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Feedback Analysis for Non-Strictly-Proper Systems

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Abstract

We consider feedback systems where the plant and compensator are governed by arbitrary rational matrices. Several conventional methods of closed-loop stability analysis are examined. These include state-space theory, rational matrix analysis, Q-parametrization, and singular system theory. We show that existing methods are inadequate to fully characterize closed-loop stability when non-strictly-proper systems are involved. Our analysis establishes a simple necessary condition under which the closed-loop system is stable. Our condition is then applied to refine the existing methods.

1 Introduction

Consider a feedback system with plant $\mathbf{P}(s)$ and compensator $\mathbf{C}(s)$ as shown in Figure 1. We assume $\mathbf{P}(s)$

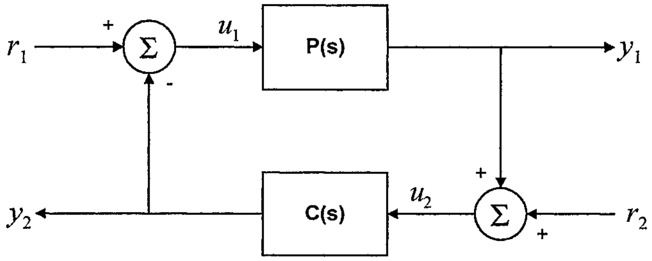

Figure 1: Feedback System

and $\mathbf{C}(s)$ are matrices of rational functions with real coefficients. Ordinarily, stability theory addresses the case where $\mathbf{P}$ is strictly proper and $\mathbf{C}$ is proper. Our intention is to conduct a careful analysis of the case where $\mathbf{P}$ and $\mathbf{C}$ are arbitrary.

Several instances of feedback systems with neither $\mathbf{P}$ nor $\mathbf{C}$ strictly proper do appear in the literature, although the relaxation of strict properness appears to be more an artifact than a serious attempt at understanding such problems. For example, a state-space approach is taken in [3], p.79, [2], p.453, and [5], p.103 which can be summarized as follows. Consider a plant

$$\begin{aligned} \dot{x} &= Ax + Bu_1 \\ y_1 &= Cx + Du_1 \\ \mathbf{P}(s) &= C(sI-A)^{-1}B + D \end{aligned} \tag{1}$$

and compensator

$$\begin{aligned} \dot{z} &= Fz + Gu_2 \\ y_2 &= Hz + Ku_2 \\ \mathbf{C}(s) &= H(sI-F)^{-1}G + K. \end{aligned} \tag{2}$$

In the closed-loop system,

$$u_1 = r_1 - (Hz + K(r_2 + Cx + Du_1)),$$

so

$$u_1 = (I+KD)^{-1}(-KCx - Hz + r_1 - Kr_2). \tag{3}$$

Similarly,

$$u_2 = (I+DK)^{-1}(Cx - DHz + Dr_1 + r_2). \tag{4}$$

Combining (1)-(4) yields

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A - B(I+KD)^{-1}KC & -B(I+KD)^{-1}H \\ G(I+DK)^{-1}C & F - G(I+DK)^{-1}DH \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B(I+KD)^{-1} & -B(I+KD)^{-1}K \\ G(I+DK)^{-1}D & G(I+DK)^{-1} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C - D(I+KD)^{-1}KC & -D(I+KD)^{-1}H \\ K(I+DK)^{-1}C & H - K(I+DK)^{-1}DH \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} D(I+KD)^{-1} & -D(I+KD)^{-1}K \\ K(I+DK)^{-1}D & K(I+DK)^{-1} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$

Consider the simple special case where the plant and compensator have order 0 (i.e x and z are 0-dimensional). Then the closed-loop system is governed by the algebraic equation

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} D(I+KD)^{-1} & -D(I+KD)^{-1}K \\ K(I+DK)^{-1}D & K(I+DK)^{-1} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{5}$$

More specifically, let $D = 2$ and $K = -1$. Then (5) becomes

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \tag{6}$$

which describes a BIBO stable mapping from the inputs of the closed-loop system to the outputs.

Another popular framework in which non-strictly-proper loop gains appear is that of rational matrix analysis. According to a straightforward calculation, the closed-loop system in Figure 1 is governed by

$$\begin{aligned} y_1 &= \mathbf{P}(r_1 - y_2) \\ y_2 &= \mathbf{C}(r_2 + y_1) \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= H \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \end{aligned}$$

where

$$H = \begin{bmatrix} \mathbf{P}(I+\mathbf{CP})^{-1} & -\mathbf{P}(I+\mathbf{CP})^{-1}\mathbf{C} \\ (I+\mathbf{CP})^{-1}\mathbf{CP} & (I+\mathbf{CP})^{-1}\mathbf{C} \end{bmatrix}. \tag{7}$$

We assume $\det(I + \mathbf{CP}) \not\equiv 0$ for well-posedness. Viewing the set of $p \times m$ BIBO stable rational matrices as a ring, the class of all stabilizing compensators $\mathbf{C}$ for a given plant $\mathbf{P}$ can be parametrized according to the Q-parametrization (e.g. see [5], p.108). Suppose $\mathbf{P}$ has right and left-coprime factorizations

$$\mathbf{P} = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}.$$

Then we may solve the left and right Bezout identities

$$\begin{aligned} XN + YD &= I \\ \widetilde{N}\widetilde{X} + \widetilde{D}\widetilde{Y} &= I, \end{aligned}$$

yielding the desired parametrization

$$S(\mathbf{P}) = \left\{ \begin{array}{c|} \left(Y - Q\widetilde{N}\right)^{-1}\left(X + Q\widetilde{D}\right) \\ Q \text{ BIBO stable rational}, \\ \det\left(Y - Q\widetilde{N}\right) \not\equiv 0 \end{array} \right\}. \tag{8}$$

For example, let $\mathbf{P} = 2$. Then we can factor $\mathbf{P}$ into $N = \widetilde{N} = 2$, $D = \widetilde{D} = 1$. One solution of the Bezout identities is $X = \widetilde{X} = 0$, $Y = \widetilde{Y} = 1$, leading to the parametrization

$$S(2) = \left\{ \frac{Q}{1-2Q} \,\middle|\, Q \text{ BIBO stable rational}, \; Q \not\equiv \frac{1}{2} \right\}.$$

In particular, $Q = 1$ gives $\mathbf{C} = -1$. Note that this result is consistent with the previous example (6).

Another setting in which the same type of issue arises is that of singular systems (e.g. see [1]). Let the plant be governed by

$$\begin{aligned} E\dot{x} &= Ax + Bu_1 \\ y_1 &= Cx \end{aligned}$$

$$\mathbf{P}(s) = C(sE - A)^{-1}B \tag{9}$$

and the compensator by

$$\begin{aligned} J\dot{z} &= Fz + Gu_2 \\ y_2 &= Hz \end{aligned}$$

$$\mathbf{C}(s) = H(sJ - F)^{-1}G. \tag{10}$$

The respective characteristic polynomials are $\Delta_p(s) = \det(sE - A)$ and $\Delta_c(s) = \det(sJ - F)$. For well-posedness of the closed-loop system, we assume that the closed-loop characteristic polynomial

$$\Delta_{cl}(s) = \det \begin{bmatrix} sE - A & BH \\ -GC & sJ - F \end{bmatrix} \not\equiv 0.$$

In this setting, closed-loop asymptotic stability is equivalent to the conditions

$$\begin{aligned} \deg \Delta_{cl} &= \operatorname{rank}(E) + \operatorname{rank}(J), \\ &\Delta_{cl} \text{ Hurwitz}. \end{aligned} \tag{11}$$

Letting $E = J = 0$, $A = C = F = G = H = 1$, and $B = -2$ yields $\mathbf{C} = 2$, $\mathbf{P} = -1$, and

$$\Delta_{cl}(s) = \det \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} = -1.$$

Since Δ_{cl} is constant and (11) is satisfied, the closed-loop system is stable.

Finally, we mention that closed-loop stability of our example system is also predicted by the simulation program Simulink. For example, one can compute the closed-loop step response by setting up the Simulink block diagram in Figure 2, yielding the output graphs in Figures 3 and 4.

In fact, any choice of inputs yields numerical outputs consistent with (6).

Hence, we have seen that all existing methods of analysis predict closed-loop stability for $\mathbf{P} = -2$, $\mathbf{C} = 1$. Reducing the example to its essential elements, both theory and simulation predict that an ideal amplifier with gain 2 will be stable under unity feedback as shown in Figure 5. (Recall that the original feedback configuration contains an extra inversion in the loop, cancelling the minus sign in $\mathbf{P}$.) Here we come face to face with a glaring contradiction: *Laboratory experience dictates that such a feedback configuration will never be stable.* For example, one could

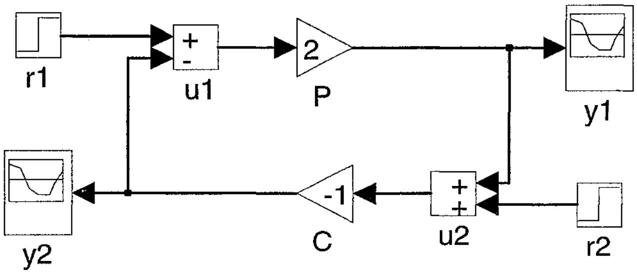

Figure 2: Simulink Diagram

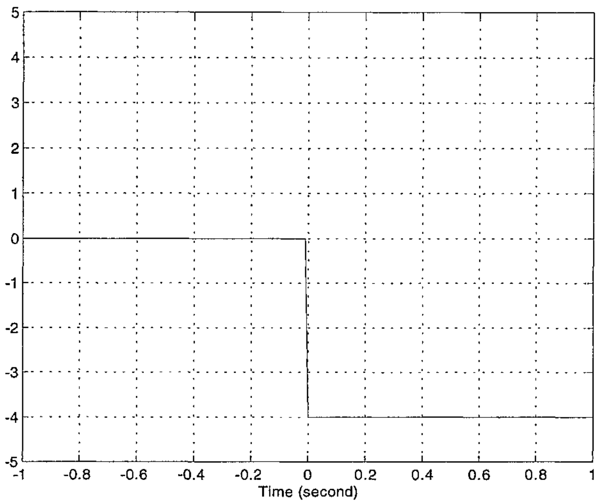

Figure 3: y1

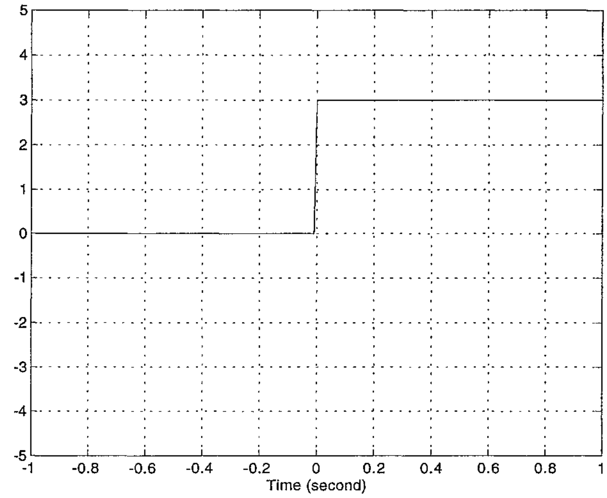

Figure 4: y2

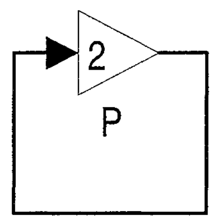

Figure 5: Positive Feedback Loop

wire up an electric circuit which closely approximates this configuration over a large bandwidth. The result would certainly be unstable because of positive feedback. It is clear that a more refined method of analysis is required in order resolve the paradox. An appropriate resolution must ultimately lead to corrections in the analysis methods discussed above.

2 Perturbation Theory

One way to treat the stability problem is to examine strictly proper perturbations of the plant and compensator and then apply conventional stability theory. That is, we first assume that we are given a pair of rational matrices $\mathbf{P}$ and $\mathbf{C}$ such that the closed-loop system is BIBO stable according to conventional analysis. Then we look at how strictly proper perturbations affect closed-loop poles. Central to our discussion will be the rational function

$$R = \det\left(I + \mathbf{CP}\right)$$

and its high-frequency limit

$$R_\infty = \lim_{\sigma \to \infty} R(\sigma) \in [-\infty, \infty].$$

Note that, for H to be BIBO stable, H must be proper, so, from (7),

$$\left(I + \mathbf{CP}\right)^{-1} = I - \mathbf{C}(\mathbf{P}\left(I + \mathbf{CP}\right)^{-1})$$

is proper. Hence, R is not strictly proper, and $R_\infty \neq 0$.

The first question we must address is "What constitutes a perturbation of the plant?" Many topologies in rational function space can been described. In the interest of obtaining the strongest possible results, we adopt the weakest topology that suits our needs. We say that a sequence of rational functions $\mathbf{P}_k \to \mathbf{P}$ *weakly* if there exists $\sigma < \infty$ such that 1) $\mathbf{P}_k$ has no pole in $[\sigma, \infty)$ for large k and 2) $\mathbf{P}_k \to \mathbf{P}$ pointwise on $[\sigma, \infty)$. Weak convergence can be related to more familiar topologies. For example, let

$$\begin{aligned} p_k(s) &= \frac{b_{qk}s^q + ... + b_{0k}}{a_{rk}s^r + ... + a_{0k}} \\ p(s) &= \frac{b_l s^l + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0}. \end{aligned}$$

We say that $p_k \to p$ *parametrically* if

$$\begin{aligned} q &\geq l, \quad r \geq n \\ a_{ik} &\to a_i; \quad i = 0, ..., n-1 \\ a_{nk} &\to 1 \\ a_{ik} &\to 0^+; \quad i = n+1, ..., r \\ b_{ik} &\to b_i; \quad i = 0, ..., l \\ b_{ik} &\to 0; \quad i = l+1, ..., q. \end{aligned}$$

For matrices, we say $\mathbf{P}_k \to \mathbf{P}$ *parametrically* if each entry converges parametrically. One can easily prove the following result.

Theorem 1 *If $\mathbf{P}_k \to \mathbf{P}$ parametrically, then $\mathbf{P}_k \to \mathbf{P}$ weakly.*

Suppose we construct weak perturbations $\mathbf{P}_k \to \mathbf{P}$ and $\mathbf{C}_k \to \mathbf{C}$. Letting $R_k = \det(I + \mathbf{C}_k\mathbf{P}_k)$, it is obvious from the definition of weak convergence that $R_k \to R$ weakly. From (7), the zeros of R are poles of the closed-loop transfer function H. Similarly, the zeros of R_k are poles of the perturbed closed-loop system. This brings us to our central result.

Theorem 2 *If $\mathbf{P}_k$ and $\mathbf{C}_k$ are strictly proper, $\mathbf{P}_k \to \mathbf{P}$ and $\mathbf{C}_k \to \mathbf{C}$ weakly, and $R_\infty < 0$, then there exist $\sigma_k \in R$ such that $\sigma_k \uparrow \infty$ and $R_k(\sigma_k) = 0$ for every k.*

Proof. Since $\mathbf{P}_k$ and $\mathbf{C}_k$ are strictly proper,

$$R_k(\infty) = \det(I + \mathbf{C}_k(\infty)\mathbf{P}_k(\infty)) = \det I = 1.$$

Since $\mathbf{P}_k \to \mathbf{P}$ and $\mathbf{C}_k \to \mathbf{C}$ pointwise, $R_k \to R$ pointwise on some $[\sigma, \infty)$. Hence, there exists a sequence of integers $k_j \uparrow \infty$ such that

$$|R_k(\sigma + j) - R(\sigma + j)| < \frac{1}{j} \quad \forall k \geq k_j, \forall j.$$

Let

$$\alpha_k = \sigma + j \quad if \quad k_j \leq k < k_{j+1}, \quad j = 1, 2, 3, ...$$

Then $\alpha_k \to \infty$ and

$$|R_k(\alpha_k) - R(\alpha_k)| = |R_k(\sigma + j) - R(\sigma + j)| < \frac{1}{j} \qquad if \quad k_j \leq k < k_{j+1},$$

so

$$R_k(\alpha_k) = (R_k(\alpha_k) - R(\alpha_k)) + R(\alpha_k) \to R_\infty.$$

For large k,

$$R_k(\alpha_k) < 0 < R_k(\infty).$$

Since $\mathbf{P}_k$ and $\mathbf{C}_k$ have no pole in $[\sigma, \infty)$, R_k is continuous on $[\alpha_k, \infty)$, so there exist $\sigma_k > \alpha_k$ such that $R_k(\sigma_k) = 0$. ∎

In short, Theorem 2 says that, if $R_\infty < 0$, then every choice of weak, strictly proper perturbations of $\mathbf{P}$ and $\mathbf{C}$ yields closed-loop instability resulting from an arbitrarily large, real, positive pole. Since weak convergence is so weak, every such feedback loop in practice must exhibit extreme instability. Thus a necessary condition for closed-loop stability is $R_\infty > 0$.

Returning to our example $\mathbf{P} = 2$, $\mathbf{C} = -1$, we obtain $R = -1$ and $R_\infty < 0$, which predicts the kind of instability observed in practice. As an example of weak perturbations, consider

$$\mathbf{P}_k(s) = \frac{2}{\frac{1}{k}s + 1}, \quad \mathbf{C}_k(s) = -\frac{1}{\frac{1}{k}s + 1}.$$

Then

$$R_k(s) = 1 - \frac{2}{\left(\frac{1}{k}s + 1\right)^2} = \frac{\frac{1}{k^2}s^2 + \frac{2}{k}s - 1}{\left(\frac{1}{k}s + 1\right)^2},$$

which has zeros $\lambda_k = (-1 \pm 2)\, k$.

3 Applications to Standard Theory

Now we may apply the condition $R_\infty > 0$ to the various frameworks mentioned in Section 1. First, consider state-space systems (1)-(4). Since $(sI - A)^{-1}$ is strictly proper, $\mathbf{P}(\infty) = D$. Similarly, $\mathbf{C}(\infty) = J$, and

$$R_\infty = \det(I + KD).$$

Thus, for stability, the state-space description of $\mathbf{P}$ and $\mathbf{C}$ must satisfy

$$\det(I + KD) > 0.$$

In the case of rational matrix analysis, the appropriate modification of the Q-parametrization follows from

$$\begin{aligned} R &= \det(I + \mathbf{CP}) \\ &= \det\left(I + \left(Y - Q\widetilde{N}\right)^{-1}\left(X + Q\widetilde{D}\right)ND^{-1}\right) \\ &= \frac{\det\left(\left(Y - Q\widetilde{N}\right)D + \left(X + Q\widetilde{D}\right)N\right)}{\det\left(Y - Q\widetilde{N}\right)\det(D)}. \end{aligned}$$

But

$$\left(Y - Q\widetilde{N}\right) D + \left(X + Q\widetilde{D}\right) N$$
$$= XN + YD + Q(\widetilde{D}N - \widetilde{N}D) = I + Q \cdot 0 = I,$$

so

$$R_\infty = \frac{1}{\lim_{\sigma \to \infty} \left(\det\left(Y(\sigma) - Q(\sigma)\widetilde{N}(\sigma)\right) \det\left(D(\sigma)\right)\right)}.$$

Hence, the set of stabilizing compensators for $\mathbf{P}$ is contained in

$$\widehat{S}(\mathbf{P}) = \left\{ \begin{array}{c} \left(Y - Q\widetilde{N}\right)^{-1}\left(X + Q\widetilde{D}\right) \Big| \\ Q \text{ BIBO stable rational,} \\ \det\left(Y(\infty) - Q(\infty)\widetilde{N}(\infty)\right) \det\left(D(\infty)\right) > 0 \end{array} \right\},$$

which is smaller than $S(\mathbf{P})$ (8).

To analyze the singular system case, we recall from [6], p.159, that

$$R(s) = \det\left(I + \mathbf{C}(s)\mathbf{P}(s)\right) = \frac{\Delta_{cl}(s)}{\Delta_p(s)\Delta_c(s)}.$$

Invoking the Weierstrass decomposition (see [4], p.28), there exist nonsingular matrices M_p, N_p, M_c, and N_c such that

$$\begin{aligned} M_p E N_p &= \begin{bmatrix} I_{n_{ps}} & 0 \\ 0 & A_f \end{bmatrix}, \; M_p A N_p = \begin{bmatrix} A_s & 0 \\ 0 & I_{n_{pf}} \end{bmatrix}, \\ M_c J N_c &= \begin{bmatrix} I_{n_{cs}} & 0 \\ 0 & F_f \end{bmatrix}, \; M_c F N_c = \begin{bmatrix} F_s & 0 \\ 0 & I_{n_{cf}} \end{bmatrix}, \end{aligned}$$

with A_f and F_f nilpotent. Then

$$\begin{aligned} \Delta_p(s) &= \det(sI - A_s)\det(sA_f - I) \qquad (12) \\ &= (-1)^{n_{pf}} \det(sI - A_s), \\ \Delta_c(s) &= \det(sI - F_s)\det(sF_f - I) \\ &= (-1)^{n_{cf}} \det(sI - F_s), \\ \Delta_{cl}(s) &= \det \begin{bmatrix} sI - A_s & B_s H_s & 0 & B_s H_f \\ -G_s C_s & sI - F_s & -G_s C_f & 0 \\ 0 & B_f H_s & sA_f - I & B_f H_f \\ -G_f C_s & 0 & -G_f C_f & sF_f - I \end{bmatrix}. \end{aligned}$$

R_∞ is completely determined by the degrees and leading coefficients of the three polynomials. Assuming closed-loop stability, it follows from (11) and (12) that

$$\deg \Delta_{cl} = \operatorname{rank}(E) + \operatorname{rank}(J) \geq n_{ps} + n_{cs} = \deg(\Delta_p) + \deg(\Delta_c)$$

with equality iff $A_f = 0$ and $F_f = 0$. Let γ be the leading coefficient of

$$\Gamma(s) = \det \begin{bmatrix} I - sA_f & -B_f H_f \\ G_f C_f & I - sF_f \end{bmatrix}.$$

Then

$$R_\infty = \begin{cases} \gamma, \text{ if } A_f = 0, \; F_f = 0 \\ \gamma \cdot \infty, \text{ else} \end{cases}.$$

Finally, for numerical simulation, it is an easy matter to compute R_∞ within any of the theoretical frameworks discussed above. Then it is a simple matter to generate an error message whenever an "algebraic loop" with $R_\infty < 0$ is encountered.

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