclusions

We have shown that m=2n-4 is the smallest integer such that stability of all m-dimensional subpolytopes of a given polytope $\mathcal{P}\subset\mathbb{R}^{n\times n}$ guarantees stability of $\mathcal{P}$. Furthermore, we have demonstrated that checking m-dimensional *faces* is always *sufficient*. This reduces the task of determining whether a polytope is stable to that of deciding whether several low-dimensional polytopes are stable. Our result has certain theoretical significance; however, more work needs to be done before it can be decided whether the result will help to reduce the computational burden inherent in robust system design.

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